E. Lefschetz Theorems

We need some algebraic & geometric preliminaries.

**Algebra**

Recall that a Lie algebra is a vector space \((K = \mathbb{R} or \mathbb{C} \text{ will do})\) \((\mathfrak{g}, \mathfrak{g})\) together with an alternating bi-linear pairing \([, ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) (the lie bracket) satisfying the Jacobi identity \([x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0\).

A homomorphism \(\mathfrak{g} \to \mathfrak{g}'\) of lie algebras is a linear map compatible with brackets, and a (finite-dimensional) representation of \(\mathfrak{g}\) on a vector space \(V\) is a lie algebra hom. \(\mathfrak{g} \to \text{End}(V)\).

**Example 1**:

\(\mathfrak{sl}_2\) is the matrix Lie algebra with generators

\[
\begin{align*}
\mathfrak{n}_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, &
\mathfrak{n}_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, &
\mathfrak{y} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

and nontrivial brackets \([\mathfrak{y}, \mathfrak{n}_+] = 2\mathfrak{n}_+\), \([\mathfrak{y}, \mathfrak{n}_-] = -2\mathfrak{n}_-\), \([\mathfrak{n}_+, \mathfrak{n}_-] = \mathfrak{y}\).

For \(k \geq 0\), the \(k\)th standard \(\mathfrak{sl}_2\) \(\mathfrak{g}_k := \text{span} \{\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{n}_+ \mathfrak{n}_-, \ldots, \mathfrak{y} \mathfrak{y} \}\) is defined by

\[
\begin{align*}
y(\mathfrak{n}_k) &= k \cdot \mathfrak{n}_k, \\
n_-(\mathfrak{n}_k) &= \begin{cases} 0, & k = 0 \\
k, & k \neq 0, \text{ otherwise} \end{cases} \\
n_+(\mathfrak{n}_k) &= \begin{cases} 0, & k = -1 \\
k+1, & k \neq -1, \text{ otherwise} \end{cases}
\end{align*}
\]

( irreducible representation)

**Lemma 1**: (i) These are all the irreps. (dim \(\mathfrak{g}_k\))

(ii) Any representation \(V\) of \(\mathfrak{sl}_2\) has a unique decomposition into isotypical components

\[
V = \bigoplus V[l], \text{ where } V[l] \cong \mathfrak{g}_l^\oplus m \text{ (mod \(\mathbb{Z}\)).}
\]

(Pf: omitted)
Considering the eigenspaces
\[ V(k) = \ker (y - k I) \subset V, \quad V(k) = V(k) \cap V(1), \]
from \[ V(k) = \bigoplus_{l=1}^{k} V(l)(k) \] we see immediately the

**Lemma 2:** For \( k \geq 0 \), \( n^+_{k} : V(-k) \to V(k) \) is an isomorphism and \( n^+_{k} : V(-k) \to V(-k+2j) \) for \( 0 \leq j \leq k \).

Defining the primitive subspaces \((k \geq 0)\)
\[ V(-k)_{pr} := \ker (n^+_{k+1}) = \ker (n^-_{k}), \]
we also have

**Lemma 3:** \[ V(-k) = \bigoplus_{j=0}^{k} n^+_j \left( V(-k-2j)_{pr} \right). \]

**Geometry**

Let \( M = \text{compact Kähler n-fold} \)
\[ \omega = \text{Kähler form} \]

Recall \( L = \wedge \omega : A^m \to A^{m+2} \)
\[ \Lambda = L^k : A^m \to A^{m-2} \]
\[ U : A^{m-n} \to A^{m-1,n+1} \]
\[ U : A^{m-m} \to A^{m-1,n-1} \]

and define \( Y = (m-n)I : A^m \to A^m \)
\[ U : A^{n,n} \to A^{m,n} \]

These are real operators of bidegrees \((1,1), (-1,-1), \text{resp.} (0,0)\).

