B. Gauss–Manin Connection

Continue to assume $\pi : X \rightarrow S$ a smooth projective morphism of $\mathbb{C}$-schemes. Proposition A.2 applies in the more general setting of filtered complexes of sheaves with $H^*$ replaced by $R^*F$. So there exists a spectral sequence of sheaves on $S$

$$E_1^{p,q} = R^p \pi_* \left( G^{p,q} \pi^* \mathcal{L}_\pi \right) = \mathcal{E}_S^p \otimes \left[ R^q \pi_* \mathcal{L}_\pi \right]$$

(E.1)

$$E_2^{p,q} = E_\infty^{p,q} = \text{Gr}^p R^p \pi_* \mathcal{L}_\pi = \left\{ \begin{array}{ll} H_{1/3}^{q+1}, & p = 0 \\ 0, & p > 0 \end{array} \right. \quad (\text{same proof})$$

(see Thm. A.1)

Definition 1: The Gauss–Manin Connection $\nabla : N_S^p \otimes \mathcal{L}_S^1 \rightarrow \mathcal{E}_S^{p+1} \otimes \mathcal{L}_S^1$ is $d_1$.

Theorem 1: $\nabla$ is flat with flat structure given by $H_{1/3}^2$.

In particular, $\nabla(H_{1/3}^2, \ast) = 0$.

Proof: $\nabla = d_1 \Rightarrow \nabla \circ \nabla = 0$.

(E.1) $\Rightarrow \left\{ \begin{array}{ll} \ker \nabla = \text{im} \nabla \text{ in } E_1^{p,q} \text{ for } p > 1 \\ \ker \nabla = H_{1/3}^2, \ast \text{ for } p = 0 \end{array} \right.$

Remark 1: We can of course consider, instead of $\nabla : H^2 \rightarrow \mathcal{E}_S^1 \otimes \mathcal{L}_S^1$, $\nabla : \Theta(\mathcal{L}_S^1) \otimes H^2 \rightarrow \mathcal{L}_S^1$,

or $\nabla_\ast : H^2 \rightarrow \mathcal{L}_S^1$ \quad ($\ast \in \Gamma(\mathcal{L}_S^1)$)
What if (instead of $\mathcal{N}$) we Leray-filtered $\mathcal{N}_{\mathcal{X}}$? Then we get

$$E_1^{-p,q} = \lim_{\to \mathcal{N}_{\mathcal{X}} (Gr^p_{\mathcal{X}} \mathcal{N}_{\mathcal{X}}^{2k-1})} = \mathcal{N}_{\mathcal{L}}^p \otimes R^{q-p} \mathcal{N}_{\mathcal{L}}^{2k-1-p} \otimes \mathcal{F}^{k-p} \mathcal{N}_{\mathcal{X}}^{2k-1/2}$$

Now $\mathcal{N}_{\mathcal{X}}^{2k} \rightarrow \mathcal{N}_{\mathcal{X}}$ induces a morphism of spectral sequences $E \rightarrow E'$, so that

$$d_1 : \mathcal{N}_{\mathcal{L}}^p \otimes \mathcal{F}^{k-p} \mathcal{N}_{\mathcal{X}}^{2k-1} \rightarrow \mathcal{N}_{\mathcal{L}}^{p+1} \otimes \mathcal{F}^{k-p-1} \mathcal{N}_{\mathcal{X}}^{2k-1/2}$$

is the restriction of $d_1$. In particular, we have

**Theorem 2 (Griffiths Transversality):** $\nabla (\mathcal{F}^{k-1} \mathcal{N}_{\mathcal{X}}^{2k}) \subset \mathcal{N}_{\mathcal{L}}^p \otimes \mathcal{F}^{k-1} \mathcal{N}_{\mathcal{X}}^{2k-1/2}$.

