F. Residue theory

Poincaré Residue map

$M =$ $\mathbb{C}$-manifold (dim. $= n$)

$N \overset{\iota}{\hookrightarrow} M$ codim.-1 $\mathbb{C}$-submanifold **N.B.**: ι_* exact on sheaves

$U := M \setminus N \overset{J}{\hookrightarrow} M$ the (open) complement

$\Omega^1_M(\log N)$:= sheaf of free $\mathcal{O}_M$-modules locally generated by

$$\frac{dz_1}{z_1}, dz_2, \ldots, dz_n$$

where $(z_1, \ldots, z_n)$ = local holo. coords. in which $N \underset{loc}{=} \{z_1 = 0\}$.

[For comparison, $\Omega^1_M(N)$ is locally generated by $\frac{dz_1}{z_1}, \frac{dz_2}{z_1}, \ldots, \frac{dz_n}{z_1}$]

Set and define maps of sheaves

$$\begin{tikzcd}
\Omega^p_M(\log N) := \Lambda^p \Omega^1_M(\log N) \arrow[r] & \iota_* \Omega^{p-1}_N \\
A^{p,q}_M(\log N) := A^{0,q}_M \otimes_{\mathcal{O}_M} \Omega^p_M(\log N) \arrow[r, "\boxed{\mathrm{Res}_N}"] & \iota_* A^{p-1,q}_N \\
A^k_M(\log N) := \sum_{p+q=k} A^{p,q}_M(\log N) \arrow[r] & \iota_* A^{k-1}_N
\end{tikzcd}$$

In each case, $\omega = \omega_1 \wedge \frac{dz_1}{z_1} + \omega_2 \longmapsto \iota^* \omega_1$.

(holomorphic or smooth)

Ex/ check this is independent of choice of local holo. coords. //

Lemma 1: Let $B^\cdot \xrightarrow{\simeq} C^\cdot$ be quasi-isomorphic complexes of Γ_M-acyclic sheaves. Then $\Gamma_M B^\cdot \xrightarrow{\simeq} \Gamma_M C^\cdot$.

[We need only the case $B^\cdot \hookrightarrow C^\cdot$, so I'll just do that:]

Proof : Consider $0 \to B^\bullet \to C^\bullet \to \mathrm{cok}^\bullet \to 0$.

- snake lemma $\Longrightarrow$ $\mathrm{cok}^\bullet$ is exact.
- The long-exact seq.s associated to $0 \to B^i \to C^i \to \mathrm{cok}^i \to 0$ $\Longrightarrow$ cok^i is Γ_M-acyclic $(\forall i)$.
- Hence, $\Gamma(M, \mathrm{cok}^\bullet)$ is exact. Now apply Γ_M (to the entire diagram. Rows stay exact by Γ_M-acyclicity of B^i.) and use the snake lemma.

$\square$

Here is what log forms are good for:

Corollary 1 : $H^q(\Gamma(M, A_M^\bullet(\log N))) \cong H^q(U, \mathbb{C})$.

Proof : To apply Lemma 1 to $A_M^\bullet(\log N) \hookrightarrow j_*(A_U^\bullet)$ (both Γ_M-acyclic b/c fine), we need to check (locally) that the inclusion of complexes is a quasi-isomorphism. Take $p \in N$ & Δ^n a polycylinder about it in M. (still thought of as $\{z_1=0\}$) Then we need $\cong$'s on cohomology:

$$H^j\{\Gamma(\Delta^n, A_M^\bullet(\log N))\} \longrightarrow H^j_{dR}(\Delta^* \times \Delta^{n-1}, \mathbb{C}) = \begin{cases} \langle 1 \rangle, & j=0 \\ \langle dz_1/z_1 \rangle, & j=1 \\ 0, & j>1 \end{cases}.$$

Clearly this is onto. Injectivity is also easy: given $\underset{d\text{-closed}}{\omega} = \omega_1 \wedge \frac{dz_1}{z_1} + \omega_2$,

$0 = d\omega = d\omega_1 \wedge \frac{dz_1}{z_1} + d\omega_2 \Rightarrow d\omega_1, d\omega_2 = 0 \underset{\substack{\text{Poincaré Lemma} \\ \text{for } \Delta^n}}{\Longrightarrow} \omega_1, \omega_2$ exact $\Rightarrow \omega$ exact.

$\square$

The Res_N maps define obvious short-exact sequences; e.g. for A^k :

(F.1)
$$0 \to A_M^\bullet \to A_M^\bullet(\log N) \xrightarrow{2\pi i\,\mathrm{Res}} \imath_* A_N^{\bullet-1} \to 0$$

with associated long-exact sequence ("Residue sequence")

(F.2)
$$\boxed{\cdots \to H^k(M, \mathbb{C}) \xrightarrow{j^*} H^k(U, \mathbb{C}) \xrightarrow[\mathrm{Res}_N]{2\pi i \times} H^{k-1}(N, \mathbb{C}) \xrightarrow[Gy]{\frac{1}{2\pi i}\times} H^{k+1}(M, \mathbb{C}) \to \cdots}$$

[The Gysin map "Gy" is computed as usual for a connecting homomorphism : given $\varphi \in A^{k-1}(N)_{d\text{-cl}}$, lift it in $\Gamma_M(\text{F.1})$ to

$\xi \in A^k(\log N)(M)$ s.t. $\mathrm{Res}_N(\xi) = \varphi$ (roughly, $\xi = \sum \eta_\alpha \tilde{\varphi}_\alpha \wedge \frac{dz_1^\alpha}{z_1^\alpha}$), then apply d; the result comes from $A^{k+1}(M)_{d\text{-cl}}$.]