**Lemma 4:** \( [Y, L] = 2L, \quad [Y, \Lambda] = -2\Lambda, \quad [L, \Lambda] = Y \).

**Proof:** The first two are easy, viz.
\[ [Y, L] \alpha = \frac{Y(d\alpha \wedge \omega) - Y(\alpha) \wedge \omega}{(2 \alpha \wedge \omega) \wedge \omega(n-1)} = 2 \alpha \wedge \omega. \]
For the 3-rd, w.m.o. (writing at some p ∈ M)

\[ \omega = i \sum \phi_j \wedge \overline{\phi}_j, \quad \text{where} \quad \phi_j = \frac{dz_j}{\sqrt{2}}. \]

It suffices to check this on a monomial \( \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_K \) (where \( \phi_i^k \) is \( \text{Hull} \) \( j \)), and as usual I'm going to forget signs/powers of \( i \) (they cancel in the end)

\[
\begin{align*}
(\omega \wedge) \wedge (\omega \wedge) \wedge \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_K &= (\omega \wedge) \wedge (\omega \wedge) \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_{\text{Hull}} \\
&= (\omega \wedge) \wedge \sum_{k \in K} \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_{\text{Hull}}(k) \\
&= (\omega \wedge) \wedge \sum_{k \in K} \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_K(k) \\
&= \sum_{k \in K} \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_{\text{Hull}}(k) + |K| \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_K
\end{align*}
\]

Similarly, \( (\omega \wedge) \wedge (\omega \wedge) \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_K = \sum_{k \in K} \phi_i^k \wedge \overline{\phi}_j^k \wedge \omega_K \).

But \( |K| - |H| = (2|k| + |z_1 + \bar{z}_1|) - (|k| + |z_1| + |k| + |z_1|) = m - n \)

and so we are done. \( \square \)

**Lemma 5**: \( \Delta \) commutes with \( L, \Lambda, Y \).

**Proof**: \( [\Delta, Y] = 0 \) is obvious since \( \Delta \) doesn't change the degree.

\[
\begin{align*}
[\Delta, L] &= 2 \left( [\Delta^* \ast, L] + [\Delta^* \ast, L] \right) \\
&\uparrow \\
\Delta &= 2\Delta_d \\
&\Rightarrow \Delta \wedge = 0 \\
&\text{(since \( \ast \) is \text{type} \( (1,1) \))}
\end{align*}
\]

\[
\begin{align*}
\delta \omega &= 0 \\
\Rightarrow \delta \wedge = 0
\end{align*}
\]

(Kähler identity)

\[-i \delta = [\Lambda, \delta] \Rightarrow -i \delta = [\Lambda^*, L] \]

\[
[\Delta, L] = 0 \quad \text{is similar.} \quad \square
\]
Hence, sending \( y \mapsto Y \) yields a real \( SL_2 \)-representation.

\[
\begin{aligned}
  n_+ &\mapsto L \\
  n_- &\mapsto \Lambda
\end{aligned}
\]

or \( \bigoplus_{m=0}^{2n} H^m(M) \cong \bigoplus_{m=0}^{2n} H^m(M,\mathbb{C}) \), with \( H^m(M) = \sqrt{m-n} \),

proving (by Lemma 2) the

**Theorem 1 (Hard Lefschetz):** For each \( k \leq n \),

\[
L^k : H^{n-k}(M,\mathbb{C}) \to H^{n+k}(M,\mathbb{C}) \quad \text{is an isomorphism,}
\]

and \( L^0 : H^{n-k}(M,\mathbb{C}) \to H^{n-k+2g}(M,\mathbb{C}) \) for \( 0 \leq j \leq k \).

**Corollary 1:**

(i) \( h^0 \leq h^1 \leq h^2 \leq \cdots \leq h^2(\frac{n}{2}) \)

(ii) \( h^1 \leq h^2 \leq \cdots \leq h^{2\left(\frac{n}{2}\right)-1} \)

(iii) \( h^{n-k} = h^{n+k} \quad \text{(obviously from Poincaré duality)} \)

**Corollary 2:** \( h^{0,q} = h^{q,0} = h^{n-p,n-q} = h^{n-q,n-p} \quad \text{(already clear from some duality)} \)

**Definition 1:** The Hodge diamond records the Hodge numbers and takes the form:

![Hodge Diamond](image)

**Example 2:** \( M \) is Calabi-Yau (\( \cong \)).

For \( n = 2 \), this is called a K3 surface.

