6. Hodge structures

Let \( V = \text{finitely-generated abelian group} \)
\[ V = V_\mathbb{Z} \oplus \mathbb{C} \quad (\text{also write } V_\mathbb{R} = V_\mathbb{Z} \oplus \mathbb{R}, \quad V = V_\mathbb{Z} \oplus \mathbb{Q}) \]

**Definition 1:** A Hodge structure (HS) of weight \( n \) on \( V \) is a decomposition

\[ V = \bigoplus_{p+q=n} V^{p,q} \]

s.t. \( V^{1,2} = V^{2,1} \). [Note: the role of a HS is that of \( V \).]

**Example 1:** \( M = \text{compact Kähler manifold} \); the Hodge decomposition theorem puts a HS of weight \( k \) on \( V = H^k(M, \mathbb{Z}) \).

**Definition 2:** A weight \( k \) HS on \( V \) is a decreasing filtration (or "flag")

\[ V = F^0 V \supset F^1 V \supset \ldots \supset F^n V = \{0\} \]

s.t. \( V = \bigoplus_{p+q=k} F^{p,q} \). [The "\( p \)-composed condition".]

**Proposition 1:** The 2 definitions are equivalent.

**Proof:** (1 \( \Rightarrow \) 2) \: \( F^p : = \bigoplus_{p' \geq p} V^{p',n-p'} \)

\[ F^{n-p+1} \supset \bigoplus_{q\geq n-p+1} V^{q,n-q} = \bigoplus_{p' < p} V^{n-p',p'} \Rightarrow F^{n-p+1} = \bigoplus_{p' < p} V^{n-p',p'} \]

**Note:** When there is no ambiguity possible, sometimes \( F^p V \) is abbreviated \( p^k \).

**Technical note:** What we are about to define is an effective HS. In general, one does allow negative \( p \).

\[ \mathbb{1} \]
(2.2) \( V^{p,n-p} := F^p \cap F^{n-p} \)

**Definition 3**: A morphism of HS is a map \( \Theta : V_\mathbb{Z} \to \tilde{V}_\mathbb{Z} \) of abelian groups s.t. \([the\ complex\ linear\ extension]\) \( \Theta_c : V_c \to \tilde{V}_c \)
satisfies \( \Theta_c (V^{p,q}) \subset \tilde{V}^{p,q} (V^{p,q}) \).

**Proposition 2**: This is equivalent to requiring \( \Theta_c (F^p) \subset \tilde{F}^p (V^p) \).

**Proof**: Because \( \Theta \) is defined \( \mathbb{Z}(\subset \mathbb{R}) \),
\[
\Theta_c (F^p) \subset \tilde{F}^p (V^p) \Rightarrow \Theta_c (F^{n-p}) \subset \tilde{F}^{n-p+1} (V^p) .
\]

**Remark 1**: One obtains \( \mathbb{Q}\)-HS resp. \( \mathbb{R}\)-HS (w/results identical to above) by replacing \( V_\mathbb{Z} \) by a \( \mathbb{Q}\)-vector sp. \( V \) or \( \mathbb{R}\)-v.s. \( V_\mathbb{R} \).

We'll use "V" to denote both the HS and the \( \mathbb{Q}\)-v.s. \( \mathbb{R}\)-HS.

**Proposition 3**: Morphisms of HS are strict for the Hodge filtrum:\n
\[
\lim_{n \to \infty} \Theta_c (V^k) = \Theta (\mathbb{P}^k) .
\]

**Proof**: (when \( \Theta (c) \in \tilde{F}^k \), with \( c = \sum x_i \), \( \Theta (c) = \sum \Theta (x_i) \).

**Example 2**: \( M \to N \) holom. \( \Rightarrow H^k (N) \to H^k (M) \) morphism of HS

\( \cong N \to M \) cohom. \( \Rightarrow H^k (N) \to H^k (M) \) morphism of HS

**Remark**: There is also the notion of a morphism of type \((r,r)\): then one demands \( \Theta_c (V^{p,q}) \subset V^{p+r,q+r} \).
Proposition 4: \( \phi \) surjective \( \Rightarrow \phi^* \) injective \( (H^k(N) \subset H^k(M)) \)

Sketch of Proof: \( \alpha \in H^k(N) \) pullback \( \beta \in H^k(M) \) such that 
\( \alpha \cap \beta = [dV_N] \), and 
\( \phi^* \mu \cap \phi^* \nu = [\phi^* dV_N \cap \omega_N^p] \neq 0 \).

