B. Gauss-Manin Connection

Continue to assume $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$ a smooth projective morphism of $\mathbb{C}$-mflds. Proposition A.2 applies in the more general setting of filtered complexes of sheaves with H^* replaced by $\mathbb{R}^* F$. So there exists a spectral sequence of sheaves on $\mathcal{S}$

$$E_1^{p,q} = \mathbb{R}^{p+q}\pi_*\big(\underbrace{Gr_{\mathcal{L}}^p \Omega^\bullet_{\mathcal{X}}}_{\pi^*\Omega^p_{\mathcal{S}} \otimes \Omega^{\bullet-p}_{\mathcal{X}/\mathcal{S}}}\big) = \Omega^p_{\mathcal{S}} \otimes \underbrace{\mathbb{R}^q \pi_* \Omega^\bullet_{\mathcal{X}/\mathcal{S}}}_{\mathcal{H}^q_{\mathcal{X}/\mathcal{S}}}$$

$$\text{(B.1)} \qquad E_2^{p,q} \underset{\substack{\Uparrow \\ \text{(same proof} \\ \text{as Thm. A.1)}}}{=} E_\infty^{p,q} = Gr^p_{\mathcal{L}} \underbrace{\mathbb{R}^{p+q}\pi_* \Omega^\bullet_{\mathcal{X}}}_{\mathbb{H}^{p+q}_{\mathcal{X}/\mathcal{S}}} = \begin{cases} \mathbb{H}^q_{\mathcal{X}/\mathcal{S},\mathbb{C}}, & p = 0 \\ 0, & p > 0 \end{cases}$$

__Definition 1__: The __Gauss-Manin connection__ $\nabla : \Omega^p_{\mathcal{S}} \otimes \mathcal{H}^q_{\mathcal{X}/\mathcal{S}} \to \Omega^{p+1}_{\mathcal{S}} \otimes \mathcal{H}^q_{\mathcal{X}/\mathcal{S}}$ is d_1.

__Theorem 1__: ∇ is flat, with flat structure given by $\mathbb{H}^q_{\mathcal{X}/\mathcal{S},\mathbb{C}}$; in particular, $\nabla(\mathbb{H}^q_{\mathcal{X}/\mathcal{S},\mathbb{Q}}) = 0$.

__Proof__: $\nabla = d_1 \implies \nabla \circ \nabla = 0$.

$$\text{(B.1)} \implies \begin{cases} \ker \nabla = \operatorname{im} \nabla \text{ in } E_1^{p,q} \text{ for } p \geq 1 \\ \ker \nabla = \mathbb{H}^q_{\mathcal{X}/\mathcal{S},\mathbb{C}} \text{ for } p = 0. \end{cases}$$

□

__Remark 1__: We can of course consider, instead of $\nabla : \mathcal{H}^q \to \Omega^1 \otimes \mathcal{H}^q$,

$$\nabla : \underbrace{\mathcal{O}(T^{1,0}_{\mathcal{S}})}_{\Theta^1_{\mathcal{S}}} \otimes \mathcal{H}^q \to \mathcal{H}^q,$$

or $\nabla_{\underline{v}} : \mathcal{H}^q \to \mathcal{H}^q \qquad (\underline{v} \in \Gamma(\Theta^1_{\mathcal{S}}))$.

□

What if (instead of $\Omega^\bullet_X$) we Leray-filtered $\Omega_X^{\bullet\geq k}$? Then we get

$$'E_1^{p,q} = \mathbb{R}^{p+q}\pi_*\big(\underbrace{Gr^p_{\mathcal{L}}\Omega_X^{\bullet\geq k}}_{\pi^*\Omega^p_S\otimes\Omega^{\bullet-p\geq k-p}_{X/S}}\big) = \Omega^p_S\otimes\underbrace{\mathbb{R}^q\pi_*\Omega^{\bullet\geq k-p}}_{\mathcal{F}^{k-p}\mathcal{H}^q_{X/S}}.$$

Now $\Omega_X^{\bullet\geq k}\hookrightarrow\Omega^\bullet_X$ induces a morphism of spectral sequences $'E\to E$, so that

(B.2) $'d_1:\Omega^p_S\otimes\mathcal{F}^{k-p}\mathcal{H}^q_{X/S}\to\Omega^{p+1}_S\otimes\mathcal{F}^{k-p-1}\mathcal{H}^q_{X/S}$

is the restriction of d_1. In particular, we have

<u>Theorem 2</u> (<u>Griffiths transversality</u>): $\nabla(\mathcal{F}^k\mathcal{H}^q_{X/S})\subset\Omega^1_S\otimes\mathcal{F}^{k-1}\mathcal{H}^q_{X/S}$.