As a consequence, or by Leray filtering $\mathcal{N}_{\mathcal{X}}^{2k} \rightarrow \mathcal{N}_{\mathcal{X}}^{2k}$, we have well-defined maps

$$\nabla : \mathcal{N}_{\mathcal{L}}^p \otimes Gr_{\mathcal{F}}^{k} \mathcal{N}_{\mathcal{X}}^{2k} \rightarrow \mathcal{N}_{\mathcal{L}}^{p+1} \otimes Gr_{\mathcal{F}}^{k-1} \mathcal{N}_{\mathcal{X}}^{2k-1/2}$$

**Theorem 3:** $\nabla$ is $\Theta_{\mathcal{L}}$-linear; in particular, for each $s \in \mathcal{L}$,

$$\nabla_s$$

is a map of vector spaces $Gr_{\mathcal{F}}^{k} H^q(X, \mathcal{E}) \rightarrow Gr_{\mathcal{F}}^{k-1} H^q(X, \mathcal{E})$.

**Proof:** Given a section $f$ of $\mathcal{F}^{k-1} \mathcal{N}_{\mathcal{X}}^{2k}$ over $U \subset \mathcal{L}$, $f \in \mathcal{F}^{k-1} \mathcal{N}_{\mathcal{X}}^{2k}$.

$$\nabla (f \sigma) = df \otimes \sigma + f \nabla \sigma$$

**Remark 2:** Sometimes $Gr_{\mathcal{F}}^{k} \mathcal{N}_{\mathcal{X}}^{2k}$ is written as $\mathcal{H}^{k,q-k}$; the problem with this is that one then thinks of it as sections of a subbundle of $\mathcal{H}^{k,q-k}$, which is wrong. In fact, $\mathcal{H}^{k,q-k} = \mathcal{F}^{k-1} \mathcal{N}_{\mathcal{X}}^{2k}$ is neither holomorphic nor anti-holomorphic. So $\mathcal{H}^{k,q-k}$ (or rather, real analytic) is obviously $Gr_{\mathcal{F}}^{k} \mathcal{N}_{\mathcal{X}}^{2k}$ is holomorphic.

So I'll avoid this notation!

$k \mathcal{N}_{\mathcal{X}}^{2k-1}$ means "shifted k places to the right" (a complex of sheaves with only one term, in degree $k$).
Henceforth we suppose \( D = \Omega \) (unit disk) with holomorphic coordinates \( t \).

Then \( L^2\mathcal{N}_D = \mathcal{O}_0 \implies \nabla \) is the connecting homomorphism in the long exact sequence associated to

\[
0 \to \pi^*\mathcal{N}_D \otimes \mathcal{N}^{-1}_D \to \mathcal{N}_D \to \mathcal{N}_D^{\times D} \to 0
\]

and \( \nabla \) the one coming from

\[
(\nabla)_1 \quad \mathcal{N}_D \times \mathcal{N}_D^{\times D} \to \mathcal{N}_D \otimes \mathcal{N}_D^{\times D}
\]

Consider the s.e.s. of sheaves on \( \mathcal{X} \)

\[
0 \to \mathcal{E}^{1}_{\mathcal{X}/\mathcal{D}} \to \mathcal{E}^{1}_{\mathcal{D}} \to \pi^*\mathcal{E}^{1}_D \to 0,
\]

with associated i.e.s. (of sheaves on \( \mathcal{D} \))

\[
\cdots \to R^0\pi^*\mathcal{N}^{1}_{\mathcal{X}/\mathcal{D}} \to \mathcal{E}_{\mathcal{D}}^{1} \to R^1\pi^*\mathcal{E}^{1}_D \to 0
\]

**Definition 2:** The Kodaira-Spencer class of \( \pi \) is \( \kappa : = \delta (\partial \phi_t) dt \in R^1\pi^* (\pi^*\mathcal{D} \otimes \Theta_{\mathcal{D}}) \).