There is a dual homology long-exact sequence ("Tube sequence"): break up $M = U \cup \tilde{N}$ where

$$\begin{cases} \tilde{N} = (\text{tubular nbhd. of } N) \underset{\text{homeo.}}{\approx} N \times \Delta \\ \tilde{N}^* := \tilde{N} \setminus N \underset{\text{homeo.}}{\approx} N \times \Delta^* \end{cases}$$

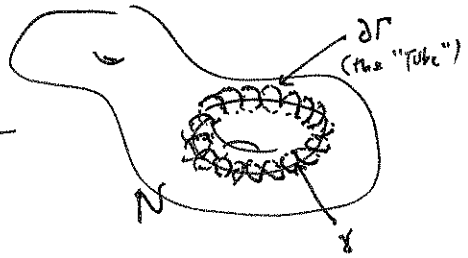
Mayer-Vietoris $\Longrightarrow$ have l.e.s.

$$\leftarrow H_k(M,\mathbb{Z}) \leftarrow H_k(U,\mathbb{Z}) \oplus \underset{\cancel{H_k(N,\mathbb{Z})}}{\underset{\cong}{H_k(\tilde{N},\mathbb{Z})}} \leftarrow \underset{\cancel{H_k(N,\mathbb{Z})} \oplus H_{k-1}(N,\mathbb{Z})\text{"}\otimes S^1\text{"}}{\underset{\cong}{H_k(\tilde{N}^*,\mathbb{Z})}} \leftarrow H_{k+1}(M,\mathbb{Z}) \leftarrow$$

$\Longrightarrow$

(F.3)

$$\boxed{\begin{array}{c} \leftarrow H_k(M,\mathbb{Z}) \xleftarrow[j_*]{} H_k(U,\mathbb{Z}) \xleftarrow{\text{Tube}} H_{k-1}(N,\mathbb{Z}) \xleftarrow{\cap N} H_{k+1}(M,\mathbb{Z}) \leftarrow \cdots \\ \partial\Gamma \longleftarrow\!\shortmid\ \gamma = \Gamma \cap N \\ (\Gamma \text{ a } (k+1)\text{-chain}) \end{array}}$$



The terms of (F.2) & (F.3) are clearly dual.
We claim that the maps are adjoint: indeed,

- $\int_{j(\beta)} \xi = \int_\beta j^* \xi$
- $\int_{\mathrm{Tube}(\gamma)} \omega = \lim_{\epsilon \to 0} \int_{\partial\Gamma_\epsilon} \sum_\alpha \overset{\text{partition of unity}}{\eta_\alpha} \left(\omega_1^\alpha \wedge \frac{dz_1^\alpha}{z_1^\alpha} + \omega_2^\alpha \right) = \lim_{\epsilon \to 0} \int_{\partial\Gamma_\epsilon} \sum \eta_\alpha \omega_1^\alpha \wedge \frac{dz_1^\alpha}{z_1^\alpha} = 2\pi i \int_\gamma \iota^* \sum \eta_\alpha \omega_1^\alpha$

  $= 2\pi i \int_\gamma \mathrm{Res}\,\omega$
- $\int_{\beta \cap N} \varphi \underset{\uparrow}{=} \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\partial\beta_\epsilon} \overset{\text{s.t. } \mathrm{Res}_N \xi = \varphi}{\xi} \underset{\text{Stokes}}{=} \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\beta_\epsilon} d\xi = \frac{1}{2\pi i} \int_\beta Gy(\varphi)$.

  (Tube/Res adjointness)

  $\beta_\epsilon = \beta \setminus \beta \cap (\text{"}\epsilon\text{-nbhd" of } N)$

  $(\partial\beta_\epsilon = \mathrm{Tube}(\beta \cap N))$

Rational differential forms We now specialize to the setting:

$X \subset \mathbb{P}^n$ smooth degree-d hypersurface
"N" "M"

$\Omega^m_{\mathbb{P}^n}(*X) :=$ sheaf of meromorphic m-forms with poles only along X (any order).

$\mathcal{A}^m_k :=$ rational m-forms on $\mathbb{P}^n$ with poles of order $\leq k$ along X (no other poles)

$\mathcal{A}^m := \bigcup_k \mathcal{A}^m_k$. We will quote the following:

<u>Chow's Theorem</u>*: $\mathcal{A}^m_k = \Gamma(\mathbb{P}^n, \Omega^m_{\mathbb{P}^n}(kX))$, $\mathcal{A}^m = \Gamma(\mathbb{P}^n, \Omega^m_{\mathbb{P}^n}(*X))$.

<u>Bott vanishing</u>**: $H^i(\mathbb{P}^n, \Omega^j(kX)) = \{0\}$ for all $i, k > 0$ and $j \geq 0$.

Set $\underset{\substack{\uparrow \\ \text{filtration by} \\ \underline{\text{pole order}}}}{\mathcal{P}_k}\Omega^m_{\mathbb{P}^n}(*X) := \Omega^m_{\mathbb{P}^n}(kX)$, $\quad Gr^{\mathcal{P}}_k \Omega^m_{\mathbb{P}^n}(*X) := \dfrac{\mathcal{P}_k \Omega^m_{\mathbb{P}^n}(*X)}{\mathcal{P}_{k-1}\Omega^m_{\mathbb{P}^n}(*X)}$.

The obvious s.e.s. $0 \to \mathcal{P}_{k-1} \to \mathcal{P}_k \to Gr^{\mathcal{P}}_k \to 0$ together with Bott + Chow $\underset{(m \geq 1)}{\Rightarrow}$

$$\text{(F.4)} \qquad \boxed{\Gamma(\mathbb{P}^n, Gr^{\mathcal{P}}_k \Omega^m_{\mathbb{P}^n}(*X)) = \mathcal{A}^m_k / \mathcal{A}^m_{k-1}}.$$

Griffiths considered the "rational dR-cohomology" groups

$\mathcal{H}^n(X) := \mathcal{A}^n / d\mathcal{A}^{n-1}$ with

- filtration $\mathcal{P}_k \mathcal{H}^n(X) := \dfrac{\mathcal{A}^n_k}{(d\mathcal{A}^{n-1}) \cap \mathcal{A}^n_k}$ $\Downarrow$
- graded pcs. $Gr^{\mathcal{P}}_k \mathcal{H}^n(X) = \dfrac{\mathcal{A}^n_k}{d\mathcal{A}^{n-1} \cap \mathcal{A}^n_k + \mathcal{A}^n_{k-1}}$

I set $\mathfrak{H}^n_k := \dfrac{\mathcal{A}^n_k}{d\mathcal{A}^{n-1}_{k-1} + \mathcal{A}^n_{k-1}}$ $\Big(\underset{\substack{\uparrow \\ \text{this will later turn out to be an } \cong.}}{\twoheadrightarrow} Gr^{\mathcal{P}}_k \mathcal{H}^n(X) \Big)$

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* also referred to as "GAGA" – global analytic $\Rightarrow$ global algebraic.

