F. Residue Theory

Poincaré Residue map

\[ M = \mathbb{C} \text{- manifold (dim. } = n) \]

\[ N \subset M \text{ codim. } 1 \text{ } C\text{-submanifold} \]

\[ U := M \setminus N \subset \mathbb{C}^n \text{ the (open) complement} \]

\[ \mathcal{L}_M^1(\log N) := \text{sheaf of free } \mathcal{O}_M \text{-modules locally generated by } \frac{dz_i}{z_i}, \frac{dz_2}{z_2}, \ldots, \frac{dz_n}{z_n} \]

where \((z_1, \ldots, z_n) = \text{local holo. coords. in which } N_0 = \{z_i = 0\} \).

[For comparison, \( \mathcal{L}_M^1(N) \) is locally generated by \( \frac{dz_i}{z_i}, \frac{dz_2}{z_2}, \ldots, \frac{dz_n}{z_n} \).]

Set

\[ \mathcal{L}_M^0(\log N) := \mathcal{L}_M^1(\log N) \]

and define maps of sheaves

\[ \mathcal{L}_M^0(\log N) \xrightarrow{\frac{dz_i}{z_i}} \mathcal{L}_M^{0-1}(\log N) \]

\[ \frac{dz_i}{z_i} \text{ holo. map (smooth)} \]

\[ \text{In each case, } \omega = \omega_1 + \omega_2 \rightarrow \omega_1. \]

\[ \text{Ex/ check this is independent of choice of local holo. coords.} \]

\[ \text{Lemma 1: Let } B^* \xrightarrow{\sim} C^* \text{ be quasi-isomorphic complexes of } \Gamma_M \text{-acyclic sheaves. Then } \Gamma_M B^* \xrightarrow{\sim} \Gamma_M C^*. \]

[We need only the case } B^* \xRightarrow{\sim} C^*, \text{ so let us just do that:} \]
Proof: Consider $0 \to B^* \to C^* \to \text{cok}^* \to 0$.

- Snake lemma $\Rightarrow \text{cok}^*$ is exact.
- The long-exact sequence associated to $0 \to B^* \to C^* \to \text{cok}^* \to 0$
  $\Rightarrow \text{cok}^*$ is $\Gamma_\text{M}$-acyclic ($\mathcal{V}$).
- Hence, $\Gamma_\mathcal{V}(M, \text{cok}^*)$ is exact. Now apply $\Gamma_\text{M}$ and use
  the Snake lemma.

Here is what log forms are good for:

Corollary 1: $H^q(C^*(M, A_m^*(\log N))) \cong H^q(U, C)$.

Proof: To apply Lemma 1 to $A_m^*(\log N) \to j_*(A_U^*)$ (both $\Gamma_\text{M}$-acyclic by $\text{cok}^*$).

we need to check (mostly) that the inclusion of complexes is a quasi-isomorphism. Take $p \in N$ & $\Delta^n$ a poly-cylinder about it in $M$.

Then we need $\varepsilon$'s on cohomology:

$$H^j\left[\Gamma(\Delta^n, A_m^*(\log N))\right] \to H^j_{\partial\bar{R}}(\Delta^n \times \Delta^{n-1}, \mathbb{C}) = \begin{cases}
\{1\}, & j = 0 \\
\{\partial \varepsilon_i\}, & j = 1 \\
0, & j > 1
\end{cases}$$

Clearly this is onto. Injectivity is also easy: given $\omega = \omega_1 + \frac{\partial \omega_2}{\partial z_i}$, 

$0 = dw = dw_1 \wedge \frac{\partial \omega_2}{\partial z_i} + dw_2 \Rightarrow dw_1, dw_2 = 0 \Rightarrow \omega_1, \omega_2$ exact $\Rightarrow \omega$ exact.

The $\text{Res}^n_m$ maps define obvious short-exact sequences: e.g., for $A^k$:

\[0 \to A_m^* \to A_m^*(\log N) \to \text{Res}^{n-1}_m A^* \to 0\]

with associated long-exact sequence (Residue sequence)

\[\cdots \to H^k(M, \mathbb{C}) \xrightarrow{\partial^*} H^k(U, \mathbb{C}) \xrightarrow{\text{Res}^n_m} H^{k-1}(N, \mathbb{C}) \xrightarrow{\partial} H^{k+1}(M, \mathbb{C}) \to \cdots\]

The Gysin map $\text{Gy}$ is computed as usual for a connecting homomorphism in: given $g \in A^{k-1}(N)_{d=1}$, lift it in $\Gamma_\text{M}(F.1)$ to
\[ \delta \in A^k(\log N)(M) \text{ s.t. } R_{\delta}(N) = 0 \text{ (roughly, } \delta = \sum_{\alpha} p_{\alpha} \cdot \frac{d\alpha}{\epsilon} ) \] 

Then apply \( d \) to the result comes from \( A^{k+1}(M)_{d-1} \).