As a consequence, or by Leray filtering* $\Omega^k_X[-k]=\dfrac{\Omega_X^{\bullet\geq k}}{\Omega_X^{\bullet\geq k+1}}$, we have well-defined maps

$$\overline{\nabla}:\Omega^p_S\otimes Gr^k_F\mathcal{H}^q_{X/S}\longrightarrow\Omega^{p+1}_S\otimes Gr^{k-1}_F\mathcal{H}^q_{X/S}.$$

<u>Theorem 3</u>: $\overline{\nabla}$ is $\mathcal{O}_S$-linear; in particular, for each $\begin{cases}s\in S\\ \underline{v}\in T^{1,0}_{S,s}\end{cases}$,

$\overline{\nabla}_{\underline{v}}$ is a map of <u>vector spaces</u> $Gr^k_F H^q(X_s,\mathbb{C})\to Gr^{k-1}_F H^q(X_s,\mathbb{C})$.

<u>Proof</u>: Given $\begin{Bmatrix}\text{section } \sigma \text{ of } F^k\mathcal{H}^q_{X/S}\\ \text{holomorphic function } f\end{Bmatrix}$ over $U\subset S$, $\nabla(f\sigma)=\underbrace{df\otimes\sigma}_{\text{still in } F^k}+f\nabla\sigma$

$\implies\overline{\nabla}(f\sigma)=f\overline{\nabla}\sigma$. □

<u>Remark 2</u>: Sometimes $Gr^k_F\mathcal{H}^q_{X/S}$ is written "$\mathcal{H}^{k,q-k}_{X/S}$"; the problem with this is that one then thinks of it as sections of a subbundle of $H^q_{X/S}$, which is <u>wrong</u>. In fact, $H^{k,q-k}_{X/S}=F^k\cap\overline{F^{q-k}}\subset H^q_{X/S}$ is neither holo. (unless $k=q$) nor anti-holo. (unless $k=0$), just C^∞ (or rather, real analytic); obviously $Gr^k_F H^q_{X/S}$ is holomorphic. So I'll avoid this notation! □

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* $\Omega^k_X[-k]$ means "shifted k places to the right" (a complex of sheaves with only one term, in degree k).

Henceforth we suppose $S = \mathbb{D}$ (unit disk) with holo. coordinate t.
Then $\mathbb{L}^2\Omega^\bullet_X = \{0\} \implies \nabla$ is the connecting homomorphism in the long-exact sequence associated to

$$0 \to \pi^*\Omega^1_{\mathbb{D}} \otimes \Omega^{\bullet-1}_{X/\mathbb{D}} \to \Omega^\bullet_X \to \Omega^\bullet_{X/\mathbb{D}} \to 0$$

and $\bar{\nabla}$ the one coming from

(B.3) $$0 \to \pi^*\Omega^1_{\mathbb{D}} \otimes \Omega^{p-1}_{X/\mathbb{D}} \to \Omega^p_X \to \Omega^p_{X/\mathbb{D}} \to 0 \quad :$$

$$\bar{\nabla} : \underbrace{R^q\pi_*\Omega^p_{X/\mathbb{D}}}_{(\dagger)\; Gr^p_F \mathcal{H}^{p+q}_{X/\mathbb{D}}} \longrightarrow \Omega^1_{\mathbb{D}} \otimes \underbrace{R^{q+1}\pi_*\Omega^{p-1}_{X/\mathbb{D}}}_{Gr^{p-1}_F \mathcal{H}^{p+q}_{X/\mathbb{D}}}$$

Consider the s.e.s. of sheaves on X ("vertical holo. vector fields" (defined by this sequence))