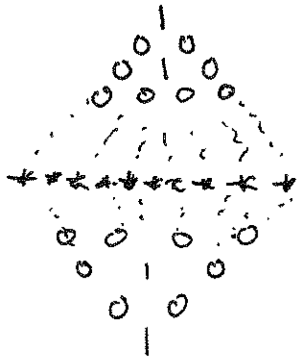
** comes from Borel-Weil-Bott theorem in representation thy of Lie groups.

<u>Remark 1</u>: (On Res, Gy, & topology of hypersurfaces)

(i) These groups will turn out to compute the primitive $H^{p,q}$-spaces of X, essentially by Poincaré residue; this means dually that the periods of closed (p,q)-forms on X (over cycles γ) are just periods of rational forms on $\mathbb{P}^n \setminus X$ (over cycles Tube(γ)). Note that we are only concerned with the case $p+q=n-1$ — i.e. middle degree cohomology. This is because of

$$\boxed{\begin{array}{ll} \underline{\text{Weak Lefschetz}}: & H^\ell(\mathbb{P}^n) \xrightarrow{\iota^*} H^\ell(X) \text{ is an } \cong \text{ for } \ell < n-1 \\ \text{dually}, & H^\ell(X) \xrightarrow{Gy} H^{\ell+2}(X) \text{ is an } \cong \text{ for } \ell > n-1 \end{array}},$$

which implies that the Hodge diamond of X takes the form



← all the interesting stuff in degree ($p+q=$) $n-1$.

(ii) By looking at the dual maps in homology, one sees that the composite

$$\begin{tikzcd} H^{n-1}(X) \arrow[r, "Gy"] \arrow[rr, bend right] & H^{n+1}(\mathbb{P}^n) \arrow[r, "\iota^*"] & H^{n+1}(X) \end{tikzcd}$$

is cup-product with $\iota^*[X] = d \cdot \iota^*[H] = d \cdot [\omega_X]$

$\uparrow$ hyperplane ($=\mathbb{P}^{n-1}$) $\qquad \uparrow$ Fubini-Study (Kähler) form

Using weak Lefschetz + Poincaré duality, ι^* is an isomorphism, so that

$$\ker(Gy) = \ker(\cup[\omega_X]) = \ker(L) = H^{n-1}_{pr}(X).$$

But then, in the Residue sequence ($\mathbb{Q}$-coeffs.)

$$\begin{tikzcd} H^{n-2}(X) \arrow[r, "Gy", "\cong"'] & H^n(\mathbb{P}^n) \arrow[r] \arrow[dr, Rightarrow] & H^n(\mathbb{P}^n \setminus X) \arrow[r, "Res"] & H^{n-1}(X) \arrow[r, "Gy"] & H^{n+1}(\mathbb{P}^n) \arrow[r] & {} \\ & & \textcircled{0} & & & \end{tikzcd},$$

($\cong$: weak Lefschetz + Poincaré duality) we see that

$$H^n(\mathbb{P}^n \setminus X) \underset{\cong}{\xrightarrow{\mathrm{Res}}} H^{n-1}_{pr}(X) \quad ;$$

similarly, $H^q(\mathbb{P}^n, \Omega^{p+1}(\log X)) \underset{\cong}{\xrightarrow{\mathrm{Res}}} H^q_{pr}(X, \Omega^p) \left(:= \ker\{ H^q(X,\Omega^p) \xrightarrow[\cup(\omega_X)]{} H^{q+1}(X,\Omega^{p+1})\} \right.$,

[where $p+q=n-1$] .

□

Lemma 2 : The sequence of sheaves on $\mathbb{P}^n$

$$\text{(F.5)} \quad \boxed{0 \to \Omega^{p+1}_{\mathbb{P}^n}(\log X) \hookrightarrow \Omega^{p+1}_{\mathbb{P}^n}(X) \xrightarrow{d} \mathrm{Gr}^P_2 \Omega^{p+2}_{\mathbb{P}^n}(*X) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{Gr}^P_{n-p} \Omega^n_{\mathbb{P}^n}(*X) \to 0}$$

is exact.

Proof: we check exactness* at the middle term of

$$\Omega^p_{\mathbb{P}^n}(\ell X) \xrightarrow{d} \frac{\Omega^{p+1}_{\mathbb{P}^n}((\ell+1)X)}{\Omega^{p+1}_{\mathbb{P}^n}(\ell X)} \xrightarrow{d} \frac{\Omega^{p+2}_{\mathbb{P}^n}((\ell+2)X)}{\Omega^{p+2}_{\mathbb{P}^n}((\ell+1)X)}$$

in local coords $(z_1, \ldots, z_n)$ s.t. $z_1 = 0$ locally defines X. First,

$$(d \circ d)\left(\frac{h\, dz_I}{z_1^\ell} \right)^{|I|=p} = d\left(\underbrace{\cancel{\frac{dh \wedge dz_I}{z_1^\ell}}}_{\substack{\equiv 0 \\ \bmod \Omega^{p+1}(\ell X)}} \pm \ell h \frac{\begin{cases} 0, & 1 \in I \\ \text{or} \\ dz_{I \cup \{1\}} \end{cases}}{z_1^{\ell+1}} \right) = \begin{cases} 0 \\ \text{or} \\ \frac{\ell\, dh \wedge dz_{I \cup \{1\}}}{z_1^{\ell+1}} \end{cases} \equiv 0 \ \bmod \Omega^{p+2}((\ell+1)X).$$

Now, given $\omega \in \Omega^{p+1}_{\mathbb{P}^n}((\ell+1)X)(U)$, $\omega = \frac{\eta_1}{z_1^{\ell+1}} + \frac{dz_1 \wedge \eta_2}{z_1^{\ell+1}}$ (no dz_1), where $\eta_1 = \sum_{1 \notin I} h_I\, dz_I$,

and we have

$$d\omega \in \Omega^{p+2}((\ell+1)X)(U) \ [\text{i.e.} \equiv 0] \iff$$

$$d\left(\frac{\eta_1}{z_1^{\ell+1}} \right) \in \Omega^{p+2}((\ell+1)X)(U) \iff$$

$$\sum \frac{h_I}{z_1^{\ell+2}} dz_{I \cup \{1\}} \in \Omega^{p+2}((\ell+1)X)(U) \iff$$

$$h_I / z_1 \in \mathcal{O}(U) \ (\forall I) \iff \frac{\eta_1}{z^{\ell+1}} \in \Omega^{p+1}(\ell X)(U) \ [\text{i.e.} \equiv 0]$$

$$\iff \omega \in d\left(\Omega^p(\ell X)\right) + \Omega^{p+1}(\ell X) .$$

□

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* locally, on some suff. small U

Bott vanishing implies that all but the 1st term of (F.4) are $\Gamma_{\mathbb{P}^n}$-acyclic, rendering (F.5) a $\Gamma_{\mathbb{P}^n}$-acyclic resolution of $\Omega^{p+1}(\log X)$.