There is a duality homology long-exact sequence ("Tube sequence") break up \( M = U \cup \widetilde{N} \) where \[ \begin{cases} \quad \widetilde{N} = \text{(tubular nbhd. of } N) \approx N \times \Delta \\ \quad \widetilde{N}^* = \text{homeo. } N \setminus \Delta \end{cases} \]

Mayer-Vietoris \( \implies \) have (i.e.):

\[ \begin{array}{c}
\left\langle H_k(M, \mathbb{Z}) \leftarrow H_k(U, \mathbb{Z}) \oplus H_k(\widetilde{N}, \mathbb{Z}) \leftarrow H_k(\widetilde{N}^*, \mathbb{Z}) \leftarrow H_k(M, \mathbb{Z}) \right. \\
\text{tube}
\end{array} \]

\[ \begin{array}{c}
\left\langle H_k(U, \mathbb{Z}) \leftarrow H_k(\mathbb{Z}) \leftarrow H_k(\mathbb{Z}) \oplus H_{k-1}(N, \mathbb{Z}) \leftarrow H_k(M, \mathbb{Z}) \right. \\
\text{tube}
\end{array} \]

\[ \left\langle H_k(N, \mathbb{Z}) \leftarrow H_k(N, \mathbb{Z}) \oplus H_{k-1}(N, \mathbb{Z}) \oplus \mathbb{Z} \right. \]

\[ \left\langle H_k(N, \mathbb{Z}) \leftarrow H_k(N, \mathbb{Z}) \oplus H_{k-1}(N, \mathbb{Z}) \oplus \mathbb{Z} \right. \]

\[ \left\langle \begin{array}{c}
\begin{array}{c}
\text{(F.3)} \\
\end{array}
\end{array} \right. \]

The terms of (F.2) and (F.3) are clearly dual.

We claim that the maps are adjoint. Indeed,

\[ \int_{\partial \delta} \omega = \lim_{\epsilon \to 0} \int_{\text{tube}(\delta)} \epsilon \omega \wedge \frac{d\alpha}{\epsilon} = \lim_{\epsilon \to 0} \int_{\text{tube}(\delta)} \epsilon \omega \wedge \frac{d\alpha}{\epsilon} = 2\pi i \int_{\gamma} \sum \omega \wedge \frac{d\alpha}{\epsilon} \]

\[ = 2\pi i \int \text{Res} \omega \]

\[ \int_{\partial \delta} \omega = \text{Res} \omega \wedge \epsilon \frac{d\alpha}{\epsilon} = \text{Res} N \frac{d\alpha}{\epsilon} = 0 \]

\[ \int \text{Res} N \frac{d\alpha}{\epsilon} = 0 \]

\[ \int \delta = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\text{tube}(\delta)} \epsilon \omega \wedge \frac{d\alpha}{\epsilon} = \frac{1}{2\pi i} \int_{\text{adjoint}} \delta \wedge \epsilon \frac{d\alpha}{\epsilon} \]

\[ \begin{array}{c}
\text{(adjoint)} \quad \beta_{\epsilon} = \text{Res}(e^{-\epsilon \text{nbhd. of } N}) \\
\end{array} \]

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\text{(adjoint)} \quad \beta_{\epsilon} = \text{Res}(e^{-\epsilon \text{nbhd. of } N}) \\
\end{array}
\end{array} \right. \]
We now specialize to the setting:

\[ X \subset \mathbb{P}^n \text{ smooth degree } d \text{ hypersurface} \]

\[ N \cong \mathbb{P}^{n-1} \]

\[ \Lambda^m_{\mathbb{P}^n}(\ast X) := \text{ sheaf of meromorphic } m\text{-forms with poles only along } X \]

\[ A^m_k := \text{ rational } m\text{-forms on } \mathbb{P}^n \text{ with poles of order } \leq k \text{ along } X \]

\[ A^m = \bigcup_k A^m_k \]

We will quote the following:

**Chow's Theorem**

\[ A^m_k = \Gamma(\mathbb{P}^n, \Lambda^m_{\mathbb{P}^n}(kX)) \]

**Bott Vanishing**

\[ H^j(\mathbb{P}^n, \Lambda^m_{\mathbb{P}^n}(kX)) = \{0\} \text{ for all } j, k > 0 \text{ and } j \geq 0 \]

Set \[ P_k \Lambda^m_{\mathbb{P}^n}(kX) := \Lambda^m_{\mathbb{P}^n}(kX) \]

Griffiths considered the "rational \( dR \) cohomology" groups

\[ h^\ast(X) := \frac{A^n}{dA^{n-1}} \text{ with filtration } P_k h^\ast(X) := \frac{A^n}{(dA^{n-1}) \cap A^n} \]

**9.4**

\[ \Gamma(\mathbb{P}^n, C_{\mathbb{P}^n}^k \Lambda^m_{\mathbb{P}^n}(kX)) = A^m_k / A^m_{k-1} \]

\[ \text{Griffiths also referred to as } "GAGA" - \text{ global analysis } \Rightarrow \text{ global algebraic} \]

**9.5**

\[ \text{comes from Bredon - Weil-Bott theorem in representation theory of Lie groups} \]
Remark: (On Res, G_, H topology of hypersurfaces)
(c) These groups will turn out to compute the primitive $H^{p+q}$-space of $X$, essentially by Poincare residue; this means dually that the periods of $\sum (p,q)$-forms on $X$ (over cycles $Y$) are just periods of rational forms on $P^n \setminus X$ (over cycles Tube($x$)). Note that we are only concerned with the case $p+q = n-1$ — i.e. middle degree cohomology. This is because of

\[ H^l(P^n) \overset{c^*}{\longrightarrow} H^l(X) \text{ is an isomorphism for } l < n-1, \]

\[ H^l(X) \overset{G_y}{\longrightarrow} H^{l+2}(X) \text{ is an isomorphism for } l > n-1, \]

which implies that the Hodge diamond of $X$ takes the form

\[ (*) \]

