E. Lefschetz Theorems

We need some algebraic & geometric preliminaries.

$\boxed{\text{Algebra}}$

Recall that a Lie algebra is a vector space ($\mathbb{R}$ or $\mathbb{C}$ will do) $\mathfrak{g}$ together with an alternating bilinear pairing $[\,,\,]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (the Lie bracket) satisfying the Jacobi identity $[x_1,[x_2,x_3]]+[x_2,[x_3,x_1]]+[x_3,[x_1,x_2]]=0$.

A homomorphism $\mathfrak{g}\to\mathfrak{g}'$ of Lie algebras is a linear map compatible with brackets, and a (finite-dimensional) representation of $\mathfrak{g}$ on a vector space V is a Lie algebra hom. $\mathfrak{g}\to \mathrm{End}(V)$.

Example 1: $\mathfrak{sl}_2$ is the $\underbrace{\text{matrix}}_{[A,B]:=AB-BA}$ Lie algebra with generators

$$n_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad n_-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad y=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and nontrivial brackets $[y,n_+]=2n_+$, $[y,n_-]=-2n_-$, $[n_+,n_-]=y$.

For $l\geq 0$, the l^{th} standard irrep (irreducible representation) $\mathcal{V}_l := \text{span}\{v_{-l}, v_{-l+2},\ldots, v_l\}$ is defined by

$$y(v_k)=k\cdot v_k$$
$$n_+(v_k)=\begin{cases}0, & k=l\\ v_{k+2}, & \text{otherwise}\end{cases}$$
$$n_-(v_k)=\begin{cases}0, & k=-l\\ v_{k-2}, & \text{otherwise}\end{cases}.$$

Lemma 1: (i) These are all the irreps. ($\dim<\infty$)

(ii) Any representation V of $\mathfrak{sl}_2$ has a unique decomposition into isotypical components

$$V=\oplus V[l],\quad \text{where } V[l]\cong \mathcal{V}_l^{\oplus m}\quad (m\in\mathbb{Z}).$$

(Pf: omitted)

Considering the eigenspaces

$$V(k) := \ker(y - k\mathbb{1}) \subset V, \quad V[l](k) := V[l] \cap V(k),$$

from $V(k) = \bigoplus_{l \geq |k|} V[l](k)$ we see immediately the

<u>Lemma 2</u>: For $k \geq 0$, $n_+^k : V(-k) \to V(k)$ is an $\cong$

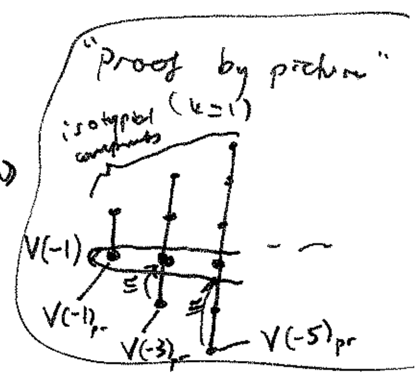
and $n_+^j : V(-k) \hookrightarrow V(-k+2j)$ for $0 \leq j \leq k$.

Defining the primitive subspaces $(k \geq 0)$

$$V(-k)_{pr} := \ker(n_+^{k+1}) = \ker(n_-),$$

we also have

<u>Lemma 3</u>: $V(-k) = \bigoplus_{j \geq 0} n_+^j \left(V(-k-2j)_{pr} \right)$. $\rightsquigarrow$



Geometry

Let M = compact Kähler n-fold
ω = Kähler form

Recall $L = \wedge \omega : A^m \to A^{m+2}$, $\quad \Lambda = L^* : A^m \to A^{m-2}$

$$\begin{array}{ccc} A^m & \to & A^{m+2} \\ \cup & & \cup \\ A^{p,q} & \to & A^{p+1,q+1} \end{array} \qquad \begin{array}{ccc} A^m & \to & A^{m-2} \\ \cup & & \cup \\ A^{p,q} & \to & A^{p-1,q-1} \end{array}$$

and define

$$Y = (m-n)\mathbb{1} : \begin{array}{ccc} A^m & \to & A^m \\ \cup & & \cup \\ A^{p,q} & \to & A^{p,q} \end{array}$$

These are <u>real</u> operators of bidegrees $(1,1)$, $(-1,-1)$, resp. $(0,0)$.

<u>Lemma 4</u>: $[Y, L] = 2L$, $[Y, \Lambda] = -2\Lambda$, $[L, \Lambda] = Y$.