So we have

$$\begin{tikzcd} \mathcal{G}^n_{n-p} := \dfrac{\mathcal{A}^n_{n-p}}{d\mathcal{A}^{n-1}_{n-p-1}+\mathcal{A}^n_{n-p-1}} \arrow[r, phantom, "\cong"] \arrow[r, bend left, "(F.4)"] & \dfrac{\Gamma(\mathbb{P}^n, Gr^p_{n-p}\Omega^n(*X))}{d\Gamma(\mathbb{P}^n, Gr^p_{n-p-1}\Omega^{n-1}(*X))} \arrow[r, phantom, "\overset{(F.5)}{\cong}"] & H^{n-p-1}(\mathbb{P}^n, \Omega^{p+1}(\log X)) \arrow[d, "\mathrm{Res}", "\cong"'] \\ & & H^{n-p-1}_{pr}(X, \Omega^p_X) \end{tikzcd}$$

, proving

<u>Corollary 2</u>: $\mathcal{G}^n_{n-p} \cong H^{p,n-p-1}_{pr}(X)$.

In fact, this "isomorphism of graded pieces"* can be lifted to a "filtered isomorphism" which is much more explicit & useful. First consider the inclusions of acyclic sheaves (the first being acyclic by Bott)

$$\Omega^\bullet_{\mathbb{P}^n}(*X) \hookrightarrow \jmath_* A^\bullet_{\mathbb{P}^n\setminus X} \overset{\simeq}{\hookleftarrow} A^\bullet_{\mathbb{P}^n}(\log X).$$

The left-hand inclusion is also a quasi-isomorphism (Ex/ repeat the proof of Lemma/Cor. 1). This means that we have isomorphisms on cohomology groups of complexes of global sections:

$$\begin{tikzcd} h^n(X) \arrow[r, "\cong"] \arrow[rrd, bend right, "\widetilde{\mathrm{Res}}", "(\cong)"'] & H^n(\mathbb{P}^n\setminus X, \mathbb{C}) & H^n\{\Gamma(\mathbb{P}^n, A^\bullet(\log X))\} \arrow[l, "\simeq"'] \arrow[d, "\mathrm{Res}", "\cong"'] \\ & & H^{n-1}_{pr}(X, \mathbb{C}) \end{tikzcd}$$

<u>Theorem 1 (Griffiths)</u>: $\widetilde{\mathrm{Res}}$ induces isomorphisms

$$(F.6) \qquad \boxed{P_{n-p}h^n(X) \xrightarrow{\cong} F^pH^{n-1}_{pr}(X,\mathbb{C})}.$$

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* of course, we haven't yet proved $\mathcal{G}^n_{n-p} \cong Gr^p_{n-p}h^n(X)$; that will come out in the wash.

Moreover, $Gr^{\mathcal{P}}_{n-p} h^n(X) = \mathcal{G}^n_{n-p}$ and the graded pieces of $\widetilde{Res}$ induce the isomorphisms of Corollary 2.

Proof: WTS: under $\widetilde{Res}$, $\mathcal{P}_{n-p}$ goes into $F^{p+1}H^n\{\Gamma(\mathbb{P}^n, A^\bullet(\log X))\}$

$$:= H^n\{\Gamma(\mathbb{P}^n, F^{p+1}A^\bullet(\log X))\}$$
$$= Res^{-1} F^p H^{n-1}_{pr}(X, \mathbb{C}).$$

[This will give a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathcal{P}_{n-p-1} h^n(X) \arrow[r, hook] \arrow[d, hook] & F^{p+1} H^{n-1}_{pr}(X) \arrow[d, hook] \\
 & \mathcal{P}_{n-p} h^n(X) \arrow[r, hook] \arrow[d, two heads] & F^p H^{n-1}_{pr}(X) \arrow[d, two heads] \\
\mathcal{G}^n_{n-p} \arrow[r, two heads, "\alpha"] \arrow[rr, bend right, "\cong\ (\mathrm{Cor.}\ 2)"'] & Gr^{\mathcal{P}}_{n-p} h^n(X) \arrow[r, "\beta"] \arrow[d] & Gr^p_F H^{n-1}_{pr}(X) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

which forces α to be injective (hence an isomorphism), then β to be an $\cong$. Then a dimension argument forces the upper $\hookrightarrow$'s to be $\cong$'s.]

NOTE: to see that the $\cong$ of Cor. 2 is compatible with $\widetilde{Res}$ (hence β) one has to check that the $\cong$ attributed to (F.5) in proof of Cor. 2 is obtained by changing a form in $\mathfrak{a}^n_{n-p}$ by a coboundary (to get a form in $Gr^{p+1}_F A^n(\log X)(\mathbb{P}^n)$). That is easily done by tensoring (F.5) $\otimes A^{0,\bullet}_{\mathbb{P}^n}$ to get a double complex.

To simplify notation, write $\boxed{k := n-p}$. Let $\omega \in \mathfrak{a}^n_k$, and again take $\mathcal{U}_\alpha$ & local coords. $\underline{z}$ s.t. $X \cap \mathcal{U}_\alpha = \{z_1 = 0\}$. On $\mathcal{U}_\alpha$,

$$\omega|_{\mathcal{U}_\alpha} = \frac{h\, dz_1 \wedge \cdots \wedge dz_n}{z_1^k} = \frac{dz_1}{z_1^k} \wedge \underbrace{(h\, dz_2 \wedge \cdots \wedge dz_n)}_{\overset{\shortparallel}{\theta_\alpha \in \Omega^{n-1}(\mathcal{U}_\alpha)}}.$$

Set $\xi_\alpha := \dfrac{\theta_\alpha}{(k-1) z_1^{k-1}} \rightsquigarrow d\xi_\alpha = \dfrac{d\theta_\alpha}{(k-1) z_1^{k-1}} - \dfrac{dz_1}{z_1^k} \wedge \theta_\alpha$

$$=: \tau_\alpha - \omega|_{\mathcal{U}_\alpha}$$

If $\mathcal{U}_\alpha \cap X = \emptyset$, then we take $\xi_\alpha =$ a primitive of $-\omega|_{\mathcal{U}_\alpha}$.