(a) By looking at the dual maps in homology, one sees that the composite

\[ H^{n-1}(X) \overset{G_y}{\longrightarrow} H^n(P^n) \overset{c^*}{\longrightarrow} H^{n+1}(X) \]

is cup product with $\ast[X] = d \cdot [\ast[H] = d \cdot [\omega_X]$

Using weak Lefschetz + Poincare duality, $c^*$ is an isomorphism, so that $\ker(G_y) = \ker(U_\ast \omega_X) = \ker(L) = H^{n-1}_{pr}(X)$.

But then, in the residue sequence ($\ast$-cohomology)

\[ H^{n-2}(X) \overset{G_y}{\longrightarrow} H^n(P^n) \overset{c^*}{\longrightarrow} H^n(P^n \setminus X) \overset{\text{Res}}{\longrightarrow} H^n(X) \overset{G_y}{\longrightarrow} H^{n+1}(P^n) \rightarrow \]

Weak Lefshetz + Poincare duality
We see that
\[ H^n(P_n^*X) \xrightarrow{\text{Res}} H^{n-1}_{pr}(X) \]

Similarly, \( H^q(P_n^*, \mathcal{L}^P_n(log X)) \xrightarrow{\text{Res}} H^q_{pr}(X, \mathcal{O}^P) \) \( (:= \ker \{ H^q(X, \mathcal{O}^P) \to H^q(X, \mathcal{O}^P) \}) \)

Lemma 2: The sequence of sheaves on \( P^n \)

\[ 0 \to \mathcal{L}^{p+1}_{\mathbb{P}^n}(\log X) \to \mathcal{L}^{p+1}_{\mathbb{P}^n}(X) \xrightarrow{d^1} \mathcal{L}^{p+2}_{\mathbb{P}^n}(X) \xrightarrow{d^2} \cdots \]

is exact.

Proof: We check exactness at the middle term by

\[ \mathcal{L}^{p}_{\mathbb{P}^n}(X) \xrightarrow{d^1} \mathcal{L}^{p+1}_{\mathbb{P}^n}(\mathcal{O}(X)) \xrightarrow{d^2} \mathcal{L}^{p+2}_{\mathbb{P}^n}(\mathcal{O}(X)) \]

in (local) coordinates \((e_1, \ldots, e_n)\) s.t. \( z_i = 0\) locally defines \( X \). First,

\[ d(\log) \left( \frac{1}{z_i} \right) = \frac{dz_i}{z_i^2} + \sum_{j \neq i} \frac{1}{z_i z_j} \cdot \frac{dz_j}{z_j} \]

Now, given \( \omega \in \mathcal{L}^{p+1}_{\mathbb{P}^n}(\mathcal{O}(X))(U) \), \( \omega = \sum \frac{\eta_i}{z_i} \frac{dz_i}{z_i^2} \), where

and we have

\[ dw \in \mathcal{L}^{p+2}(\mathcal{O}(X))(U) \ [\text{i.e.}\ \in 0] \iff \]

\[ d\left( \frac{\eta_i}{z_i^2} \right) \in \mathcal{L}^{p+2}(\mathcal{O}(X))(U) \iff \]

\[ \sum \frac{\eta_i}{z_i^2} dz_i \in \mathcal{L}^{p+2}(\mathcal{O}(X))(U) \iff \]

\[ h_{\frac{1}{z_i^2}} \in \mathcal{O}(U) \ (\forall i) \iff \]

\[ \frac{\eta_i}{z_i^2} \in \mathcal{L}(\mathcal{O}(X)) \ [\text{i.e.}\ \in 0] \]

for locally, in some affine, small \( U \)

\[ \iff \omega \in \mathcal{L}(\mathcal{O}(X)) + \mathcal{L}^{p+1}(\mathcal{O}(X)) \]

\[ \Box \]
But recalling imply this oh but the 1st term of (F.4) are 
$\Gamma_{p^n}$-acyclic, rendering (F.5) a $\Gamma_{p^n}$-acyclic resolution of $\mathcal{I}^{p^n}(\log X)$. 

So we have 

$$
\begin{align*}
\mathcal{Y}^n_{n-p} : & \quad \frac{\mathcal{N}^n}{dA^n_{n-p}} \\
\text{(F.4)} & \quad \Gamma(p^n, Gr_{n-p}^{p^n} \mathcal{N}^n(*)X) \\
\text{(F.5)} & \quad H^{n-p-1}(p^n, \mathcal{I}^{p^n}(\log X)) \\
\sim & \quad \text{Res} \\
& \quad H^{n-p-1}(X, \mathcal{I}^p_X), \text{ proving}
\end{align*}
$$

Corollary 2: $\mathcal{Y}^n_{n-p} \cong H^{p,n-p-1}_{pr}(X)$.

