III. Variations of Hodge structure (VHS)

We shall start with VHS coming from geometry (families of compact Kähler manifolds or smooth projective varieties). The abstract setting will come later. First of all we expand somewhat massively on the blurb (pp. 105-6) on deformations of complex structure.

A. Leray spectral sequence

It will be useful to have in hand the notion of hypercohomology (not just for this section). More generally, there is the

Definition 1: Given a left-exact functor $F : \mathcal{A} \to \mathcal{B}$ and a bounded complex $\mathcal{C}^\bullet$ of objects in $\mathcal{A}$, the <u>right hyper-derived functors</u> are given by

(cohomology sheaves)

quasi-isomorphism (isom. on cohom.)

$$\mathbb{R}^k F(\mathcal{C}^\bullet) := H^k\{F(\mathcal{I}^\bullet)\} \quad \text{where} \quad \mathcal{C}^\bullet \xrightarrow{\simeq} \mathcal{I}^\bullet$$

$\mathcal{I}^\bullet$ = cx. of injectives, or more generally F-acyclic objects.

Example 1: M manifold, $\mathcal{A} = Sh(M)$, $\mathcal{B} = Ab$, $F = \Gamma_M$.

The <u>hypercohomology</u> of a complex $\mathcal{C}^\bullet$ of sheaves on M is then

$$\mathbb{H}^k(M, \mathcal{C}^\bullet) := \mathbb{R}^k\Gamma_M(\mathcal{C}^\bullet) \underset{\substack{\uparrow \\ \text{good op.} \\ \text{cover}}}{=} H^k\{\underbrace{\mathrm{Tot}^\bullet}_{}(\check{C}^\bullet(M, \mathcal{C}^\bullet))\}$$

simple cx. associated to double cx. (differential $D = \delta + (-1)^p d$)

$$\text{(A.1)} \quad \boxed{\underset{\substack{\uparrow \\ \text{if } \mathcal{C}^\bullet \text{ is } \Gamma_M\text{-acyclic}}}{\cong} H^k\{\Gamma(M, \mathcal{C}^\bullet)\}}$$

Ex/ (M Kähler) $\Omega_M^{\bullet \geq p} \xhookrightarrow{\simeq} F^p A_M^\bullet$ // [N.B.: these cxs. start in degree p]

$$\Rightarrow \mathbb{H}^k(M, \Omega^{\bullet \geq p}) \simeq \underbrace{H^k(F^p A^\bullet(M))}_{\text{using (A.1)}} \underset{\substack{\uparrow \\ \text{Hodge thy.}}}{\cong} F^p H^k(M, \mathbb{C}).$$
(Corollary II.D.1)

The setting we shall care about is that of a proper holomorphic submersion of complex manifolds

$$\pi : \mathcal{M} \longrightarrow \mathcal{S}$$

with (smooth, compact) Kähler fibers $M_s := \pi^{-1}(s)$. Recall that (essentially by Frobenius) we have for a contractible submanifold $0 \in B \subset \mathcal{S}$ that

(A.2) $$\boxed{\pi^{-1}(B) \xleftarrow[\mathcal{Y}]{\cong} M_0 \times B}$$

where $\mathcal{Y}(M_0 \times \{b\}) = M_b$ $(\forall b \in B)$. Warning: this does not preserve Hodge type, because $\mathcal{Y}$ is a diffeomorphism, not a holomorphic map.

Example 2: $\mathcal{A} = Sh(\mathcal{M})$, $\mathcal{B} = Sh(\mathcal{S})$, $F = \pi_*$ ← defined in I.F ("vertical sections functor")

We have for $U \in Op(\mathcal{S})$ contractible* that

(A.3) $$(\mathbb{R}^k \pi_* \mathbb{C}^\bullet)(U) \cong \mathbb{H}^k(\pi^{-1}U, \mathbb{C}^\bullet)$$

For example, $\mathbb{R}^k \pi_* A_{\mathcal{M}}^\bullet \simeq \mathbb{R}^k \pi_* \Omega_{\mathcal{M}}^\bullet \simeq R^k \pi_* \mathbb{C} =: \mathbb{H}^k_{\mathcal{M}/\mathcal{S}, \mathbb{C}}$ ← Can also do with $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ coefficients

has sections over $0 \in U$ (contractible!) $\cong H^k(\pi^{-1}(U), \mathbb{C}) \cong H^k(M_0, \mathbb{C})$

and stalks at $s \in U$ $\cong H^k(M_s, \mathbb{C})$ $(\cong \quad " \quad)$.