Set $\xi := \sum \eta_\alpha \Xi_\alpha \rightsquigarrow d\xi = \sum(d\eta_\alpha \wedge \xi_\alpha + \eta_\alpha \tau_\alpha) - \omega$

$$=: \tau - \omega$$

where $\tau \in \ker(d) \subseteq \Gamma(\mathbb{P}^n, F^{n-1}A^n((k-1)X))$.

So we have reduced the pole-order at the cost of losing type $(n,0)$.

Keep reducing until you reach $\Gamma(\mathbb{P}^n, F^{\overset{p+1}{\boxed{n-k+1}}} A^n(\underbrace{X}))_{d\text{-cl}}$.

i.e. order-1 pole along X

The result has local form

$$\underbrace{\frac{\boxed{\eta_1}}{z_1} + \frac{\boxed{\eta_2} \wedge dz_1}{z_1}}_{C^\infty,\ \text{no } dz_1} \xmapsto{d} \frac{d\eta_1}{z_1} - \frac{dz_1 \wedge \eta_1}{z_1^2} + \frac{d\eta_2 \wedge dz_1}{z_1} = 0$$

$$\Downarrow$$

η_1/z_1 is C^∞.

Hence it has only a log pole along X. ☐

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Jacobian rings

We have
$$\boxed{\mathcal{A}^n_k = \Gamma(\mathbb{P}^n, \underbrace{K_{\mathbb{P}^n} \otimes \mathcal{O}(kX)}_{\cong \mathcal{O}(kd-n-1)}) \cong S^{kd-n-1}_{n+1}}\ .\qquad (F.7)$$

How to realize this $\cong$ in such a way as to make the rational forms explicit?

Given $\omega \in \mathcal{A}^n_k$, by definition of a rational form, it lifts under

$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$

N.B. Think of π as quotient by the action $t.(z_0, \ldots, z_n) = (tz_0, \ldots, tz_n)$, which is the flow along the euler vector field $\boxed{e := \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}}$.

to

$$\pi^*\omega =: \Psi = \frac{1}{F^k} \sum_{|I|=m} A_I(\underline{z})\, d\underline{z}_I \in \tilde{\mathcal{A}}^m := \left(\frac{\text{rational } m\text{-forms}}{\text{on } \mathbb{C}^{n+1}}\right),$$

where $F \in S^d_{n+1}$ defines $X\ (= \{F = 0\})$. We now characterize those $\Psi \in \tilde{\mathcal{A}}^m$ that *descend to* $\mathbb{P}^n$ (i.e. arise in this fashion).

Let $\psi \in \tilde{\mathcal{A}}^m$ be a monomial, and define its *degree* by

$$(\deg \psi)\cdot\psi := [d, e\lrcorner]\psi := e \lrcorner d\psi + d(e \lrcorner \psi).$$

Example 1: (i) $\deg(d\underline{Z}_I) = |I|$

(ii) $\deg(P) = k$, for $P \in S^k$

(iii) writing $\boxed{\Omega := e \lrcorner d\underline{Z} = \sum (-1)^i Z_i\, dZ_0 \wedge \cdots \wedge \widehat{dZ_i} \wedge \cdots \wedge dZ_n}$

$$(\deg \Omega)\Omega = d(\underbrace{e \lrcorner (e \lrcorner d\underline{Z})}_{0}) + e \lrcorner d(e \lrcorner d\underline{Z})$$

$$= 0 + e \lrcorner \underbrace{(\deg d\underline{Z})}_{n+1} d\underline{Z}$$

$$= (n+1)\,\Omega.$$

(iv) Ex/ $\deg(\alpha \wedge \beta) = \deg(\alpha) + \deg\beta$ $\left(\Rightarrow \deg\left(\frac{\alpha}{F^k}\right) = \deg(\alpha) - kd\right)$.

This leads to the crucial

Lemma 3: (i) For a homogeneous polynomial $P \in S^a_{n+1}$,

$$\deg \underbrace{\left(\frac{P\Omega}{F^k}\right)}_{\tau \in \tilde{\mathcal{A}}^n} = 0 \iff a = kd - n - 1;$$ (this is clear)

and (ii) π^* identifies $\mathcal{A}^n_k$ with $\frac{\Omega}{F^k} \cdot S^a_{n+1}$.

Sketch of Proof (ii): A form Ψ descends to $\mathbb{P}^n$ $\iff$ it is *invariant* $(\deg \Psi = 0)$ and *horizontal* $(e \lrcorner \Psi = 0)$.

For $\frac{P\Omega}{F^k}$, horizontality is clear since $e \lrcorner \Omega = e \lrcorner (e \lrcorner dZ) = 0$.

But then $\frac{\Omega}{F^k} \cdot S^a_{n+1} \hookrightarrow \pi^* \mathcal{A}^n_k$, and by *equality of dimensions* (cf. (F.7)) this must be onto. □

writing $\frac{P\Omega}{F^{k-1}} = \frac{(F\cdot P)\Omega}{F^k}$, we see that the inclusion $\mathcal{A}^n_{k-1} \hookrightarrow \mathcal{A}^n_k$ corresponds (under the identification of Lemma 3) to

$$S^{(k-1)d-n-1} \hookrightarrow S^{kd-n-1}$$
$$P \longmapsto F\cdot P \quad .$$

Now our goal is to use Lemma 3 to identify the $Gr^p_k \mathcal{H}^n(X)$, for which we need the holomorphic pole-reduction statement:

Lemma 4: If, for some j, $P (\in S^{kd-n-1}_{n+1})$ has a factor of $\partial F/\partial z_j$, then $\frac{P\Omega}{F^k} \in \mathcal{A}^n_{k-1} + d\mathcal{A}^{n-1}_{k-1}$. More precisely, under the identification $\mathcal{A}^n_k \cong S^{kd-n-1}_{n+1}$, we have

$$\mathcal{A}^n_{k-1} + d\,\mathcal{A}^{n-1}_{k-1} \cong \underset{\text{"Jacobian ideal"}}{J_F^{kd-n-1}} := \underbrace{\overbrace{\left(\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}\right)}^{\text{ideal in } S = \oplus S^m}_{kd-n-1}}_{\text{ideal} \cap S^{kd-n-1}}$$

Remark 2: The idea (say, on U_0) should be that if $P = P_0 \frac{\partial f}{\partial z_j}$, then

local defining eqn. for X

writing $\Theta := \frac{P\,dz_1\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_n}{\partial f/\partial z_j} = P_0\, dz_1\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_n$,

$$d\left(\frac{\Theta}{(k-1)f^{k-1}}\right) = \frac{d\Theta}{(k-1)f^{k-1}} - \frac{df\wedge\Theta}{f^k} = \frac{\partial P_0/\partial z_j}{(k-1)f^{k-1}}\,d\underline{z} - \underbrace{\frac{P_0\,\partial f/\partial z_j}{f^k}\,d\underline{z}}_{\frac{P\,d\underline{z}}{f^k}}$$

(reduce pole, still mero.)