In fact, this "isomorphism of graded pieces" can be lifted to a 
"filtered isomorphism" which is much more explicit elsewhere. First 
consider the inclusion of acyclic schemes (the first being acyclic by (F.4))

$$
\mathcal{N}^n_{p^n}(*)X \hookrightarrow j_* A^n_{p^n \setminus X} \subseteq A^n_{p^n}(\log X).
$$

The left-hand inclusion is also a quasi-isomorphism (repeat the proof of 
Lemma/Cor. 1). This means that we have isomorphisms on cohomology 
groups of complexes of global sections:

$$
\begin{align*}
\mathcal{Y}^n(X) & \cong H^n(\mathcal{I}^{p^n}(\log X)) \\
\text{(F.5)} & \quad \text{Res} \\
& \quad H^{n-1}_{pr}(X, \mathcal{I}^p_X).
\end{align*}
$$

Theorem 1 (Brittin): $\text{Res}$ induces isomorphisms

$$
\begin{align*}
& \quad (F.6) \\
& \quad p^{-p} h^n(X) \cong F^p h^{n-1}_{pr}(X, \mathcal{I}^p_X).
\end{align*}
$$

* Of course we haven’t yet proved $\mathcal{Y}^n_{n-p} \cong Gr_{n-p}^{p^n} \mathcal{N}^n(*)X$; that will 
come out in the wash.
Moreover, \( Cr_{n-p} h^n(x) = y^n_{n-p} \) and the graded pieces of \( \tilde{R} \)

induce the isomorphisms of Corollary 2.

Proof: WTS: under \( \tilde{R} \), \( p_{n-p} \) goes into \( F^{p+1} H^n \{ \Gamma (R^n, A^\ast(\log X)) \} = H^n \{ \Gamma (R^n, F^{p+1} A^\ast(\log X)) \} = \tilde{R}^{-1} F^p H^{n-1} (X) \).

This will give a diagram

\[
\begin{array}{ccc}
\mathcal{A}_n & \mathcal{A}_n^{\beta} & \mathcal{A}_n^{\alpha} \\
\downarrow & \downarrow & \downarrow \\
\mathcal{A}_{n-p} & \mathcal{A}_{n-p}^{\beta} & \mathcal{A}_{n-p}^{\alpha} \\
\end{array}
\]

which forces \( \alpha \) to be injective (hence an isomorphism), then \( \beta \) to be in \( \mathcal{A}_n^{\alpha} \). Then a dimension argument forces the upper \( \mathcal{A}_n^{\beta} \) to be \( \mathcal{A}_n^{\alpha} \) too.

To simplify notation, write \[ x = n-p \]. Let us \( \mathcal{A}_x \), and again take \( \mathcal{U}_x \) & local wards. \( z \) s.t. \( x \cap \mathcal{U}_x = \{ z = 0 \} \). On \( \mathcal{U}_x \),

\[
\omega |_{\mathcal{U}_x} = \frac{dz_1 \wedge \cdots \wedge dz_k}{z_1^{k-1}} \Rightarrow \frac{dz_1 \wedge (dz_2 \wedge \cdots \wedge dz_k)}{z_1^{k-1}} = \frac{dz_1}{z_1^{k-1}} \wedge \theta_1 \in \Omega^{n-1} (\mathcal{U}_x)
\]

Set \( g_x := \frac{\theta_1}{(k-1)z_1^{k-1}} \Rightarrow d z_x = \frac{d \omega_x}{(k-1)z_1^{k-1}} - \frac{dz_1}{z_1^{k-1}} \wedge \omega_x = \omega |_{\mathcal{U}_x}
\]

If \( \mathcal{U}_x \cap X = \emptyset \), then we take \( g_x \) a primitive of \( -\omega |_{\mathcal{U}_x} \).
\[ \omega = \sum_{i=1}^{n} \frac{\partial x_i}{\partial z_i} d\bar{z}_i \quad \text{on} \quad \Gamma (\mathbb{P}^n, \mathcal{O}(1)) \]

where \( \omega \in \Omega^1 (\mathbb{P}^n, \mathcal{O}(1)) \).

So we have reduced the pole-order of the cost of losing type \((n, 0)\).

Keep reducing until you reach \( \Gamma (\mathbb{P}^n, \mathcal{O}(1)) \) d-dec.

The result has local form

\[ \frac{d\bar{z}_1}{z_1^a} + \frac{d\bar{z}_2}{z_2^b} \quad \text{and} \quad \frac{d\bar{z}_1}{z_1^c} + \frac{d\bar{z}_2}{z_2^d} = 0 \]

Hence it has only a log pole along \( \mathcal{C} \).

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**Jacobian rings**

We have

\[ \mathcal{A}_k^n = \Gamma (\mathbb{P}^n, \mathcal{O}) \cong S^{kd-n-1} \]

\[ \mathcal{A}_k^n \cong \mathcal{O}(kd-n-1) \]

(F.7)

How to realize this in such a way as to make the rational forms explicit?

Given we \( \mathcal{A}_k^n \), by definition of a rational form, it lifts under

\[ \Pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n \]

N.B. Think of \( \Pi \) as quotient by the action

\[ t (z_0, \ldots, z_n) = (e^{z_0}, \ldots, e^{z_n}), \]

which is the flow along the Euler vector field

\[ e := \frac{\partial}{\partial z_i} \]

\[ \Pi^* \omega = \frac{1}{F^k} \sum_{|I|=m} A_I \otimes d\bar{z}_I \in \Omega^m (\mathbb{P}^n, \mathcal{O}(1)) \]

on \( \mathbb{C}^{n+1} \).
where \( F \in S_{\text{Alt}}^d \) defines \( X(= \{ F = 0 \}) \). We now characterize
those \( \Psi \in \hat{A}^n \) that descend to \( \mathbb{P}^n \) (i.e. arise in this fashion).