(B.4) $$0 \to \Theta^1_{X/\mathbb{D}} \to \Theta^1_X \to \pi^*\Theta^1_{\mathbb{D}} \to 0 ,$$

with associated l.e.s. (of sheaves on $\mathbb{D}$)

$$\cdots \to \underbrace{R^0\pi_*\pi^*\Theta^1_{\mathbb{D}}}_{\Theta^1_{\mathbb{D}}} \xrightarrow{\delta} R^1\pi_*\Theta^1_{X/\mathbb{D}} \to 0$$

<u>Definition 2</u>: The <u>Kodaira-Spencer class</u> of π is $\kappa_\pi := \delta(\partial/\partial t) \otimes dt \in R^1\pi_*(\pi^*\Omega^1_{\mathbb{D}} \otimes \Theta^1_{X/\mathbb{D}}) \cong \Omega^1_{\mathbb{D}} \otimes R^1\pi_*\Theta^1_{X/\mathbb{D}}$.

<u>Theorem 4</u>: $\bar{\nabla}$ is computed by cup-product with κ_π.

<u>Sketch</u>: Dualizing (B.4) and tensoring with $\Omega^{p-1}_{X/\mathbb{D}}$ gives

$$\begin{array}{ccccccccc}
0 \to & \pi^*\Omega^1_{\mathbb{D}} \otimes \Omega^{p-1}_{X/\mathbb{D}} & \to & \Omega^1_X \otimes \Omega^{p-1}_{X/\mathbb{D}} & \to & \Omega^1_{X/\mathbb{D}} \otimes \Omega^{p-1}_{X/\mathbb{D}} & \to 0 \\
& \| & & \uparrow \sum dz_j \otimes \left(\frac{\partial}{\partial z_j} \lrcorner (\cdot)\right) & & \uparrow & \\
((B.3) =)\; 0 \to & \pi^*\Omega^1_{\mathbb{D}} \otimes \Omega^{p-1}_{X/\mathbb{D}} & \to & \Omega^p_X & \longrightarrow & \Omega^p_{X/\mathbb{D}} & \to 0
\end{array}$$

(which still has $\delta = \cup \kappa_\pi$)

from which the result is basically clear. $\square$

(†) $$0 \to \Omega^{\bullet \geq p+1}_{X/\mathbb{D}}[p] \to \Omega^{\bullet \geq p}_{X/\mathbb{D}}[p] \to \Omega^p_{X/\mathbb{D}} \to 0$$ (shift by $-p$) induces (taking $R^q\pi_*$)

$$\to F^{p+1}\mathcal{H}^{p+q} \hookrightarrow F^p\mathcal{H}^{p+q} \to R^q\pi_*\Omega^p_{X/\mathbb{D}} \xrightarrow{0} F^{p+1}\mathcal{H}^{p+q+1} \hookrightarrow F^p\mathcal{H}^{p+q+1} \to$$

(from 3A)

Now for a more geometric point of view on ∇, one would like to see how it acts on families of de Rham cohomology classes given by C^∞ forms.

From

(short exact sequences)

$$\begin{array}{ccccccc}
\pi^*\Omega^1_{\mathbb{D}}\otimes\Omega^{p-1}_{\mathcal{X}/\mathbb{D}} & \xleftarrow{\cong} & \ker(\Omega^\bullet_{\mathcal{X}}\to\Omega^\bullet_{\mathcal{X}/\mathbb{D}}) & \to & \Omega^\bullet_{\mathcal{X}} & \to & \Omega^\bullet_{\mathcal{X}/\mathbb{D}}\\
\downarrow & & \downarrow & & \simeq\downarrow & & \downarrow\\
\pi^*A^1_{\mathbb{D}}\otimes A^{p-1}_{\mathcal{X}/\mathbb{D}} & \twoheadleftarrow & \ker(A^\bullet_{\mathcal{X}}\to A^\bullet_{\mathcal{X}/\mathbb{D}}) & \to & A^\bullet_{\mathcal{X}} & \to & A^\bullet_{\mathcal{X}/\mathbb{D}}
\end{array}$$