To make this precise over all of $\mathbb{P}^n$, we'll need to again employ $\mathbb{C}^*$-invariant forms on $\mathcal{U} = \mathbb{C}^{n+1}\setminus\{0\}$. Let $\mathcal{Q}^m :=$ polynomial m-forms on $\mathbb{C}^{n+1}$.

Proof (Lemma 4): ① Let $\omega = \frac{P\Omega}{F^k} \in \mathcal{A}^n_{k-1}$. Then $P = FQ$ $(Q \in S^{(k-1)d-n-1})$

viewed as subspace of $\mathcal{A}^n_k$

$= \frac{1}{d}\,e(F)\,Q \in J_F^{kd-n-1}$.

$e(F) = \sum z_i \frac{\partial F}{\partial z_i}$ $(= d\cdot F)$

② Next, consider $\eta \in \mathcal{A}^{n-1}_{k-1}$; then $\pi^*\eta =: \Psi = \frac{\Psi_0}{F^{k-1}}$ with $\Psi_0 \in \Omega^{n-1}$.

Using $e \lrcorner \Psi = 0 = e \lrcorner d\Psi$, one has* $\Psi_0 = e \lrcorner \underbrace{\frac{1}{(k-1)d}\, d\Psi_0}_{=:\varphi \in \Omega^n}$, with $\deg(\varphi) = (k-1)d$.

So $\deg\left(\frac{\varphi}{F^{k-1}}\right) = 0$ and

$$d\Psi = d\left(\frac{e \lrcorner \varphi}{F^{k-1}}\right) = d\left(e \lrcorner \frac{\varphi}{F^{k-1}}\right) = \deg\left(\frac{\varphi}{F^{k-1}}\right)\frac{\varphi}{F^{k-1}} - e \lrcorner d\left(\frac{\varphi}{F^{k-1}}\right)$$
$$= \frac{e \lrcorner \left((k-1)\, dF \wedge \varphi - F d\varphi\right)}{F^k}.$$

Now the only options for φ (= holo. n-form on $\mathbb{C}^{n+1}$ of degree $(k-1)d$) are to take $d\underline{Z}_{\hat{i}} := dZ_0 \wedge \cdots \wedge \widehat{dZ_i} \wedge \cdots \wedge dZ_n$ and any $R_i \in S^{(k-1)d-n}$ and write

$$\varphi = \sum_{i=0}^{n} (-1)^i R_i \, d\underline{Z}_{\hat{i}} \quad \text{so that}$$

$$d\varphi = \sum_{i=0}^{n} \frac{\partial R_i}{\partial Z_i} d\underline{Z}, \quad dF \wedge \varphi = \sum_{i=0}^{n} R_i \frac{\partial F}{\partial Z_i} d\underline{Z}, \text{ and}$$

$$(\pi^* d\eta =)\, d\Psi = \frac{e \lrcorner \left(\sum_i \left\{(k-1) R_i \frac{\partial F}{\partial Z_i} - F \frac{\partial R_i}{\partial Z_i}\right\} d\underline{Z}\right)}{F^k}$$

$$= \frac{\left(\sum_i \left\{(k-1) R_i \frac{\partial F}{\partial Z_i} - F \frac{\partial R}{\partial Z_i}\right\}\right) \Omega}{F^k} \quad \text{is the general}$$

form of an element of $d\mathcal{A}^{n-1}_{k-1}$. But the bracketed polynomials $\in J_F$.

③ Conversely, given any $P \in J_F^{kd-n-1}$, say $P = \sum (k-1) R_i \frac{\partial F}{\partial Z_i}$, clearly subtracting the class $d\Psi \in d\mathcal{A}^{n-1}_{k-1}$ will "reduce the poles" of $\frac{P\Omega}{F^k}$ to $\frac{\left(\sum \frac{\partial R}{\partial Z_i}\right)\Omega}{F^{k-1}} \in \mathcal{A}^n_{k-1}$.

$\square$

$*$ $0 = F^k e \lrcorner d\Psi = e \lrcorner \left(F\, d\Psi_0 - (k-1) dF \wedge \Psi_0\right) = F \cdot e \lrcorner d\Psi_0 - (k-1) e \lrcorner (dF \wedge \Psi_0)$
$= F \cdot e \lrcorner d\Psi_0 - (k-1)(e \lrcorner dF) \wedge \Psi_0 = F \cdot \deg(\Psi_0)\Psi_0 - (k-1)d \cdot F \cdot \Psi_0$

Substituting back in $k=n-p$, we may now conclude the

<u>Theorem 2 (Griffiths)</u>: The map (F.6) has graded pieces

(F.8)
$$\boxed{R_F^{\overset{(n-p)d-n-1}{\textcircled{kd-n-1}}} := \frac{S_{n+1}^{kd-n-1}}{J_F^{kd-n-1}} \xrightarrow{\cong} Gr^{p}_{\underset{n-p}{\textcircled{k}}}\, h^n(X) \xrightarrow{\cong} H_{pr}^{\overset{p,n-p-1}{\textcircled{n-k,k-1}}}(X)}$$
$$P \longmapsto \frac{P\Omega}{F^k} \longmapsto Res_X\Big(\underbrace{\frac{P\Omega}{F^k}+d\xi}\Big)$$

$d\xi$ chosen so that this is a log form.

<u>Remark 3</u>: (i) One should really think of the $H_{pr}^{n-k,k-1}(X)$ as $Gr_F^{n-k}H_{pr}^{n-1}(X,\mathbb{C})$.

(ii) $R_F := \frac{S_{n+1}}{J_F}$ is called the "<u>Jacobian ring</u>" (we need homogeneous pieces of it). In general computing it can be hard (cf require Gröbner bases).