Let \( \Psi \in \hat{A}^n \) be a monomial, and define its degree by
\[
(\text{deg} \, \Psi) \cdot \Psi := \left[ d, e - 1 \right] \Psi := e \int d\Psi + d(e - 1)\Psi.
\]

Example 1: (i) \( \text{deg} (d\mathbb{Z}_{1}) = 1 \)

(ii) \( \text{deg} (P) = k \), for \( P \in S^k \)

(iii) writing \( \mathcal{M} := e \int d\mathbb{Z} = \sum (-1)^i \mathbb{Z}_i \, dt_0 \wedge \cdots \wedge \mathbb{Z}_i \wedge \cdots \wedge dt_n \)

\[
\text{deg} \mathcal{M} = d(e - 1)(d\mathbb{Z}) + e \int d(e - 1)\mathbb{Z}
\]

\[
= 0 + e \int (\text{deg} d\mathbb{Z}) \, d\mathbb{Z}
\]

\[
= (n + 1) \mathcal{M}.
\]

(iv) \( \text{Ex} / \text{deg} (a \wedge \beta) = \text{deg} (a) + \text{deg} \beta \) \( (= \text{deg} (\frac{\mathbb{Z}}{F_F}) = \text{deg}(a) - kd) \).

This leads to the crucial

Lemma 3: (i) For a homogeneous polynomial \( P \in S^a_{\text{Alt}} \),
\[
\text{deg} \left( \frac{\nabla a}{F_k} \right) = 0 \iff a = kh - n - 1;
\]

(\( \text{this is clear} \))
and (ii) \( \pi^* \) identifies \( \hat{A}^n \) with \( \frac{\mathcal{M}}{F_k} \cdot S^a_{\text{Alt}} \).

Sketch of Proof: A form \( \Psi \) descends to \( \mathbb{P}^n \) \( \iff \) it is \textit{invariant} \( (\text{deg} \Psi = 0) \)
and \textit{horizontal} \( (e - 1)\Psi = 0) \).

For \( \frac{\nabla a}{F_k} \), horizontality is clear since \( e \int \mathbb{Z} = e \int (e - 1)dt = 0 \).

But then \( \frac{\mathcal{M}}{F_k} \cdot S^a_{\text{Alt}} \rightarrow \pi^* \hat{A}^n \), and by equality of dimensions

\( \text{this must be} \) \( \text{onto} \).
Writing \( \frac{PD}{F^{n-1}} = \frac{(FP)R}{F_n} \), we see that the inclusion

\[ A^h_{k-1} \to \mathbb{A}^h \]

corresponds (under the identification of Lemma 3)

\[ S^{(h-1)d-n-1} \to S^{kd-n-1} \]

\[ P \to F \cdot P \]

Now our goal is to use Lemma 3 to identify the \( \text{Gr}_{k} \mathbb{A}^n(x) \), for which we need the holomorphic pole-reduction statement:

**Lemma 4:** If, for some \( j \), \( P(\in S^{kd-n-1}) \) has a factor of \( \frac{dF / \partial z_j}{\partial \bar{z}_j} \), then

\[ \frac{PA}{F^{n-1}} = A^h_{k-1} \cdot dA^h_{k-1}. \]

More precisely, under the identification \( A^h_{k-1} = S^{kd-n-1} \), we have

\[ A^h_{k-1} \cdot dA^h_{k-1} = J_{kd-n-1} \]

\[ \frac{\text{ideal in } S = F(S^n)}{\text{ideal in } S^{kd-n-1}} \]

"Jacobian ideal"

**Remark 2:** The idea (say, in \( U_0 \)) should be that if \( P = P \cdot \frac{dF}{\partial z_j} \), then the local defining eqn. for \( x \)

writing \( \Theta = \frac{P \partial z_1 \cdots \partial z_j \cdots \partial z_n}{\partial F / \partial z_j} = \frac{P \partial z_1 \cdots \partial z_j \cdots \partial z_n}{\partial F / \partial z_j} \),

\[ \frac{d}{(k-1)F^{n-1}} = \frac{d\Theta}{(k-1)F^{n-1}} - \frac{dt \cdot \Theta}{F^{n-1}} = \frac{dP / \partial z_j}{(k-1)F^{n-1}} \frac{dz_j}{F^{n-1}} - \frac{P \partial F / \partial z_j}{F^{n-1}} \frac{dz_j}{F^{n-1}} \]

(reduce poles)

(\( \text{still not clear} \))

To make this precise over all of \( P^n \), we'll need to again employ \( C^\infty \)-invariant forms on \( U = C^{n+1} \setminus \{0\} \). Let \( \mathbb{A}^n := \text{polynomial forms on } C^{n+1} \).