For \( n = 1 \), this is just an elliptic curve (a genus 1 curve).

\[
\begin{aligned}
  &\text{compact Kähler:} \\
  &K_M = \Omega_M \quad (\Leftrightarrow h^{0,0} = 1) \\
  &h^{1,0} = h^{2,0} = \cdots = h^{n-1,0} = 0
\end{aligned}
\]
The Hodge diamonds become:

\[
\begin{array}{cccc}
(n=1) & (n=2) & (n=3) & (n=4) \\
& 1 & 0 & 0 & 1 \\
& 1 & 0 & 1 & 0 \\
& 0 & 0 & 0 & 0 \\
& 0 & 1 & 0 & 0 \\
& 0 & 0 & 0 & 0 \\
\end{array}
\]

...and one can show \( r = 20 \).

Defining:

\[
(k \leq n) \quad H^k_{pr}(M, \mathbb{C}) := \ker (L^{n-k+1}) \subset H^k_{pr}(M, \mathbb{C})
\]

and

\[
(p+q=k) \quad H^p_{pr}(M) := H^{p,q}(M) \cap H^k_{pr}(M, \mathbb{C})
\]

Lemma 3 \( \Rightarrow \)

**Theorem 2 (Lefschetz decomposition):**

\[
H^k(M, \mathbb{C}) = \bigoplus_{0 \leq j \leq \lfloor \frac{k}{2} \rfloor} L^j H^{k-2j}_{pr}(M, \mathbb{C})
\]

\[
H^p_{pr}(M) = \bigoplus_{j} L^j H^{p-j,q+j}_{pr}(M).
\]

(Note: these depend on the choice of \( \omega \), unlike the Hodge decomposition.)

**Definition 2:** The Weil operator

\[
C : A^k \to A^k
\]

is the unique \( \mathbb{C} \)-linear map with \( C_{|A^k} = i^{p-1} I \) (for each \( p+q=k \)). Clearly it commutes with \( \Delta (= \Delta^g) \), hence operates on \( \mathbb{R}^k/\mathbb{Z}^k \).

**Lemma 6:** On \( A^k_{pr} := \ker (L^{n-k+1}) \subset A^k \),

\[
L^{n-k} = (-1)^{\frac{k(km)}{2}} (n-k)! C
\]

(Will prove at the end. Note that it is false on the whole \( A^k \), so just checking on manifolds would work. You need to check on generators of \( A^k_{pr} \), which makes this one ugly.)
Now define, for \( 0 \leq k \leq N \), a bilinear form

\[
Q_k : H^k(M, \mathbb{C}) \times H^k(M, \mathbb{C}) \to \mathbb{C}
\]

by

\[
Q_k(\alpha, \beta) := (-1)^{\frac{k(k+1)}{2}} \int_M \alpha \wedge \beta \wedge \omega^{n-k}.
\]

Clearly this is well defined and symmetric for \( k \) even, alternating for \( k \) odd.

**Theorem 3 (Hodge–Riemann bilinear relations):**

(HRBR I) \( Q_k(H^{p,q}(M), H^{p',q'}(M)) = 0 \) unless \( p' = q \) and \( q' = p \).

(HRBR II) \((-1)^{\frac{k(k+1)}{2}} Q_k(\cdot, C(\cdot)) > 0 \) on the primitive space, that is, for \( \alpha \in H^{p,q}(M) \) holomorphic,

\[
(-1)^{\frac{k(k+1)}{2}} \gamma^{p-p'} Q_k(\alpha, C(\alpha)) > 0.
\]

**Proof:** (I) clear; otherwise the integral vanishes.

(II) Lemma (I) \( \implies \)

\[
(-1)^{\frac{k(k+1)}{2}} \int_M \omega^{n-k} \wedge \beta = \gamma^{-(n-k)!} C(\beta)
\]

for \( \beta \in A^k_p(M) \).