<u>Proof</u>: The 1st 2 are easy, viz. $(\alpha \in A^m)$

$$[Y, L]\alpha = \underbrace{Y(\alpha \wedge \omega)}_{(m+2-n)(\alpha \wedge \omega)} - \underbrace{Y(\alpha)}_{(m-n)\alpha} \wedge \omega = 2\alpha \wedge \omega = 2L\alpha.$$

For the 3$^{\text{rd}}$, w.m.a. (working at some $p \in M$)

$$\omega = i \sum \underbrace{\phi_j \wedge \bar\phi_j}_{=:\omega_j} \quad , \text{where } \phi_j := \frac{dz_j}{\sqrt{2}} .$$

It suffices to check this on a monomial $\phi_I \wedge \bar\phi_J \wedge \omega_K$ (where $\{1,\dots,n\} = H \amalg I \amalg J \amalg K$),

and as usual I'm going to forget signs / powers of i (they cancel in the end)

$$\begin{aligned}
(\omega\wedge) * (\omega\wedge) * \phi_I \wedge \bar\phi_J \wedge \omega_K &= (\omega\wedge) * (\omega\wedge)\, \phi_I \wedge \bar\phi_J \wedge \omega_H \\
&= (\omega\wedge) * \sum_{h \in K} \phi_I \wedge \bar\phi_J \wedge \omega_{H \cup \{h\}} \\
&= (\omega\wedge) \sum_{h \in K} \phi_I \wedge \bar\phi_J \wedge \omega_{K \setminus \{h\}} \\
&= \underbrace{\sum_{\substack{h \in K, \\ k \in H}} \phi_I \wedge \bar\phi_J \wedge \omega_{K \cup \{k\} \setminus \{h\}}} + |K|\, \phi_I \wedge \bar\phi_J \wedge \omega_K
\end{aligned}$$

Similarly, $* (\omega\wedge) * (\omega\wedge)\, \phi_I \wedge \bar\phi_J \wedge \omega_K = \quad '' \quad + |H|\, \phi_I \wedge \bar\phi_J \wedge \omega_K$.

But $|K| - |H| = (2|K| + |I| + |J|) - (|K| + |I| + |J| + |H|) = m - n$

and so we are done. $\square$

<u>Lemma 5</u>: Δ commutes with L, Λ, Y.

<u>Proof</u>: $[\Delta, Y] = 0$ is obvious since Δ doesn't change the degree.

$$[\Delta, L] \underset{\substack{\uparrow \\ \Delta = 2\Delta_\partial}}{=} 2\left([\partial\partial^*, L] + [\partial^*\partial, L]\right) \underset{\substack{\uparrow \\ d\omega = 0 \\ \Rightarrow \partial\omega = 0 \\ \text{(since of type} \\ (1,1))}}{=} 2\left(\partial[\partial^*, L] + [\partial^*, L]\partial\right)$$

$$\underset{\substack{\uparrow \\ \text{Kähler identity} \\ -i\bar\partial^* = [\Lambda, \partial] \xRightarrow[\text{take adjoints}]{} -i\bar\partial = [\partial^*, \Lambda]}}{=} -2i\left(\partial\bar\partial + \bar\partial\partial\right) = 0 .$$

$[\Delta, \Lambda] = 0$ is similar. $\square$

Hence, sending $\begin{cases} y \mapsto Y \\ n_+ \longmapsto L \\ n_- \longmapsto \Lambda \end{cases}$ yields a real $\mathfrak{sl}_2$-representation

on $V \equiv \bigoplus_{m=0}^{2n} \mathcal{H}^m(M) \cong \bigoplus_{m=0}^{2n} H^m(M,\mathbb{C})$, with $\mathcal{H}^m(M) = V(m-n)$,

proving (by Lemma 2) the

Theorem 1 ("Hard Lefschetz"): For each $k \leq n$,

$$L^k : H^{n-k}(M,\mathbb{C}) \to H^{n+k}(M,\mathbb{C}) \quad \text{is an } \cong,$$

$$\text{and } L^j : H^{n-k}(M,\mathbb{C}) \hookrightarrow H^{n-k+2j}(M,\mathbb{C}) \quad \text{for } 0 \leq j \leq k.$$

Corollary 1: (i) $h^0 \leq h^2 \leq h^4 \leq \cdots \leq h^{2\lfloor \frac{n}{2} \rfloor}$