**Proof (Lemma 4):** Let \( \omega = \frac{PD}{F^{n-1}} \in A^h_{k-1} \). Then \( P = \mathfrak{F} \Theta \) (\( Q \in S^{(h-1)d-n-1}) \)

\[ e(P) = \sum z_i \frac{dF}{\partial z_i} (+d \cdot F) \]

\[ \text{viewed as subspace of } A^h_{k-1} \]

\[ \frac{1}{d \cdot F} Q \in J^{kd-n-1} \]

\[ e(P) = \sum z_i \frac{dF}{\partial z_i} (+d \cdot F) \]

\[ \text{viewed as subspace of } A^h_{k-1} \]

\[ \frac{1}{d \cdot F} Q \in J^{kd-n-1} \]
2) Next, consider $\eta \in \mathbb{A}^{n-1}_{k-1}$; then $\pi^* \eta = 0 = e - d\psi$, one has $\psi_0 = e - \frac{1}{(k-1)!} d\psi$.

Using $e_1 \psi = 0 = e_1 d\psi$, one has $\psi_0 = e_1 - \frac{1}{(k-1)!} d\psi$.

So $\deg \left( \frac{\varphi}{F^{k-1}} \right) = 0$ and

$$d\psi = d \left( e_1 \varphi \right) = d \left( e_1 \frac{\varphi}{F^{k-1}} \right) = \deg \left( \frac{\varphi}{F^{k-1}} \right) \frac{d\varphi}{F^{k-1}} = e_1 \left( \frac{1}{(k-1)!} dF \varphi - F d\varphi \right) = e_1 \left( \frac{1}{(k-1)!} dF \varphi - F d\varphi \right).$$

Now the only options for $\varphi = \text{h.o.}$ $n$-form on $\mathbb{A}^n$, of degree $(k-1)d$ are to take $d\varphi := d\varphi_0 \wedge \ldots \wedge d\varphi_n$ and any $R_i \in S^{(k-1)d-n}$ and with

$$\varphi = \sum_{i=0}^n (-1)^i R_i d\varphi_i \text{ so that}$$

$$d\varphi = \sum_{i=0}^n \frac{\partial R_i}{\partial \varphi_i} d\varphi_i, \quad \text{and} \quad dF \wedge \varphi = \sum_{i=0}^n R_i \frac{\partial F}{\partial \varphi_i} d\varphi_i.$$

$$(\pi^* d\eta) \psi = e_1 \left( \sum_{i=0}^n \left( (k-1) R_i \frac{\partial F}{\partial \varphi_i} - F \frac{\partial R_i}{\partial \varphi_i} \right) d\varphi_i \right)$$

$$= \left( \sum_{i=0}^n \left( (k-1) R_i \frac{\partial F}{\partial \varphi_i} - F \frac{\partial R_i}{\partial \varphi_i} \right) d\varphi_i \right) \frac{\varphi}{F^{k-1}}$$

is the ground form of an element of $\mathbb{A}^{n-1}_{k-1}$. But the homogeneous polynomials $\in \mathbb{J}_F$.

3) Conversely, given any $P \in \mathbb{J}_F^{k-1}$, say $P = \sum_{i=0}^n (k-1) R_i \frac{\partial F}{\partial \varphi_i}$, clearly abstract the class $d\psi \in \mathbb{A}^{n-1}_{k-1}$ until "reduce the pole" of

$$\frac{Pd\varphi_0}{F^{k-1}} \in \mathbb{A}^n_{k-1}.$$
Substituting back in $k = \nu - p$, we may now conclude the

**Theorem 2 (Griffiths):** The map $(F \circ G)$ has graded pieces

\[
R_F^{(n-p)d-1} \rightarrow S \rightarrow G^p \otimes H^n(X) \rightarrow H_{pr}^{\nu-k}(X)
\]

\[P \rightarrow \frac{PR}{P^k} \rightarrow \text{Res}_X \left( \frac{PR}{P^k} + ds \right)\]

**Remark 3:**
(i) One should really think of the $H_{pr}^{\nu-k}(X)$ as $G_{pr}^p H^n(X, \mathbb{C})$.

(ii) $R_F := \frac{S}{J_F}$ is called the "Jacobian ring" (we need nonsingular piece of it). In general computing it can be hard (of require Grothendieck bases).

(iii) To put this all in perspective, suppose we now want to compute $\int_X \omega$ for $[\omega] \in H_{pr}^n(X, \mathbb{Z})$, $[\omega] \in F^p H_{pr}^{\nu-k}(X, \mathbb{C})$.

Then up to cobordism on $X$, $\omega = \frac{1}{2\pi i} \text{Res}_X(\beta)$ for $\beta \in \Gamma(F^p, F^{p+h}(\log X))_{d=1}$, and up to cobordism on $\mathbb{P}^n \setminus X$, $\beta \in \frac{PR}{P^k} \in H^0(\mathcal{O}^{\nu-k}(p-1)X) = A_{\nu-k}^n$.

Consequently by Stokes's theorem

\[
\int_X \omega = \frac{1}{2\pi i} \int_{\text{tube}(y)} \beta = \frac{1}{2\pi i} \int_{\text{tube}(y)} \frac{PR}{P^p - P^n}
\]

for some $P \in S^{kd-n-1}(k = \nu - p)$. So we have a tight relationship, via Poincaré residues and pole reduction, between "Hodge-theoretic periods" on $X$ and rational integrals on $\mathbb{P}^n$ (which is some sense were the "original" object).