gets rid of $d\bar{t}$ terms

one has

∇

$$\mathcal{H}^q_{\mathcal{X}/D}\cong\mathbb{R}^q\pi_*\Omega^\bullet_{\mathcal{X}/\mathbb{D}}\xrightarrow{\delta}\mathbb{R}^{q+1}\pi_*\ker(\cdots)\xrightarrow{\cong}\Omega^1_D\otimes\mathbb{R}^q\pi_*\Omega^\bullet_{\mathcal{X}/\mathbb{D}}\cong\Omega^1_D\otimes\mathcal{H}^q_{\mathcal{X}/\mathbb{D}}$$

$$\Omega^1_D\otimes\mathcal{H}^q_{\mathcal{X}/\mathbb{D}}\hookrightarrow A^{1,0}_{\mathbb{D}}\otimes C^\infty(H^q_{\mathcal{X}/\mathbb{D}})\hookrightarrow A^1_{\mathbb{D}}\otimes C^\infty(H^q_{\mathcal{X}/\mathbb{D}})$$

$$\mathcal{H}^q_{\mathcal{X}/D}\hookrightarrow C^\infty(H^q_{\mathcal{X}/D})$$

$$C^\infty(H^q_{\mathcal{X}/D})\simeq\mathbb{R}^q\pi_*A^\bullet_{\mathcal{X}/\mathbb{D}}\xrightarrow{\delta^\infty}\mathbb{R}^{q+1}\pi_*\ker(\cdots)\longrightarrow A^1_D\otimes\mathbb{R}^q\pi_*A^\bullet_{\mathcal{X}/\mathbb{D}}\cong A^1_{\mathbb{D}}\otimes C^\infty(H^q_{\mathcal{X}/\mathbb{D}})$$

$$=:\nabla^\infty=\nabla^{(1,0)}+\nabla^{(0,1)}$$

I will often denote $\nabla^{(1,0)}$ simply by ∇.

Now let $\mathcal{Y}: X_0\times D\xrightarrow{\cong}\mathcal{X}$ be a C^∞ diffeomorphism as in (A.2), $\left(\begin{array}{ccc} \cup & & \cup \\ X_0\times\{t\} & \xrightarrow[\cong]{\mathcal{Y}_t} & X_t\end{array}\right)$, and let*

$\widetilde{\partial/\partial t}$:= the corresponding lift (to $\mathcal{X}$) of $\partial/\partial t$, and (given $\alpha\in A^q_{\mathcal{X}/D}(\mathcal{X})$)

$\tilde{\alpha}\in A^q_{\mathcal{X}}(\mathcal{X})$ the unique lift with $\widetilde{\partial/\partial t}\lrcorner\tilde{\alpha}=0$.

As on p. 187, any section of $C^\infty(\mathcal{H}^q_{\mathcal{X}/D})$ is of the form $[\alpha]$ (family of relative cohom. classes) with $\begin{cases}\alpha\in A^q_{\mathcal{X}/D}(\mathcal{X})\\ d_{rel}\alpha=0\end{cases}$. Let $\hat{\alpha}\in A^q_{\mathcal{X}}(\mathcal{X})$ be <u>any</u> lift of α — e.g., one in the same Hodge filtrand (cf. p. 187).

<u>Theorem 5</u>: $\nabla_{\partial/\partial t}[\alpha]=\left[\mathcal{L}_{\widetilde{\partial/\partial t}}\,\alpha\right]$, where the Lie derivative is computed by the <u>Cartan formula</u> $\mathcal{L}_{\widetilde{\partial/\partial t}}\,\alpha:=\left\{\begin{array}{l}\text{image in}\\ A^q_{\mathcal{X}/D}(\mathcal{X})\text{ of}\end{array}\right\}\left(\widetilde{\partial/\partial t}\lrcorner\, d\hat{\alpha}+\underbrace{d\{\widetilde{\partial/\partial t}\lrcorner\,\hat{\alpha}\}}_{\text{exact, so not important}}\right)$

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* with respect to π, vertical vectors + horizontal forms are defined in the absence of a C^∞ trivialization $\mathcal{Y}$. What $\mathcal{Y}$ provides is a notion of vertical forms + horizontal vectors.