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* otherwise the statement is false

This is an example of a <u>local system</u> (cf. p. 79), i.e. a sheaf locally $\cong$ to a constant sheaf. Note that

$$\mathbb{H}^k_{M/S} \otimes \mathcal{O}_S =: \mathcal{H}^k_{M/S} \cong \mathcal{O}_S\left(H^k_{M/S}\right), \quad \text{where } H^k_{M/S} \text{ has fibers } H^k(M_s,\mathbb{C}).$$

$\mathcal{H}^k_{M/S}$: loc. free $\mathcal{O}_S$-mod, hence must be sections of a holomorphic vector bundle $H^k_{M/S}$.

$\square$

Now, given $(K^{\bullet,\bullet}, d, \delta)$ = a <u>double</u> complex of abelian groups, how in practice does one compute the cohomology of $K^\bullet := \mathrm{Tot}^\bullet(K^{\bullet,\bullet})$?

<u>Definition 3</u>: The <u>spectral sequence</u> associated to $K^{\bullet,\bullet}$ is the family of complexes

0th <u>page</u> : $E_0^{p,q} := K^{p,q}$, $d_0 := (-1)^p d : E_0^{p,q} \to E_0^{p,q+1}$ $\quad\uparrow$

1st <u>page</u> : $E_1^{p,q} := H^\circ(E_0^{p,q+\circ}, d_0)$, $d_1 :=$ "δ" $: E_1^{p,q} \to E_1^{p+1,q}$ $\quad$ i.e. $H^0(\delta)$ $\quad\rightarrow$

2nd <u>page</u> : $E_2^{p,q} := H^\circ(E_1^{p+\circ,q}, d_1)$, $d_2 :=$ "$\delta \circ d^{-1} \circ \delta$" $: E_2^{p,q} \to E_2^{p+2,q-1}$ $\quad\searrow$

$\vdots \qquad\qquad \vdots$

(in general, $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ and $\underbrace{d_r \circ d_r = 0}_{\text{this has to be proved}}$)

Clearly, $K^{\bullet,\bullet}$ bounded $\implies d_r = d_{r+1} = \cdots = 0$ (for some r)

$$\implies E_r^{\bullet,\bullet} = E_{r+1}^{\bullet,\bullet} = \cdots =: E_\infty^{\bullet,\bullet}.$$

We say "$E_\bullet^{\bullet,\bullet}$ <u>degenerates</u> at E_r".

<u>Proposition 1</u>: Set $F^p H^k(K^\bullet) := \mathrm{Im}\left(H^k(\mathrm{Tot}^\bullet K^{\bullet \geq p,\bullet}) \to H^k(K^\bullet)\right)$

Then $E_\infty^{p,q} \cong \mathrm{Gr}^p_F H^{p+q}(K^\bullet)$, and so

$$H^k(K^\bullet) \underset{\substack{\text{non-}\\\text{canonically}}}{\cong} \bigoplus_{p+q=k} E_\infty^{p,q}.$$

There is one more kind of spectral sequence we shall need (which actually contains the above as a special case).

Definition 4: A filtered complex is a complex $(\mathcal{C}^\bullet, d)$ together with a decreasing filtration $\mathcal{F}^\bullet$ on its terms s.t. $d(\mathcal{F}^p \mathcal{C}^q) \subset \mathcal{F}^p \mathcal{C}^{q+1}$ $(\forall p, q)$.
We set $\mathcal{F}^p H^k(\mathcal{C}^\bullet) := \mathrm{im}\{H^k(\mathcal{F}^p \mathcal{C}^\bullet) \to H^k(\mathcal{C}^\bullet)\}$.

Proposition 2: There exists a spectral sequence as above, but with

$$E_0^{p,q} = Gr_F^p \mathcal{C}^{p+q}, \quad d_0 = (Gr_F^p) d$$

$$E_1^{p,q} = H^{p+q}(Gr_F^p \mathcal{C}^\bullet), \quad d_1 : H^{p+q}(Gr_F^p \mathcal{C}^\bullet) \to H^{p+q+1}(Gr_F^{p+1} \mathcal{C}^\bullet)$$

the connecting homomorphism induced by

$$0 \to Gr_F^{p+1} \mathcal{C}^\bullet \to \frac{F^p \mathcal{C}^\bullet}{F^{p+2}\mathcal{C}^\bullet} \to Gr_F^p \mathcal{C}^\bullet \to 0$$

$\vdots$

$$E_\infty^{p,q} = Gr_F^p H^{p+q}(\mathcal{C}^\bullet). \qquad \left(\text{“} E_\bullet^{\bullet,\bullet} \underset{\text{converges to}}{\Longrightarrow} H^*(\mathcal{C}^\bullet) \text{”} \right)$$

Good references for both kinds of spectral sequences are [Bott + Tu] and [Voisin]. (But [Bott + Tu] gives a better feel for how to compute the d_r's and work with them.)