* i.e. exact forms

**Important point!**
We quote the following result, which follows from a standard upper semi-continuity theorem for the dimension of the kernel for a continuous family of elliptic differential operators.

**Theorem 3:** Let $\mathcal{X} \to \mathcal{B}$ be a family of compact complex manifolds and $\mathcal{V} \to \mathcal{X}$ a holomorphic vector bundle. Then for each $b_0 \in \mathcal{B}$, there exists $U \subset \mathcal{B}$ s.t.:  
\[ \dim H^q(X_b, \mathcal{V}|_{X_b}) \leq \dim H^q(X_{b_0}, \mathcal{V}|_{X_{b_0}}) \quad (\forall b \in U). \]

**Corollary 3:** For a family of compact Kähler manifolds, the Hodge numbers $h^{p, q}$ are constant.

**Proof:** Theorem 3 says (taking $\mathcal{V} = \mathcal{L}^p$) they can only jump at isolated points. But $(\forall b \in U)$ the sum is $\sum h^{p, q}(X_b) = h^i(X_b)$, which (all $X_b$ being diffeomorphic) is constant in $b$.

Since varying $F \in S_{d, 1}$ (avoiding the choices that make $X = \{ F = 0 \} \subset \mathbb{P}^n$ singular) yields all degree-$d$ smooth projective hypersurfaces (in $\mathbb{P}^n$) in a connected family,

(a) The Hodge $h^{p, q}(X)$ depend only on $d$ and $n$.

(b) We may select the most convenient $F = z_0^d + \ldots + z_n^d$ for computing the

\[ h^{p, n-p-1}_{(d)} = \dim H^p_{(d)}(X) = \dim (R^jF)_{n-p-1}(F) \]

(this can be omitted except for $n-p-1-p$)

---

More precisely, $X'$ and $\mathcal{B}$ are $\mathbb{P}$-manifolds; if $\pi$ a proper holomorphic submersion.

Write $X_b = \pi^{-1}(b)$ for the (compact $\mathbb{P}$-fiber) fibers.
Theorem 4: Let \( X \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( d \). We have

\[
\begin{align*}
&h^{n-1,0} = \dim (\mathcal{N}^{n-1}(X)) = \binom{d-1}{n} , \\
&\mathcal{N}^{n-1}(X) = \left\{ \rho \in \mathcal{P} \left( \frac{\mathcal{R}}{\mathcal{P}} \right) \mid \rho \in \mathcal{S}^{d,n-1} \right\}.
\end{align*}
\]

Proof: The generators of \( J_F \) have degree \( d-1 \); hence \( J_F^{d-1} = \{ 0 \} \), and \( R_{d,n-1} = \mathcal{S}^{d,n-1} \), whose dimension is computed by the borel-berg formula. \( \square \)

Corollary 4: For \( X \subset \mathbb{P}^2 \) a smooth degree \( d \) curve, \( g = \frac{(d-1)(d-2)}{2} \).

Corollary 5: For \( X \) as in Thm. 4, \( X \) is Calabi-Yau \( \iff d = n+1 \).

\[\begin{align*}
&\text{If } d > n+1 \text{, then } h^{n-1,0} > 1 \\
&\text{If } d < n+1 \text{, then } h^{n-1,0} = 0
\end{align*}\]

(Pf.: The C-Y assertion uses adjunction: \( K_X = K_{\mathbb{P}^n} \otimes N_{X/\mathbb{P}^n} = \mathcal{O}(n-1) \otimes \mathcal{O}(d) \).

The rest is by Thm. 4.)

Finally, we shall compute the Hodge diamonds in a few special cases.

Proposition 1: For complex surfaces in \( \mathbb{P}^3 \) one has

\[
\begin{array}{ccc}
(d=3) & (d=4) & (d=5) \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 20 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

and for quintic 3-folds \( \subset \mathbb{P}^9 \)

\[
\begin{array}{cccc}
(d=5) & \\
1 & 101 & 101 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
Proof: For the surfaces ($n=3$), we only need to compute $h^{1,1} (p=1)$. So $n(n-1) = 2d-4$, and

$$H^{1,1}_p(X) \leq \frac{\sum_{d=1}^{2d-4}}{(2,0,1,2^5)^d_{d-1}}$$

where we have used (6) from below p. 115 to compute $J_F$ (take $F: z^2 + \cdots + z^d$)

$d=3$:

$$\frac{\sum_2}{(2,0,1,2^5)^3} = \mathcal{C} \left( \left\{ z_i z_j \mid 0 \leq i < j \leq 2 \right\} \right)$$

and work on left.

$$\Rightarrow \quad h^{1,1} = h^{1,1}_p + 1 = \binom{4}{2} + 1 = 7$$

$k=4$:

$$\frac{\sum_4}{(2,0,1,2^5)^4} = \mathcal{C} \left( \left\{ z_0^2 z_2 z_3, \left\{ z_i^2 z_j^2 \mid 0 \leq i < j \leq 3 \right\}, \left\{ z_i z_j z_k \mid 0 \leq j < k \leq 3, \quad i = 0, 1, 2 \right\} \right) \right)$$

$$\Rightarrow \quad h^{1,1} = h^{1,1}_p + 1 = \left( 1 + \binom{4}{2} + 2 \binom{4}{3} \right) + 1 = 20$$

Ex: Compute the proof. (For the CY 3-fold, compute $h^{1,1}$.)