**Theorem 4:** \( \nabla \) is computed by cup-product with \( \kappa \).

Sketch: Dualizing (2.4) and tensoring with \( \mathcal{N}^{-1}_{\mathcal{X}/\mathcal{D}} \) gives

\[
0 \to \pi^*\mathcal{N}^{-1}_D \otimes \mathcal{N}^{-1}_{\mathcal{X}/\mathcal{D}} \to \mathcal{N}_D \otimes \mathcal{N}^{-1}_{\mathcal{X}/\math{D}} \to \mathcal{N}_D^{\times D} \otimes \mathcal{N}^{-1}_{\mathcal{X}/\math{D}} \to 0
\]

from which the result is basically clear.

\[
(\nabla)_1 \quad \mathcal{N}_D \times \mathcal{N}_D^{\times D} \to \mathcal{N}_D \otimes \mathcal{N}_D^{\times D}
\]
Now for a more geometric point of view on $\mathcal{D}$, one would like to see how it acts on families of de Rham cohomology classes given by $C^\infty$ forms.

From
\[
\pi^*\mathcal{J}_{\mathcal{D}} \otimes \mathcal{J}^{\mathcal{D}}_{\lambda_{\Omega_0}} \xrightarrow{\Delta} \ker (\mathcal{J}_{\lambda_x} \to \mathcal{J}_{\lambda_{\Omega_0}}) \to \mathcal{J}_{\lambda_x} \to \mathcal{J}_{\lambda_{\Omega_0}}
\]
\[
\downarrow \quad \downarrow \quad \downarrow \\
\pi^*\mathcal{J}_{\mathcal{D}} \otimes \mathcal{J}^{\mathcal{D}}_{\lambda_{\Omega_0}} \xrightarrow{\Delta} \ker (\mathcal{J}_{\lambda_x} \to \mathcal{J}_{\lambda_{\Omega_0}}) \to \mathcal{J}_{\lambda_x} \to \mathcal{J}_{\lambda_{\Omega_0}}
\]
\[
\text{one has}
\]
\[
H^q_{\lambda_{\Omega_0}} \cong R^q\pi_* S_{\lambda_{\Omega_0}} \to R^{q+1}\pi_* \ker (\ldots) \to \mathcal{J}_{\lambda_{\Omega_0}} \otimes R^q\pi_*\mathcal{J}_{\lambda_x} \to \mathcal{J}_{\lambda_{\Omega_0}} \otimes R^q\pi_*\mathcal{J}_{\lambda_x}
\]
\[
\text{or}
\]
\[
C^\infty(H^{q-1}_{\lambda_{\Omega_0}}) \cong R^q\pi_* A \to \ker (\ldots) \to A \otimes R^q\pi_* A \cong A \otimes C^\infty(H^{q-1}_{\lambda_{\Omega_0}})
\]
\[
= \nabla^\lambda = \nabla^{(1,0)} + \nabla^{(1,1)}
\]

I will often denote $\nabla^{(1,0)}$ simply by $\nabla$.

Now let $\gamma : x_0 \times D \to X$ be a $C^\infty$ diffeomorphism as in (A.2), and let
\[
\gamma^*: (U \subseteq X) \to (U \times X)
\]
\[
\gamma^*: \text{the corresponding lift of } \gamma, \quad (x_0 \times t) \mapsto \gamma(t)
\]
\[
\gamma^* \in A^\lambda_x(\gamma(x)) \quad \text{the unique lift with } \gamma^* \gamma = 0.
\]
As on p. 187, any section of $C^\infty(H^{q-1}_{\lambda_{\Omega_0}})$ is of the form $[\alpha]$ with
\[
\alpha \in A^\lambda_x(\gamma(x))
\]
\[
\text{Let } \gamma^* \in A^\lambda_x(\gamma(x)) \text{ be any lift of } \gamma, \quad \text{e.g., one in family of relative cohom. class.}
\]

The same Hodge deRham (cf. p. 187).