Proof: $\nabla^\infty[\alpha] = (\widetilde{\partial/\partial t} \lrcorner d\tilde{\alpha}) \otimes dt + (\widetilde{\partial/\partial \bar{t}} \lrcorner d\tilde{\alpha}) \otimes d\bar{t}$ is clear,
(↑ compute δ^∞; $d\tilde{\alpha}$ in $\mathcal{I}^1$ since $d_{rel}\tilde{\alpha} = 0$)

and (thinking, via $\mathcal{G}^*$, on $X_0 \times \mathbb{D}$) the RHS is d_{rel}-closed. Moreover,

$\tilde{\alpha} = \hat{\alpha} - \pi^* dt \wedge (\widetilde{\partial/\partial t} \lrcorner \hat{\alpha})$; and so

$$\nabla_{\partial/\partial t}[\alpha] = \widetilde{\partial/\partial t} \lrcorner d\hat{\alpha} \underbrace{- \widetilde{\partial/\partial t} \lrcorner d\{\pi^* dt \wedge (\widetilde{\partial/\partial t} \lrcorner \hat{\alpha})\}}$$

$$= + \widetilde{\partial/\partial t} \lrcorner \pi^* dt \wedge d(\widetilde{\partial/\partial t} \lrcorner \hat{\alpha})$$

$$\underset{\text{mod } \mathcal{I}^1}{\equiv} d(\widetilde{\partial/\partial t} \lrcorner \hat{\alpha}).$$ □

From this perspective, Griffiths transversality is abundantly clear: for $[\alpha] \in \Gamma(\mathcal{F}^p \mathcal{H}^k_{\mathcal{X}/\mathbb{D}})$, choose $\hat{\alpha} \in F^p A^k(\mathcal{X}) \Rightarrow \underbrace{\widetilde{\partial/\partial t}}_{\text{removes a "}dt\text{"}} \lrcorner \underbrace{d\hat{\alpha}}_{\text{in } F^p A^{k+1}} \in F^{p-1} A^k(\mathcal{X})$

$\Rightarrow \nabla_{\partial/\partial t}[\alpha] \in \Gamma(\mathcal{F}^{p-1}\mathcal{H}^k_{\mathcal{X}/\mathbb{D}})$. Also, we see quite concretely the holomorphicity of the $\mathcal{F}^p \mathcal{H}^k_{\mathcal{X}/\mathbb{D}}$: for the same $[\alpha]$, $\nabla^{0,1}[\alpha]$ is given by $\mathcal{L}_{\widetilde{\partial/\partial \bar{t}}}$ hence $\underbrace{\widetilde{\partial/\partial \bar{t}}}_{\text{removes a "}d\bar{t}\text{"}} \lrcorner d\hat{\alpha} \in F^p A^k(\mathcal{X})$.

We can now additionally confirm that Theorem 1 $(\nabla(\mathbb{H}^k_{\mathcal{X}/\mathbb{D},\mathbb{Q}}) = 0)$ says that ∇ differentiates the periods of cohomology classes.

Proposition 1: Given $\begin{cases} \{\varphi_t\}_{t\in D} \text{ a } C^\infty \text{ family of } k\text{-cycles} \\ \{\omega_t\}_{t\in D} \text{ a } C^\infty \text{ family of closed } k\text{-forms} \end{cases}$ on X_t,

$$\text{(B.5)} \qquad \boxed{\int_{\varphi_t} \mathcal{L}_{\widetilde{\partial/\partial t}}\, \omega_t = \frac{\partial}{\partial t} \int_{\varphi_t} \omega_t}$$

[Remark 1: Given (periods of) a holomorphic section of $\mathcal{F}^k \mathcal{H}^k$, this can be used to produce (periods of) a holomorphic section of $\mathcal{F}^{k-1}\mathcal{H}^k$, $\mathcal{F}^{k-2}\mathcal{H}^k$, etc.]