(**)(iii) To put this all in perspective, suppose we now want to compute $\int_\gamma \omega$ for $[\gamma]\in H_{n-1}(X,\mathbb{Z})$, $[\omega]\in F^pH^{n-1}_{pr}(X,\mathbb{C})$.
Then up to coboundary* on X, $\omega = \frac{1}{2\pi i}Res_X(\beta)$ for $\beta\in\Gamma(\mathbb{P}^n, F^{p+1}A^n(\log X))_{d\text{-}cl}$, and up to coboundary on $\mathbb{P}^n\setminus X$, $\beta \equiv \frac{P\Omega}{F^{n-p}} \in H^0(\Omega^n_{\mathbb{P}^n}((n-p)X)) = \mathcal{U}^n_{n-p}$.
Consequently by Stokes's theorem

(F.9)
$$\boxed{\int_\gamma \omega = \frac{1}{2\pi i}\int_{Tube(\gamma)}\beta = \frac{1}{2\pi i}\int_{Tube(\gamma)}\frac{P\Omega}{F^{n-p}}}$$

for some $P\in S^{kd-n-1}$ $(k=n-p)$. So we have a tight relationship, via Poincaré residues and pole-reduction, between "Hodge-theoretic periods" on X and <u>rational integrals on $\mathbb{P}^n$</u> (which in some sense were the "original" object).

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* i.e. exact forms

(**) important point!

Computing Hodge numbers

We quote the following result, which follows from a standard upper semi-continuity theorem for the dimension of the kernel for a continuous family of elliptic differential operators:

Theorem 3: Let $\mathcal{X} \xrightarrow{\pi} \mathcal{B}$ be a family of compact complex manifolds* and $\mathcal{V} \to \mathcal{X}$ a holo. vector bundle. Then for each $b_0 \in \mathcal{B}$, $\exists$ nbhd. $U \subset \mathcal{B}$ s.t. $\dim H^q(X_b, \mathcal{V}|_{X_b}) \leq \dim H^q(X_{b_0}, \mathcal{V}|_{X_{b_0}})$ $(\forall b \in U)$.

Corollary 3: For a family of compact Kähler manifolds, the Hodge #'s $h^{p,q}$ are constant.

Proof: Theorem 3 says (taking $\mathcal{V} = \Omega^p$) they can only jump up (at isolated points). But $(\forall b \in U)$ the sum is $\sum_{p+q=k} h^{p,q}(X_b) = h^k(X_b)$, which (all X_b being diffeomorphic) is constant in b. □

Since varying $F \in S^d_{n+1}$ (avoiding the choices that make $X = \{F=0\} \subset \mathbb{P}^n$ singular) yields all degree-d smooth projective hypersurfaces (in $\mathbb{P}^n$) in a connected family,

(a) the Hodge #'s of X depend only on d & n

(b) we may select the most convenient $F \left(\overset{\text{always}}{=} Z_0^d + \dots + Z_n^d \right)$

for computing the

$$h^{p,n-p-1}_{pr} = \dim H^{p,n-p-1}_{pr}(X) = \dim \left(R_F^{(n-p)d-n-1} \right).$$

↑ (this can be omitted except for $n-p-1 = p$)

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* more precisely: $\mathcal{X}$ & $\mathcal{B}$ are $\mathbb{C}$-mflds; π a proper holo. submersion. Write $X_b := \pi^{-1}(b)$ for the (compact $\mathbb{C}$-mfld.) fibers.

Theorem 4: Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. We have

$$\begin{cases} h^{n-1,0} = \dim(\Omega^{n-1}(X)) = \binom{d-1}{n} . \\ \Omega^{n-1}(X) = \left\{ \mathrm{Res}_X\left(\frac{P\Omega}{F}\right) \;\middle|\; P \in S_{n+1}^{d-n-1} \right\} \end{cases}$$

Proof: The generators of J_F have degree $d-1$; hence $J_F^{d-n-1} = \{0\}$, and $R_F^{d-n-1} = S_{n+1}^{d-n-1}$, whose dimension is computed by the box-bar formula. $\square$

Corollary 4: (degree-genus formula) For $X \subset \mathbb{P}^2$ a smooth degree d curve, $g = \frac{(d-1)(d-2)}{2}$.

Corollary 5: For X as in Thm. 4, X is Calabi-Yau $\Leftrightarrow$ $d = n+1$.

$$\begin{cases} \text{If } d > n+1, \text{ then } h^{n-1,0} > 1 \\ \text{If } d < n+1, \text{ then } h^{n-1,0} = 0 . \end{cases}$$

(Pf.: The C-Y assertion uses adjunction: $K_X = K_{\mathbb{P}^n} \otimes N_{X/\mathbb{P}^n} = \mathcal{O}(-n-1) \otimes \mathcal{O}(d)$
The rest is by Thm. 4.)

Finally, we shall compute the Hodge diamonds in a few special cases.

Proposition 1: For complex surfaces in $\mathbb{P}^3$ one has

$(d=3)$

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

$(d=4)$

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

$(d=5)$

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 4 & & 45 & & 4 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

and for quintic 3-folds $\subseteq \mathbb{P}^4$ $(d=5)$

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

<u>Proof</u>: For the surfaces (n=3), we only need to compute $h^{1,1}$ $(p=1)$.

So $(n-p)d-n-1 = 2d-4$, and

$$H^{1,1}_{pr}(X) \cong \frac{S^{2d-4}}{(z_0^{d-1},\dots,z_3^{d-1})_{2d-4}} \longleftarrow$$ where we have used (6) from bottom p. 171 to compute J_F (take $F = z_0^d + \dots + z_3^d$)

$$d=3: \quad \frac{S^2}{(z_0^2,\dots,z_3^2)_2} = \mathbb{C}\left\langle \{ z_i z_j \mid 0 \le i < j \le 3\} \right\rangle$$

$\uparrow$ # of 6 such elts.

$$\Rightarrow \quad h^{1,1} = h^{1,1}_{pr} + 1 = \binom{4}{2} + 1 = 7.$$

$$d=4: \quad \frac{S^4}{(z_0^3, z_1^3, z_2^3, z_3^3)_4} = \mathbb{C}\left\langle \begin{array}{c} z_0 z_1 z_2 z_3,\ \{ z_i^2 z_j^2 \mid 0 \le i < j \le 3\}, \\ \left\{ z_i^2 z_j z_k \;\middle|\; \begin{array}{l} 0 \le j < k \le 3, \\ i \in \{0,1,2,3\}\setminus\{j,k\} \end{array} \right\} \end{array} \right\rangle$$

$$\Rightarrow \quad h^{1,1} = h^{1,1}_{pr} + 1 = \left(1 + \binom{4}{2} + 2\binom{4}{2}\right) + 1 = 20.$$

Ex/ Complete the proof. (for the CY 3-fold, compute $h^{2,1}$!) $\square$

Euler integrals

Residue theory gives a technique for directly computing certain kinds of periods as power series. This is very important for mirror symmetry (a point of contact between string theory and algebraic geometry).