\[ \square \]

Polarized Hodge Structures

Definition 4: A polarization of a weight \( n \) Hodge structure \( V \), is a \( \mathbb{Q} \)-symmetric bilinear form 
\[ Q : V^*_q \times V^*_{-q} \rightarrow \mathbb{Q} \]

Satisfying
\( Q \) vanishes for \( \mu \) and \( \nu \) with \( \mu \cap \nu \neq 0 \), and \( \mu \cap \nu = 0 \) for \( \mu \) and \( \nu \) with \( \mu \cap \nu = 0 \), and \( \mu \cap \nu = 0 \) for \( \mu \) and \( \nu \) with \( \mu \cap \nu = 0 \).

A polarized Hodge structure (PHS) is often written as a triple \( (V, \Phi, Q) \).

Example 3: For \( X \) projective, \( \Phi \) integral Kähler class (pullback of \( \omega_{FS} \) from \( \mathbb{P}^N \))

\[ \Rightarrow \text{(by Theorem 3.3)} \]

\[ Q_k(x, y) = \int_X \langle e^x \omega_{FS} - h \rangle \text{ are integral, hence } Q_k \text{ polarizes } H^k(X). \]

In fact, writing \( H^k(X) = \bigoplus_{a+b=k} H^{a-b}(X) \), the orthogonal direct sum of the \( (-1)^{\frac{k(k+1)}{2}}Q_{k-2j} \) polarizes \( H^k(X) \).

The category of \( n \) Hodge structures is not only abelian but semisimple (up to tensoring with \( \mathbb{Q} \)).

Proposition 5: Given a \( \mathbb{Q} \)-PHS \( V \) with \( \Phi \), \( \Phi \)-Hodge \( V' \). Then \( V' \) is polarized and \( V \cong V' \oplus V' \) as a Hodge structure.

Remark 2: There are actually a lot of non-polarizable Hodge structures, even in weight 2.
Proof: Since $V'$ defined $Q_{\mathcal{Q}}$, $\overline{V}_{\mathcal{Q}} = V_{\mathcal{Q}}' \subseteq V_{\mathcal{Q}}$. Hence, HR II for $Q$ on $V_{\mathcal{Q}} \Rightarrow Q_{\mathcal{Q}}$ is nondegenerate (hence polarizing $V'$). Now take $V'' = (V')^\perp_{\mathcal{Q}}$, nondegenerate of $Q_{\mathcal{Q}}$ on $V'' \Rightarrow V = V' \oplus V''$, and rationality of $Q \Rightarrow V''$ defined $Q_{\mathcal{Q}}$. Finally, since $Q$ is nondegenerate on $V_{\mathcal{Q}}' \times V_{\mathcal{Q}}''$ and $W_{\mathcal{Q}}' \times (V'')^\perp_{\mathcal{Q}}$, $V_{\mathcal{Q}}' \otimes (V'')^\perp_{\mathcal{Q}} \Rightarrow V'' \cong V'' \oplus V_{\mathcal{Q}}'' \Rightarrow V''$ sub-Hecke.

### Classifying Spaces for PHS

Let $V_{\mathbb{Z}} = \text{f.g. abelian group} \ (\text{rank} \ r)$

$$Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z} \quad (-1)^{\text{symmetric}} \text{-symmetric nondegenerate bilinear form}$$

$$h^{0, 0}, h^{1, 0}, \ldots, h^{r, r} \in \mathbb{Z}_{\geq 0} \quad \text{s.t.} \quad h^{r, r} = h^{0, 0} \quad \text{and} \quad \sum h^{r, r} = r.$$

Clearly, a classifying space should parametrize weight $n$ HSS on $V_{\mathbb{Z}}$ polarized by $Q$.

**Definition 5:** The period domain $D$ associated to the above data is

$$D = \{ \text{decomposition } V_{\mathcal{Q}} = \oplus V_{\mathcal{Q}}^{p, q} \text{ s.t.} \begin{cases} \overline{V}_{\mathcal{Q}}' = V_{\mathcal{Q}}^{q, p} \\ \dim V_{\mathcal{Q}}^{p, q} = h^{p, q} \end{cases} \} \quad \text{(i.e. HR I II held)}$$

In terms of the Hodge filtration $F^\cdot$, the Hodge-Riemann condition I in Defn. 4 can be rewritten

$$(HR I') \quad Q(F^p, F^{n-p+1}) = 0 \quad (\forall p)$$

One can rewrite (HR II) as

$$Q(\mathcal{L}(c, c)) > 0 \quad (\text{weil})$$
Definition 6: The "compact dual" $\tilde{D}$ of $D$ is a projective algebraic variety containing $D$, defined by

$$
\tilde{D} = \{ \text{flags } F^0 \subseteq V_c \text{ s.t. } \begin{cases} 
\dim_c F^0 = \sum h^p \cdot 1^{n-p} \\
(\text{HRI}') \text{ holds}
\end{cases} \}
$$

(we do not impose the "$p$-opposed condition" — these will not all correspond to decompositions, and the ones that do may not be polarized by $Q$.)