On \( H^k(M) \), \( C^2 = ( -1 )^{k+1} = \gamma^2 \), so taking \( \beta = C(\alpha) \),

\[
Q(\alpha, C(\alpha)) = (-1)^{\frac{k(k+1)}{2}} \int_M \alpha \wedge C(\alpha) \wedge \omega^{n-k}
\]

\[
= (n-k)! \left( \int_M \alpha \wedge \gamma^{-(k-1)} C^{k-1} \right)
\]

\[
= (n-k)! \left( \int_M \alpha \wedge \gamma^{-(k-1)} \right)^2
\]

\[
\| \alpha \|^2 > 0.
\]
Example 3: For a curve \((\gamma, 1)\), this says that for \(Q_{\gamma} = \varphi\cdot\text{product}\),
\[ Q_{\gamma}(H^{1,0}, H^{1,0}) = 0 \]
\[ Q_{\gamma}(x, \overline{x}) > 0 \quad \text{for} \quad x \in H^{1,0}\backslash \{0\} \]
which are forms of the Riemann bilinear relations proved above.

Example 4: For \(M\) projective, we have a \(\text{unique} \quad \text{Kähler} \quad \text{class} \quad \text{[\(\omega\)]},\)
and so \(Q_k\) is defined \(\text{/}\(Q\). This leads to the concept
of a polarized Hodge structure (which we won't discuss now).

Remark 1: The reason why HRBR II won't work on all of
\[ H^k / (H^k)^* \quad \text{is that in the Lefschetz decomposition} \]
\[ (-1)^{[k/2]} Q_k(C(\cdot), C(\cdot)) \quad \text{will be} \quad \begin{cases} > 0 & \text{on terms with} \ j \ \text{even} \\ < 0 & \text{on terms with} \ j \ \text{odd} \end{cases} \]
This is basically because \((-1)^{[k/2]} \cdot (-1)^{(k-2j)(k-2j)+1} = (-1)^j \cdot (-1)^{\frac{k(k+1)}{2}} \quad \text{(a)}\)
One can take the \(\mathbb{C}\)-linear extension of the \((-1)^j Q_k\) \(\text{to fix this, if desired.}\)

Idea of pf. of Lemma 6: In general \(A_{pr}^k = \ker \Lambda \quad \text{for } \mathbb{C} \quad \text{hen.} \quad \text{say} \quad n = k = 2\). Then the identity says that
\[ \star = -C : A_{pr}^2 \to A_{pr}^2 \]
\[ \text{Let's compare } \star \text{ on the basis (0, 1) in Hodge metric) } \]
\begin{align*}
\phi_1 \wedge \phi_2 & \mapsto -\phi_1 \wedge \phi_2 \\
\phi_2 \wedge \phi_1 & \mapsto -\phi_2 \wedge \phi_1 \\
\omega_1 & \mapsto \omega_2 \\
\omega_2 & \mapsto \omega_1 \\
\phi_1 \wedge \phi_2 & \mapsto \phi_1 \wedge \phi_2 \\
\overline{\phi}_1 \wedge \overline{\phi}_2 & \mapsto \overline{\phi}_1 \wedge \overline{\phi}_2
\end{align*}

Now on $A^{2,0} \oplus A^{0,2}$, $-C = 1$; while

on $A^{1,1}$, $-C = -1$.

So this will only work on the span of $\phi_1 \wedge \phi_2$, $\phi_2 \wedge \phi_1$, and $\omega_1 - \omega_2$, which is precisely the kernel of $\Lambda$ (i.e. of $\mathfrak{ke}$, or nearly just of $L$ in this case).

See Vorisin for the full proof. The $(-1)^{k(k+1)/2}$ comes from shuffling $dz_I \wedge dz_I$'s to get $\omega_I$. 