Proof: wolog $\varphi_t = \mathcal{G}_t(\varphi_0)$ $(\forall t \in \mathbb{D})$, so [and the calculation that follows is actually valid for even nonclosed φ_0!]

$$\frac{\partial}{\partial t}\left(\int_{\varphi_t}\omega_t\right)_{t=0} = \frac{\partial}{\partial t}\left(\int_{\varphi_0} g_t^{*}\omega_t\right)_{t=0} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{\int_{\varphi_0} g_\epsilon^{*}\omega_\epsilon - \int_{\varphi_0} g_0^{*}\omega_0\right\}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\underbrace{\varphi_0\times\{\epsilon\}-\varphi_0\times\{0\}}_{=\partial(\varphi_0\times[0,\epsilon])+\partial\varphi_0\times[0,\epsilon]}} g^{*}\tilde{\omega} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\varphi_0\times[0,\epsilon]} d(g^{*}\tilde{\omega}) + \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\partial\varphi_0\times[0,\epsilon]} g^{*}\omega$$

$$= \int_{\varphi_0}\partial/\partial t \lrcorner\, d(g^{*}\tilde{\omega}) + \int_{\partial\varphi_0}\left(\partial/\partial t \lrcorner\, g^{*}\tilde{\omega}\right)$$

$$= \int_{\varphi_0}\underbrace{\left\{\widetilde{\partial/\partial t}\lrcorner\, d\tilde{\omega} + d\left(\widetilde{\partial/\partial t}\lrcorner\,\tilde{\omega}\right)\right\}}_{=\left(\mathcal{L}_{\widetilde{\partial/\partial t}}\,\omega_t\right)_0}$$

□

The really key point is now to see how this links up with the representation of primitive cohomology of a hypersurface by residues. Suppose

$$\begin{cases} X_t = \left\{\underbrace{F-\tfrac{t}{n-p}G}_{\text{call this } F_t}=0\right\}, & F,G\in S^d, \quad 1\le p\le n-1, \\ [\omega_t] = \mathrm{Res}_{X_t}\left(\left[\frac{P_t\Omega}{F_t^{\,n-p}}\right]\right)\in\Gamma\left(\mathbb{D}, F^p\mathcal{H}^{n-1}_{\mathcal{X}/\mathbb{D},pr}\right), & P_t\in S^{(n-p)d-n-1} \\ \varphi_t \text{ family of } (n-1)\text{-cycles} \end{cases}$$

Then

$$\int_{\varphi_t}\nabla[\omega_t] = \frac{\partial}{\partial t}\int_{\varphi_t}\mathrm{Res}\left(\left[\frac{P_t\Omega}{F_t^{\,n-p}}\right]\right) = \frac{\partial}{\partial t}\int_{\mathrm{Tube}(\varphi_t)}\frac{P_t\Omega}{\left(F-\frac{t}{n-p}G\right)^{n-p}} = \int_{\mathrm{Tube}(\varphi_t)}\left(\frac{G P_t\Omega}{F_t^{\,n-p+1}} + \frac{(\partial P_t/\partial t)\Omega}{F_t^{\,n-p}}\right) = \int_{\varphi_t}\mathrm{Res}([\;\cdot\cdot\;])$$

[this ($\mathrm{Tube}(\varphi_t)$) can be taken to be locally constant in $\mathbb{P}^n\setminus X_t$]

$\Rightarrow \nabla[\omega_t] \equiv \mathrm{Res}_{X_t}\left(\left[\frac{G P_t\Omega}{F_t^{\,n-p+1}}\right]\right) \bmod F^p$

$\Rightarrow \overline{\nabla}$ is computed by mult. by G, i.e. (recalling $R^q = \frac{S^q}{J_F^q} = \frac{S^q}{(\partial F/\partial z_0,\ldots,\partial F/\partial z_n)}$)

<u>Theorem 6</u>: The diagram

$$\begin{array}{ccc} R^{(n-p)d-n-1} & \xrightarrow{\cong} & Gr^p_{F^\bullet}H^{n-1}_{pr}(X_0) \\ \cdot G\downarrow & & \downarrow\overline{\nabla}_{(p)} \\ R^{(n-p+1)d-n-1} & \xrightarrow{\cong} & Gr^{p-1}_{F^\bullet}H^{n-1}_{pr}(X_0) \end{array}$$

commutes. □

Now, the local system $\mathbb{H}^{n-1}_{\mathcal{X}/\mathbb{D}}$ over $\mathbb{D}$ is trivial, so we have a Hodge flag $F_t^\bullet$ varying over the constant vector space $H^{n-1}(X_0,\mathbb{C})$, giving a period map

$$\Phi : \mathbb{D} \to \mathcal{D} \,(= \text{period domain}).$$

The $\{\bar{\nabla}_{(p)}\}_{1\le p\le n-1}$ capture the infinitesimal changes in this flag (by Thm. 2), hence

(B.6) $\boxed{\Phi \text{ is locally injective} \iff d\Phi|_0 \neq 0 \iff \text{some } \bar{\nabla}_{(p)} \neq 0}$.