Euler integrals

Residue theory gives a technique for directly computing certain kinds of periods as power series. This is very important for mirror symmetry (a point of contact between string theory and algebraic geometry).

Let \( \{ X_t \} \) be the 1-parameter family of (degree \( n+1 \))

\( \text{C} = (t \cdot z_0, t \cdot z_1, t \cdot z_3, t \in \mathbb{C}) \)

Calabi-Yau hypersurfaces in \( \mathbb{P}^n \) given by
\( O = F_e ( \overline{z} ) = t \frac{n}{n+1} \prod_{i=0}^{n} z_i - t \sum_{i=0}^{n} z_i^{n+1} \)

or in affine coordinates \((t; z_1, \ldots, z_n)\) (and dividing out by \(z_1 z_2 \cdots z_n\))

\[
O = 1 - t \phi(\overline{z}), \quad \phi(\overline{z}) = \frac{1 + z_1 z_2 + \cdots z_1^{n+1}}{\prod z_i} = z_1^{-1} z_2^{-1} + z_1^{-1} z_2^{-1} z_3^{-1} + \cdots + z_1^{-1} z_2^{-1} z_3^{-1} \cdots z_n^{-1}.
\]

One has for the constant terms in powers of \(\phi(\overline{z})\):

\[
[\phi^k]_0 = \begin{cases} 0, & k \neq n+1 \\ \frac{(k+1)!}{(n+1)! m!}, & k = m(n+1) \end{cases}
\]

**Lemma 5:** For \(0 < t < \frac{1}{n+1}\), the family of topological \((n-1)\)-cycles \(Y_t\) (with class \([Y_t] \in H_{n-1}(X_t, \mathbb{Z})\)) such that \( \text{Tub}_n([Y_t]) \subset H_n(\partial Y_t, \mathbb{Z}) \) is represented by \(\tau_t = \{(z_1, \ldots, z_n) \mid |z_j| = 1, (V_j) \approx (S^1)^{n-1}\} \).

**Proof:** Let \(\Gamma_n = \{(z_1, \ldots, z_n) \mid |z_j| = 1, (V_j) \approx (S^1)^{n-1}\} \rightarrow \partial \Gamma_n = \tau_t \).

Put \( |\phi(\overline{z})| = (t) \left| \frac{1 + z_1 z_2 + \cdots + z_1^{n+1}}{\prod z_i} \right| \leq \frac{(n+1)|t| < 1}{1 - t^n} \) on \(\tau_t\),

\(|t| < \frac{1}{n+1} \rightarrow \tau_t \cap X_t = \emptyset \)

\(\Rightarrow \) \(\gamma_t = \Gamma_t \cap X_t \) is a cycle (\(3\)-closed)

\(\therefore \) \(\text{why?} \)

Consider the family of nowhere-vanishing holomorphic \((n-1)\)-forms given by (cf. Theorem 4) \(\omega_t := \text{Res}_{X_t} \left( \frac{F}{F_t} \right) \in \Lambda^{n-1}(X_t) \)
(here $P \lessgtr \frac{d}{d t} = 0$ is $\leq 1$) or in affine coordinates,
\[
\text{Res}_{x_t} \left( \frac{\frac{d x_1, \ldots, d x_n}{(1 - x_t) x_1, \ldots, x_n}}{F_t} \right) = \text{Res}_{x_t} \left( \frac{\text{deg} x_1, \ldots, \text{deg} x_n}{1 - x_t} \right).
\]

Then we compute the period $\left( |t| < \frac{1}{n+1} \right)$
\[
\begin{align*}
\Phi(t) &= \sum_{k=0}^{n} \frac{c_k}{(2\pi i)^k} t^k = \sum_{k=0}^{n} \frac{1}{(2\pi i)^k} \left( \Phi(0)^k \text{ deg} x_1, \ldots, \text{deg} x_n \right) \\
&= \sum_{k=0}^{n} \Phi^k(0) t^k = \sum_{m=0}^{n+1} \frac{(n+1)! m!}{(m)! (n+1)^m} t^{n+1}
\end{align*}
\]

Show that $\Phi(t)$ satisfies the (generalized hypergeometric) ODE
\[
0 = \left( D^{n+1} + (n+1)^{n+1} \left( D + (n+1) \right) \cdots \left( D + (n+1) \cdot (n+1) \right) \right) \Phi(t)
\]

where $D := t \frac{d}{d t}$. This is an example of a Picard-Fuchs equation.

Remark: The monodromy of the analytic continuation of $\Phi(t)$, which still satisfies (F.11), reflects monodromy of the cycle class $\gamma_t$ as $t$ goes around points in the discriminant locus $\Delta = \{ t \in \mathbb{P}^1 \mid \chi_t \text{ singular} \}$.

Perversely **this monodromy acts irreducibly** on $H_{n-1}(X_t, \mathbb{Z})$. It is then clear that the remaining periods of $\omega_t$ about $t = 0$ will be given by the other
\[
\begin{cases}
\log x + \text{holo} \\
(\log x + \log (\text{holo}) + \text{holo}) \\
\log \text{holo} + \\
\log t + \\
\log x + \\
\ldots
\end{cases}
\]