(ii) $h^1 \leq h^3 \leq \cdots \leq h^{2\lfloor \frac{n+1}{2} \rfloor - 1}$

(iii) $h^{n-k} = h^{n+k}$ ← (we already know this from Poincaré duality)

Corollary 2: $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$ ← (already clear from Serre duality)

Definition 1: The Hodge diamond records the Hodge #'s and takes the form:

$$\begin{array}{lccccccc}
 & & & & h^{n,n} & & & \\
 & & & \cdot & & \cdot & & \\
 & h^{0,n} & & & & & & h^{n,0} \\
 & & & & \vdots & & & \\
m = h^2 & & h^{0,2} & & h^{1,1} & & h^{2,0} & \\
m = h^1 & & & h^{0,1} & & h^{1,0} & & \\
h^0 = & & & & h^{0,0} & & &
\end{array}$$

e.g. for a compact Kähler 1, 2, or 3-fold we have (taking into account Cor. 2)

$$(n=1)\quad \begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array} \qquad (n=2)\quad \begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ g & & r & & g \\ & q & & q & \\ & & 1 & & \end{array} \qquad (n=3)\quad \begin{array}{ccccccc} & & & 1 & & & \\ & & c & & c & & \\ & d & & a & & d & \\ g & & b & & b & & g \\ & d & & a & & d & \\ & & c & & c & & \\ & & & 1 & & & \end{array}$$

Example 2: M is Calabi-Yau $\iff$ $\begin{cases} \text{compact Kähler} \\ K_M \cong \mathcal{O}_M \ (\Rightarrow h^{n,0} = 1) \\ h^{1,0} = h^{2,0} = \cdots = h^{n-1,0} = 0 \end{cases}$

For $n=2$, this is called a K3 surface.

For $n=1$, this is just an elliptic curve (= genus 1 RS).

The Hodge diamonds become

(n = 1)
$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

(n = 2)
$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & r & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(n = 3)
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & a & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(one can show $r = 20$) $\square$

Defining

$(k \leq n)$ $$H^k_{pr}(M, \mathbb{C}) := \ker\left(L^{n-k+1}\right) \subset H^k(M, \mathbb{C})$$

$(p+q = k)$ $$H^{p,q}_{pr}(M) := H^{p,q}(M) \cap H^k_{pr}(M, \mathbb{C}),$$

Lemma 3 $\Rightarrow$

<u>Theorem 2 (Lefschetz decomposition)</u>: $$H^k(M, \mathbb{C}) = \bigoplus_{0 \leq j \leq \lfloor \frac{k}{2} \rfloor} L^j H^{k-2j}_{pr}(M, \mathbb{C})$$

$$H^{p,q}(M) = \bigoplus_j L^j H^{p-j, q-j}_{pr}(M).$$

(Note: these depend on the choice of ω, unlike the Hodge decomposition.)

<u>Definition 2</u>: The Weil operator

$$C : A^k \to A^k$$

is the unique $\mathbb{C}$-linear map with $C|_{A^{p,q}} = i^{p-q} \cdot \mathbb{1}$ (for each $p+q = k$). Clearly it commutes with Δ $(= \Delta_J)$, hence operates on $\mathcal{H}^k / H^k$.

<u>Lemma 6</u>: On $A^k_{pr} := \ker(L^{n-k+1}) \subset A^k$,

$$* L^{n-k} = (-1)^{\frac{k(k+1)}{2}} (n-k)! \, C$$

(Will prove at the end in a special case. Note that it is false on the whole A^k, so just checking on monomials won't work. You need to check on generators of A^k_{pr}, which makes this one ugly.)

Now define, for $0 \leq k \leq N$, a bilinear form

$$Q_k : H^k(M, \mathbb{C}) \times H^k(M, \mathbb{C}) \to \mathbb{C}$$

by $Q_k([\alpha],[\beta]) := (-1)^{k(k+1)/2} \int_M \alpha \wedge \beta \wedge \omega^{n-k}$. Clearly this is well defined and $\begin{cases} \text{symmetric for } k \text{ even} \\ \text{alternating for } k \text{ odd} \end{cases}$.

<u>Theorem 3 (Hodge–Riemann bilinear relations):</u>

(HRBR I) $Q_k(H^{p,q}(M), H^{p',q'}(M)) = 0$ unless $p' = q$ and $q' = p$.