Assume for the moment the

Lemma (Macaulay): Set $e = (n+1)(d-2)$, and assume $k, e-k \geq 0$. Then $(R^k)^\vee \cong R^{e-k}$, with the duality given by multiplication
$$R^k \times R^{e-k} \to R^e \cong (R^0)^\vee \cong \mathbb{C}. \quad //$$

Griffiths used this together with Theorem 6 to prove the

Local Torelli Theorem: The period map for degree-d hypersurfaces in $\mathbb{P}^n$ (modulo the action* of $PGL_{n+1}(\mathbb{C})$) is locally injective for $d>2$ & $n>1$, except for the case of cubic surfaces ($d=n=3$).

This says that locally** the Hodge flag (or Φ) recovers the isomorphism class of X_t (as a complex manifold)!!

Proof:
$$G \equiv 0 \in R^d \iff G \in J_F$$
$$\iff G = \sum A_{ij} z_i \frac{\partial F}{\partial z_j}$$
$$\iff \left\{F - \frac{t}{n-p} G = 0\right\} \text{ is tangent to the action of } PGL_{n+1}(\mathbb{C})$$

(use 1-parameter subgroup given by $\exp(t[A_{ij}])$.)

WTS: $G \not\equiv 0 \in R^d \Rightarrow$ some $\bar{\nabla}_{(p)} \neq 0$ (by Theorem 6). It suffices to check the

Claim (i): $R^d \times R^{(n-p)d-n-1} \to R^{(n-p+1)d-n-1}$ has no left kernel

$\Updownarrow$ lin. alg.

Claim (ii): $R^d \hookrightarrow R^{(n-p+1)d-n-1} \otimes (R^{(n-p)d-n-1})^\vee$

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* which produces $\cong$'s of hypersurfaces | ** not just in $\mathbb{D}$ but in any ball inside $\mathcal{S} = \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))$

⇕ lin. alg.

Claim (iii): $\left(R^{(n-p+1)d-n-1}\right)^\vee \otimes R^{(n-p)d-n-1} \twoheadrightarrow (R^d)^\vee$

⇕ (Macaulay Lemma)

Claim (iv): $R^{\overbrace{e-(n-p+1)d+n+1}^{A}} \otimes R^{\overbrace{(n-p)d-n-1}^{B}} \twoheadrightarrow R^{e-d}$

But this last version just follows from the surjectivity of polynomial multiplication $S^A \otimes S^B \twoheadrightarrow S^{A+B}$ provided $A, B \geq 0$.

Ex/ show $A, B \geq 0 \iff p \in \left[\frac{n+1}{d}, n - \frac{n+1}{d}\right] (\cap \mathbb{Z}) \iff \begin{matrix} d > 2 \\ n > 1 \end{matrix}$ and not $d = n = 3$. // $\square$

Proof of the Lemma (due to M. Green): let $W = \mathbb{C}\langle \partial/\partial z_0, \ldots, \partial/\partial z_n \rangle$.

From the map of sheaves on $\mathbb{P}^n$: $W \otimes \mathcal{O} \to \mathcal{O}(d-1)$

$$\partial/\partial z_i \otimes 1 \longmapsto \partial F/\partial z_i$$

one builds a long-exact sequence (of sheaves on $\mathbb{P}^n$) called a Koszul complex.