We can cook up a couple of nice applications by introducing relative differential forms:

$$\text{s.e.s.} \quad 0 \to \pi^* \Omega^1_{\mathcal{S}} \to \Omega^1_{\mathcal{M}} \to \Omega^1_{\mathcal{M}/\mathcal{S}} \to 0 \quad , \quad \Omega^\ell_{\mathcal{M}/\mathcal{S}} := \Lambda^\ell \Omega^1_{\mathcal{M}/\mathcal{S}}$$

$$\rightsquigarrow \text{s.e.s.} \quad 0 \to \mathrm{im}\{\pi^* \Omega^1_{\mathcal{S}} \otimes \Omega^{\ell-1}_{\mathcal{M}} \xrightarrow{\wedge} \Omega^\ell_{\mathcal{M}}\} \to \Omega^\ell_{\mathcal{M}} \to \Omega^\ell_{\mathcal{M}/\mathcal{S}} \to 0$$

this becomes a complex with d_{rel} := differential induced by d on $\Omega^\bullet_{\mathcal{M}}$.

think: "vertical differentiation"

and then the <u>Leray filtration</u> on complexes of forms:

$$\mathcal{L}^\ell \Omega^\bullet_{\mathcal{M}} := \mathrm{im}\left\{\pi^*\Omega^\ell_{\mathcal{S}} \otimes \Omega^{\bullet-\ell}_{\mathcal{M}} \to \Omega^\bullet_{\mathcal{M}}\right\}$$

$$\Downarrow$$

$$Gr^\ell_{\mathcal{L}}\, \Omega^\bullet_{\mathcal{M}} \cong \pi^*\Omega^\ell_{\mathcal{S}} \otimes \Omega^{\bullet-\ell}_{\mathcal{M}/\mathcal{S}}.$$

Now $$\mathbb{R}^k\pi_*(\Omega^\bullet_{\mathcal{M}/\mathcal{S}}) \cong R^k\pi_*(\pi^{-1}\mathcal{O}_{\mathcal{S}}) \cong R^k\pi_*\mathbb{C}\otimes\mathcal{O}_{\mathcal{S}} \cong \mathcal{H}^k_{\mathcal{M}/\mathcal{S}},$$

and similarly we can re-do all of this with C^∞ forms ($A^\bullet$ in lieu of $\Omega^\bullet$), to obtain

$$\mathbb{R}^k\pi_* A^\bullet_{\mathcal{M}/\mathcal{S}} \cong C^\infty(H^k_{\mathcal{M}/\mathcal{S}}).$$

By Theorem II.F.3 and its Corollary, $F^pH^k_{\mathcal{M}/\mathcal{S}}$ gives an (a priori C^∞) sub-bundle; harmonic theory associates to $[\sigma]\in\Gamma(U, F^pH^k_{\mathcal{M}/\mathcal{S}})$, a d_{rel}-closed section $\sigma\in\Gamma(\pi^{-1}U, F^kA^\ell_{\mathcal{M}/\mathcal{S}})$ which automatically lifts* to $\tilde\sigma \in \Gamma(\pi^{-1}(U), F^kA^\ell_{\mathcal{M}})$ (<u>not</u> d-closed). The upshot is that

$$\underbrace{\mathbb{R}^k\pi_*F^pA^\bullet_{\mathcal{M}/\mathcal{S}}}\ \underset{\text{harmonic theory}}{\overset{\text{[pull back to fibers]}}{\rightleftarrows}}\ C^\infty(F^pH^k_{\mathcal{M}/\mathcal{S}})$$

$$\uparrow\cong$$

$$\mathbb{R}^k\pi_*\overbrace{(\Omega^{\bullet\geq p}_{\mathcal{M}/\mathcal{S}}\otimes\pi^{-1}C^\infty_{\mathcal{S}})}$$

$$\|$$

$$\underbrace{(\mathbb{R}^k\pi_*\Omega^{\bullet\geq p}_{\mathcal{M}/\mathcal{S}})}_{\text{locally free}/\mathcal{O}_{\mathcal{S}}}\otimes C^\infty_{\mathcal{S}} \implies F^pH^k_{\mathcal{M}/\mathcal{S}} \text{ must be holo. and}$$

$$\boxed{\mathbb{R}^k\pi_*\Omega^{\bullet\geq p}_{\mathcal{M}/\mathcal{S}} \cong \mathcal{O}_{\mathcal{S}}(F^pH^k_{\mathcal{M}/\mathcal{S}}) =: \mathcal{F}^p\mathcal{H}^k_{\mathcal{M}/\mathcal{S}}.}$$

for a different approach to this you could also use Voisin 10.10 (but then you have to prove that)

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* using $0\to F^k_{\mathrm{im}}(\pi^*A^1_{\mathcal{S}}\otimes A^{\ell-1}_{\mathcal{M}}\to A^\ell_{\mathcal{M}}) \to F^kA^\ell_{\mathcal{M}} \to F^kA^\ell_{\mathcal{M}/\mathcal{S}}\to 0$ + the fact that these sheaves are <u>fine</u> on $\mathcal{M}$ — hence have no H^1.