(HRBR II) $(-1)^{\frac{k(k+1)}{2}} Q_k((\cdot), C\overline{(\cdot)}) > 0$ on <u>the primitive space</u>, that is, for $\alpha \in H^{p,q}_{pr}(M) \setminus \{0\}$, $\underline{(-1)^{\frac{k(k+1)}{2}} i^{q-p} Q_k(\alpha, \bar{\alpha}) > 0}$.

<u>Proof</u>: (I) clear: otherwise the integrand vanishes

(II) Lemma 6 $\Rightarrow$ $(-1)^{k(k+1)/2} \omega^{n-k} \wedge \beta = *^{-1}(n-k)!\, C(\beta)$ for $\beta \in A^k_{pr}(M)$.

On $H^k(M)$, $C^2 = (-1)^k \mathbb{1} = *^2$, so taking $\beta = C(\bar{\alpha})$,

$$Q(\alpha, C\bar{\alpha}) = (-1)^{k(k+1)/2} \int_M \alpha \wedge C(\bar{\alpha}) \wedge \omega^{n-k}$$

$$= (n-k)! \int_M \alpha \wedge \underbrace{*^{-1}}_{(-1)^k *} \underbrace{C^2}_{(-1)^k} \bar{\alpha}$$

$$= (n-k)! \underbrace{\int_M \alpha \wedge * \bar{\alpha}}_{\|\alpha\|^2}$$

$$> 0.$$

$\square$

Example 3: For a curve ($n=1$), this says that for Q_1 = cup-product,

$$Q_1(H^{1,0}, H^{1,0}) = 0$$
$$iQ_1(\alpha, \bar{\alpha}) > 0 \quad \text{for } \alpha \in H^{1,0}\setminus\{0\}$$

which are forms of the Riemann bilinear relations proved above.

Example 4: For M projective, we have a rational Kähler class $[\omega]$, and so Q_k is defined $/\mathbb{Q}$. This leads to the concept of a polarized Hodge structure (which we won't discuss now).

Remark 1: The reason why HRBR II won't work on all of $H^k / H^{p,q}$, is that in the Lefschetz decomposition

$$(-1)^{\frac{k(k+1)}{2}} Q_k((\cdot), C\overline{(\cdot)}) \text{ will be } \begin{cases} >0 & \text{on terms with } j \text{ even} \\ <0 & \text{on terms with } j \text{ odd} \end{cases}.$$

(This is basically because $(-1)^{\frac{(k-2j)((k-2j)+1)}{2}} = (-1)^j(-1)^{\frac{k(k+1)}{2}}$.)

One can take the $\mathbb{C}$-linear extension of the $(-1)^j Q_k\big|_{L^j H^{k-2j}_{pr}}$ to fix this, if desired.

Idea of pf. of Lemma 6: In general $A^k_{pr} = \ker \Lambda$ for $0 \le k \le n$.

Say $n=k=2$. Then the identity says that

$$* = -C \;:\; A^2_{pr} \to A^2_{pr}.$$

Let's compute $*$ on the basis (o.n. in Hodge metric)

$$
\left.\begin{array}{ccc}
\phi_1 \wedge \bar\phi_2 & \overset{*}{\longmapsto} & -\phi_1 \wedge \bar\phi_2 \\
\phi_2 \wedge \bar\phi_1 & \longmapsto & -\phi_2 \wedge \bar\phi_1 \\
\omega_1 & \longmapsto & \omega_2 \\
\omega_2 & \longmapsto & \omega_1
\end{array}\right\} A^{1,1}
$$

$$
\left.\begin{array}{ccc}
\phi_1 \wedge \phi_2 & \longmapsto & \phi_1 \wedge \phi_2
\end{array}\right\} A^{2,0}
$$

$$
\left.\begin{array}{ccc}
\bar\phi_1 \wedge \bar\phi_2 & \longmapsto & \bar\phi_1 \wedge \bar\phi_2
\end{array}\right\} A^{0,2}
$$

Now on $A^{2,0} \oplus A^{0,2}$, $-C = 1$; while

on $A^{1,1}$, $-C = -1$.

So this will only work on the span of $\phi_1 \wedge \bar\phi_2$, $\phi_2 \wedge \bar\phi_1$, and $\omega_1 - \omega_2$, which is precisely the kernel of Λ (i.e. of $*^{-1} \underline{L*}$, or really just of L in this case).

See Voisin for the full proof. The $(-1)^{k(k+1)/2}$ comes from shuffling $dz_I \wedge d\bar{z}_I$'s to get ω_I.