Ex/ show
$$\underbrace{0 \to \underbrace{\Lambda^{n+1} W}_{\cong \mathbb{C}} \otimes \mathcal{O}(\ell-(n+1)(d-1)) \to \underbrace{\Lambda^n W}_{\cong W^\vee} \otimes \mathcal{O}(\ell - n(d-1)) \to \cdots \to W \otimes \mathcal{O}(\ell-(d-1)) \to \mathcal{O}(\ell) \to 0}_{\text{all this } (\mathcal{C}^\bullet, d)}$$
is exact. //

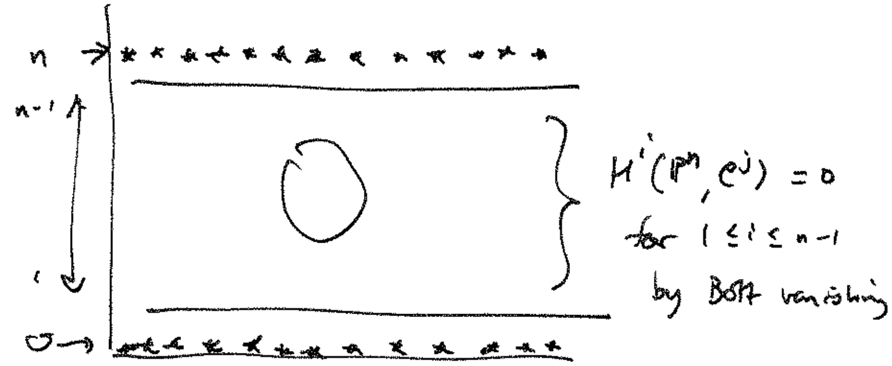
In computing hypercohomology $\mathbb{H}^*(\mathbb{P}^n, \mathcal{C}^\bullet) = H^*(\mathrm{Tot}^\bullet\{\check{C}^{\bullet,\bullet}(\mathbb{P}^n, \mathcal{C}^\bullet)\})$ by spectral sequence, there are 2 ways to proceed:

(a) $\uparrow d_0 = \delta$, $\to d_1$, $\searrow d_2 \ldots$ and (b) $\to d_0 = d$, $\uparrow d_1$, $\nwarrow d_2 \ldots$

Of these, (b) gives $E_1^{\bullet,\bullet} = 0$ by exactness of $\mathcal{C}^\bullet$.

(a) gives

$E_1^{\bullet,\bullet} =$



Since they must give the same answer, and in (a) d_1 & d_{n+1} are evidently the only nonzero differentials, we must have in particular that

$$\text{(B.7)} \qquad \boxed{E_{n+1}^{0,n} \xrightarrow[d_{n+1}]{\cong} E_{n+1}^{n+1,0}}$$

where

$$(\text{LHS (B.7)})^\vee = \left(\ker E_1^{0,n} \xrightarrow{d_1} E_1^{1,n}\right)^\vee$$

$$= \left(\ker\{H^n(\mathbb{P}^n, \mathcal{C}^0) \to H^n(\mathbb{P}^n, \mathcal{C}^1)\}\right)^\vee$$

$$= \left(\ker\{H^n(\mathbb{P}^n, \underset{\mathcal{O}(-(n+1))}{K_{\mathbb{P}^n}} \otimes \mathcal{O}(l-e)) \to H^n(\mathbb{P}^n, W^\vee \otimes K_{\mathbb{P}^n} \otimes \mathcal{O}(l-e+(d-1)))\}\right)^\vee$$

$$\underset{\text{Serre duality}}{=} \operatorname{coker}\{\underbrace{H^0(\mathbb{P}^n, W \otimes \mathcal{O}(e-l-(d-1)))}_{} \to \underbrace{H^0(\mathbb{P}^n, \mathcal{O}(e-l))}_{S^{e-l}}\}$$

$$= R^{e-l}, \quad \text{and}$$

$$\text{RHS (B.7)} = \operatorname{coker}\{\underbrace{H^0(\mathbb{P}^n, W \otimes \mathcal{O}(l-(d-1)))}_{} \to \underbrace{H^0(\mathbb{P}^n, \mathcal{O}(l))}_{S^l}\}$$

$$= R^l.$$

(The compatibility of the duality with multiplication comes from the fact that cup products of sections of $\mathcal{O}(a)$ & $\mathcal{O}(b)$ are computed by multiplying polynomials, and the compatibility of Serre duality with cup products.)

□

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