Let $\{X_t\}_{t\in\mathbb{P}^1}$ be the 1-parameter family of (degree $n+1$)
$(t = t_1/t_0,\ [t_0:t_1] \in \mathbb{P}^1)$
Calabi-Yau hypersurfaces in $\mathbb{P}^n$ given by

$$0 = F_t(\underline{Z}) := t_0 \prod_{i=0}^{n} Z_i - t_1 \sum_{i=0}^{n} Z_i^{n+1}$$

or in affine coordinates $(t;\underbrace{z_1,\ldots,z_n}_{z_i = Z_i/Z_0})$ (and dividing out by $z_1 z_2 \cdots z_n$)

$$0 = 1 - t\phi(\underline{z}), \quad \phi(\underline{z}) := \frac{1+\sum_{i=1}^{n} z_i^{n+1}}{\prod z_i} = z_1^{-1}\cdots z_n^{-1} + z_1^{n} z_2^{-1}\cdots z_n^{-1} + \cdots + z_1^{-1}\cdots z_{n-1}^{-1} z_n^{n}.$$

One has for the constant terms in powers of $\phi(\underline{z})$:

Ex/ $$[\phi^k]_0 = \begin{cases} 0, & n+1 \nmid k \\ \dfrac{((n+1)m)!}{(m!)^{n+1}}, & k = m(n+1) \end{cases}$$

<u>Lemma 5</u>: For $0 < |t| < \frac{1}{n+1}$, $\exists$ a family of topological $(n-1)$-cycles γ_t (with class $[\gamma_t] \in H_{n-1}(X_t, \mathbb{Z})$) s.t. $\mathrm{Tube}([\gamma_t]) \in H_n(\mathbb{P}^n \setminus X_t, \mathbb{Z})$ is represented by $\tau_n := \{(z_1,\ldots,z_n) \mid |z_j| = 1\ (\forall j)\} \approx (S^1)^{\times n}$.

<u>Proof</u>: Let $\Gamma_n := \left\{(z_1,\ldots,z_n) \,\middle|\, \begin{array}{l} |z_j| = 1\ (\forall j > 1) \\ |z_1| \le 1 \end{array}\right\} \rightsquigarrow \partial\Gamma_n = \tau_n$.

But $$1 \underset{\text{on } X_t}{=} |t\phi(\underline{z})| = |t| \frac{|1+\sum z_i^{n+1}|}{\underset{1 \text{ on } \tau_n}{\underbrace{\prod |z_i|}}} \le (n+1)|t| \underset{\overset{\uparrow}{|t| < \frac{1}{n+1}}}{<} 1$$

$\Longrightarrow \tau_n \cap X_t = \emptyset$

$\Longrightarrow \underbrace{\gamma_t := \Gamma_t}_{\text{Tube}} \cap X_t$ is a cycle (∂-closed)

$\blacksquare$

Consider the family of nowhere-vanishing $\overset{\text{Ex/ (why?)}}{}$ holomorphic $(n-1)$-forms given by (cf. Theorem 4) $\omega_t := \mathrm{Res}_{X_t}\left(\frac{\Omega}{F_t}\right) \in \Omega^{n-1}(X_t)$

(here $P \in S^{\cancel{d-n-1} \to 0}$ is $\equiv 1$) or in affine coordinates,

$$\mathrm{Res}_{X_t}\left(\frac{\overset{=\Omega,}{\overbrace{dz_1\wedge\cdots\wedge dz_n}}}{\underset{F_t}{\underbrace{(1-t\phi)z_1\cdots z_n}}}\right) = \mathrm{Res}_{X_t}\left(\frac{d\log z_1\wedge\cdots\wedge d\log z_n}{1-t\phi}\right).$$

Then we compute the period $\left(|t| < \frac{1}{n+1}\right)$

(F.10)
$$\begin{cases} \mathcal{P}(t) = \int_{\gamma_t} \frac{\omega_t}{(2\pi i)^{n-1}} \overset{\boxed{\text{using adjointness of Tube \& Res}}}{=} \sum_{k\geq 0} t^k \frac{1}{(2\pi i)^n} \int_{\tau_n} (\phi(\underline{z}))^k \, d\log z_1 \wedge \cdots \wedge d\log z_n \\ \qquad \text{Tube} \\ \underset{\text{Cauchy}}{=} \sum_{k\geq 0} [\phi^k]_0 t^k = \sum_{m \geq 0} \frac{((n+1)\ m)!}{(m!)^{n+1}} t^{(n+1)m} \end{cases}$$

Ex/ show that $\mathcal{P}(t)$ satisfies the (generalized hypergeometric) ODE

(F.11)
$$\boxed{0 = \left(\delta^{n+1} + (n+1)^{n+1} t^{n+1} (\delta + (n+1)) \circ (\delta + n) \circ \cdots \circ (\delta + 1)\right) \mathcal{P}(t)}$$

where $\delta := t\frac{d}{dt}$. This is an example of a Picard-Fuchs equation.

Remark 9: The monodromy of the analytic continuation of $\mathcal{P}(t)$, which still satisfies (F.11), reflects monodromy of the cycle class γ_t as t goes around points t_0 in the discriminant locus $\Delta = \{t_0 \in \mathbb{P}^1 \mid X_{t_0} \text{ singular}\} \subset \mathbb{P}^1$.

Provided this monodromy (or rather, $\pi_1(\mathbb{P}^1 \setminus \Delta)$) acts irreducibly on $H_{n-1}(X_t, \mathbb{Z})$ (finite set), it is then clear that the remaining periods of ω_t about $t=0$ will be given by the other

$$\begin{cases} \log t + \text{holo}. \\ \log^2 t + \log t \cdot (\text{holo}) + \text{holo}) \\ \vdots \\ \log^{n-1} t + \ \cdots \end{cases}$$

local solutions of (F.11).