**Theorem 5:** \[\nabla^\lambda [\alpha] = \left[ L_{\gamma^*} \alpha \right], \] where the Lie derivative is computed by the Cartan formula
\[
\left[ L_{\gamma^*} \alpha \right] := \left( \text{image in } \gamma^* A^\lambda_x(\gamma(x)) \otimes (d + d\gamma^* (\gamma^* + 1)) \right)
\]

*With respect to $\pi$, vertical vectors + horizontal forms are defined in the absence of a $C^\infty$ trivialization $\gamma$. What $\gamma$ provides is a notion of vertical forms + horizontal vectors.*
Proof: \( \nabla^0 [\xi] = \left( \frac{d}{dt} \cdot \xi \right) \otimes dt + \left( \frac{d}{dt} \cdot \xi \right) \otimes \delta t \) is clear, compute \( S^0 \) since \( \partial_{\xi} \zeta = 0 \) and \( \bar{S}^{\infty} \).

From this perspective, Griffiths transversality is abundantly clear: for \( [\xi] \in \Gamma(\mathbb{F}^p H^{k}_{x_{10}}) \), choose \( \hat{\xi} \in \mathbb{F}^p A^{k}(X) \Rightarrow \hat{\xi} \rangle \Rightarrow \mathbb{F}^p A^{k}(X) \Rightarrow \mathbb{F}^p H^{k}_{x_{10}} \rangle \). Also, we use quite concretely the holomorphicity of the \( \mathbb{F}^p H^{k}_{x_{10}} \): for the same \( [\xi] \), \( \nabla^0 [\xi] \) is given by \( \mathbb{F}^p A^{k}(X) \Rightarrow \mathbb{F}^p A^{k}(X) \). 

We can now additionally confirm that Theorem 1 \( (\mathbb{F}^p H^{k}_{x_{10}}, q) = 0 \) says that \( \nabla \) differentiates the periods of cohomology classes.

**Proposition 1:** Given \( \{\{P_t\}_{t \in D} \) a \( C^\infty \) family of \( k \)-cycles on \( X \), \( \{W_t\}_{t \in D} \) a \( C^\infty \) family of closed \( k \)-forms on \( X \).

\[
(3.5) \quad \int_{P_t} \omega_t = \frac{d}{dt} \int_{P_t} \omega_{t_0}
\]

\( \text{(Remark 1): Given a holomorphic section of } \mathbb{F}^k H^k \text{, this can be used to produce a holomorphic section of } \mathbb{F}^{k-1} H^k, \mathbb{F}^{k-2} H^k, \text{ etc.} \)

**Proof:** Write \( p_t = q_t (p_0) \) \((t \in D)\), so \( \text{and the calculation that follows is actually valid for even non-closed } p_0 \! ? \)
\[ \frac{d}{dt} \left( \int_{F_t} w_t \right)_{t=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathcal{O}_{t_x}^{\text{wd}}} \left( \int_{q_0}^{q_0, \varepsilon} \frac{1}{3} \partial_{s} \lambda^* \tilde{\omega} \right) \] 

\[ = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{q_0, \varepsilon} \left( \partial_{s} \lambda^* \tilde{\omega} \right) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathcal{O}_{t_x}^{\text{wd}}} \left( \int_{q_0, \varepsilon} \frac{1}{3} \partial_{s} \lambda^* \tilde{\omega} \right) \] 

\[ = \int_{q_0, \varepsilon} \left( \frac{1}{3} \partial_{s} \lambda^* \tilde{\omega} \right) + \int_{\mathcal{O}_{t_x}^{\text{wd}}} \left( \frac{1}{3} \partial_{s} \lambda^* \tilde{\omega} \right) \] 

\[ = \int_{\mathcal{O}_{t_x}^{\text{wd}}} \left( \frac{1}{3} \partial_{s} \lambda^* \tilde{\omega} \right). \]