Example 3: (Leray spectral sequence) Put

$$\mathcal{L}^\ell(A^\bullet(\mathcal{M})) := (\mathcal{L}^\ell A^\bullet)(\mathcal{M})$$

$$\mathcal{L}^\ell H^k(\mathcal{M},\mathbb{C}) = \mathrm{im}\left\{\mathbb{H}^k(\mathcal{M}, \mathcal{L}^\ell A^\bullet) \to \mathbb{H}^k(\mathcal{M}, A^\bullet)\right\}$$

Prop. 2 $\Rightarrow$ $\exists$ s.s. with $E_1^{p,q} = H^{p+q}(\underbrace{Gr^p_{\mathcal{L}} A^\bullet}_{\pi^* A^p_{\mathcal{S}} \otimes A^{\bullet-p}_{\mathcal{M}/\mathcal{S}}}(\mathcal{M})) = A^p_{\mathcal{S}}(H^q(A^\bullet_{\mathcal{M}/\mathcal{S}})) \cong A^p_{\mathcal{S}}(\mathbb{H}^q_{\mathcal{M}/\mathcal{S},\mathbb{C}})$.

$$\Rightarrow E_2^{p,q} = H^p(\mathcal{S}, \mathbb{H}^q_{\mathcal{M}/\mathcal{S},\mathbb{C}}).$$

"cohomology with twisted coeffs."

Theorem 1 (Deligne): If π is projective*, then the Leray s.s. degenerates at E_2.

Proof (Sketch): π projective $\Rightarrow$ $\underbrace{\exists\, L : \mathbb{H}^q_{\mathcal{M}/\mathcal{S}} \to \mathbb{H}^{q+2}_{\mathcal{M}/\mathcal{S}}}_{\text{cup-product with relative Fubini-Study form; the usual hard Lefschetz} \cong\text{'s obtain}}$ $\underbrace{\text{commuting with all } d_r}_{\text{essentially because } \omega \text{ is a global d-closed form on } \mathcal{M}}$

$$\Rightarrow \begin{array}{ccc} E_2^{p,q} & \xrightarrow{d_2} & E_2^{p+2,q-1} \\ \big\downarrow L^{n-q+1} & & \cong \big\downarrow L^{n-q+1} \\ E_2^{p,2n-q+2} & \xrightarrow{d_2} & E_2^{p+2,2n-q+1} \end{array}$$

but this is zero on $H^p(\mathcal{S}, \mathbb{H}^q_{\mathcal{M}/\mathcal{S},\mathbb{C},pr})$,

and so $\therefore$ is the upper d_2.

By the same argument for $q-2$, $q-4$, etc. together with $[d_2, L] = 0$, d_2 is identically zero. $\square$

Corollary 1: For $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$ projective, $Gr^p_{\mathcal{L}} H^{p+q}(\mathcal{X},\mathbb{C}) \cong H^p(\mathcal{S}, \mathbb{H}^q_{\mathcal{X}/\mathcal{S},\mathbb{C}})$.

Hence we have (non-canonically) $\boxed{H^k(\mathcal{X},\mathbb{C}) \cong \bigoplus_{p+q=k} H^p(\mathcal{S}, \mathbb{H}^q_{\mathcal{X}/\mathcal{S},\mathbb{C}})}$.

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* this means that the fibration π factors thru $\mathbb{P}^N \times \mathcal{S} \twoheadrightarrow \mathcal{S}$; in particular, the M_s are projective algebraic.

Remark 1: The efficient description of d_1 as connecting homomorphism doesn't make it clear why $d_1 \circ d_1 = 0$. A better description uses not one but two short-exact sequences, splicing their long-exact sequences together.

$$0 \to F^{p+1}C^\bullet \to F^pC^\bullet \to Gr_F^p C^\bullet \to 0$$

and

$$0 \to F^{p+2}C^\bullet \to F^{p+1}C^\bullet \to Gr_F^{p+1} C^\bullet \to 0$$

Ex/ Use the associated l.e.s.'s to define a map

$$d_1 : H^{p+q}(Gr_F^p C^\bullet) \to H^{p+q+1}(Gr_F^{p+1} C^\bullet).$$

Then check that $d_1 \circ d_1 = 0$. //