Example 4: \( V = \mathbb{Z} \langle Y_1, Y_2 \rangle \)

\[
\begin{cases}
Q(Y_i, Y_j) = (-1)^i \delta_{ij} \\
h^{1,0} = 1 = h^{0,1}
\end{cases}
\]

(Insipred by $H'(\mathbb{E}_q, \mathbb{Q})$)

Write $V_c = \mathbb{C} \langle \omega \rangle \oplus \mathbb{C} \langle \bar{\omega} \rangle$

( for (def. 5) $\Rightarrow U \begin{array}{c}
V_1 \omega \\
V_0 \omega \\
\end{array}$

for (lem. 6) $\Rightarrow C\langle \omega \rangle \cong F^1$)

Since $Q$ is hermitian, $Q(\omega, \omega) = 0 \Rightarrow (\text{HRI}')$ automatic

$\Rightarrow \tilde{D} = \mathbb{P}^1$

HRI II says $0 < iQ(\omega, \bar{\omega})$

$= i(\bar{z}_1 \bar{z}_2 - z_2 \bar{z}_1)$

$\Rightarrow 0 < \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{(z_1 \bar{z}_2)^2} = i(z_1 \bar{z} - \bar{z} z_1) = 2 \text{Im}(z_1)$

$\Rightarrow \tau \in \mathbb{H}$

So $D = \tau$. 

$\blacksquare$
Example 5: There are 1-to-1 correspondences

\[ \text{Weight 1 HS } \leftrightarrow \text{compact complex tori} \}\text{ cf. the Appendix below} \]

\[ \text{Weight 1 PHS } \leftrightarrow \text{abelian varieties} \]

\( \text{for } \text{rk}(V) = 2 \), if \( \mathbf{Q} = \mathbb{Q}(\sqrt{-1}) \) then these are parameterized by \( \mathbf{H}_g = \text{D}, \text{ cf. pp. 101-3} \).

Remark 3: There are general formulas for period domains which come from setting

\[ G_{\mathbb{R}} = \text{Aut}(V, \mathbb{R}, Q) \equiv \begin{cases} \text{Sp}(r, \mathbb{R})/\{\pm 1\} & \text{if } n \text{ odd} \\ \text{SO}(h_{\text{odd}}, h_{\text{even}}) & \text{if } n \text{ even} \end{cases} \]

and noting that this acts transitively on \( \text{D} \). Fixing a "base PHS" \( \mathbf{P} \in \text{D}, \) the subgroup fixing it is denoted

\[ H_\mathbf{P} \equiv \begin{cases} \prod_{p \leq n} U(h^{n/p}) & \text{if } n = 2m + 1 \\ \prod_{p \mid m} U(h^{n/p}) \times \text{SO}(h^{n/m}) & \text{if } n = 2m. \end{cases} \]

and so

\[ \text{D} \cong G_{\mathbb{R}}/H_\mathbf{P}. \]

More on this later.

Easy examples:

\[ \text{H}_n = \text{SL}_n(\mathbb{R})/\text{U}(n), \quad \text{H}_g = \text{Sp}_{2g}(\mathbb{R})/\text{U}(g) \]

(Gen. to Ex. 3 above)

Ex/Show that \( G_{\mathbb{R}} \) acts transitively on \( \text{D} \).

Remark 4: I forgot. you can also take tensors & duals of (P)HS's.

[Draw diagram]

Remark 5: One can also define a PHS on \( V \) (polarized by \( \mathbf{Q} \)) as a real representation \( \varphi: \mathbb{S}^1 \rightarrow \text{Aut}(V, \mathbb{R}, \mathbf{Q}) \) [think: \( \varphi(z) \big|_{V_{\mathbb{R}}} = z^{1-2} \)]

The smallest \( \mathbb{Q} \)-algebraic group containing \( \varphi(\mathbb{S}^1) \) is called \( \varphi \)'s Mumford-Tate group.