The really key point is now to see how this links up with the representation of primitive cohomology of a hypersurface by residues. Suppose

\[ X_t = \{ F - \frac{r}{n-p} G = 0 \}, \quad F, G \in \mathcal{S}^d, \quad 1 \leq s \leq n-1. \]

\[ [w_t] = \text{Res}_{X_t} \left( \left[ \frac{P_t}{F^{n-p}} \right] \right) \in F^* \mathcal{R}^{n-n-1}(D), \quad F \in \mathcal{S}^{n-p-d-1} \]

\[ \text{Res}_{X_t} \text{ family of (n-1)-cycles} \]

Then

\[ \int_{q_0, \varepsilon} \left( \partial_{s} \lambda^* \tilde{\omega} \right) = \frac{d}{dt} \int_{q_0, \varepsilon} \text{Res} \left( \left[ \frac{P_t}{F^{n-p}} \right] \right) = \frac{d}{dt} \int_{q_0, \varepsilon} \frac{P_t}{F^{n-p} \lambda} \frac{G P_t \lambda}{F^{n-p} \lambda} + \frac{P_t}{F^{n-p} \lambda} \frac{(\partial P_t \lambda)}{F^{n-p} \lambda} \]

\[ = \int_{q_0, \varepsilon} \text{Res} \left( \left[ \frac{P_t}{F^{n-p}} \right] \right) \]

\[ \Rightarrow \nabla [w_t] \equiv \text{Res}_{X_t} \left( \left[ \frac{G P_t \lambda}{F^{n-p} \lambda} \right] \right) \bmod F^p \]

\[ \Rightarrow \nabla \] is computed by multiplication by \( G \), i.e. (replacing \( R^{\bullet} = \sum_{i=0}^{n-1} R^{\bullet} \mathbb{Z}^i \mathcal{R}^{n-i}(X_0) \))

\[ R^{(n-p)d-1, n-1} \longrightarrow \mathcal{R}^{n-1} \mathcal{R}^{n-1}(X_0) \]

\[ \nabla \downarrow \]

\[ R^{(n-p+1)d-1, n-1} \longrightarrow \mathcal{R}^{n-1} \mathcal{R}^{n-1}(X_0) \] \text{commutes} \]

Theorem 6: The diagram

\[ R^{(n-p)d-1, n-1} \longrightarrow \mathcal{R}^{n-1} \mathcal{R}^{n-1}(X_0) \]

\[ \nabla \downarrow \]

\[ R^{(n-p+1)d-1, n-1} \longrightarrow \mathcal{R}^{n-1} \mathcal{R}^{n-1}(X_0) \] \text{commutes} \]

Now, the local system \( \mathcal{R}^{n-1} \mathcal{R}^{n-1}(X_0) \) over \( D \) is trivial, so we have a Hodge flag \( F \) varying over the constant vector space \( \mathcal{R}^{n-1}(X_0, \mathbb{C}) \), giving a
\[ \Phi : D \to D \text{ (period domain).} \]

The \( \{ \overline{D}_{(p)} \} \) depend capture the infinitesimal changes in this map (by Thm. 2), have

\[ (B.6) \quad \overline{\Phi} \text{ is locally injective } \iff \text{d}\Phi^0 \neq 0 \iff \text{some } \overline{D}_{(p)} \neq 0. \]

Assume for the moment the

Lemma (Macaulay): Set \( x = (n+1)(d-2) \), and assume \( k \geq x \).

Then \( (R^k)^* \cong (R^{d-k})^{*} \), with the duality given by multiplication

\[ R^k \times R^{d-k} \to R \cong (R^0)^* \cong \mathbb{C}. \]

Griffiths used this together with Theorem 6 to prove the

Local Torus Theorem: The period map for degree \( d \) hypersurfaces

in \( \mathbb{P}^n \) (modulo the action of \( \text{PGL}_n(\mathbb{C}) \)) is locally injective for

\[ d \geq 3 \quad \text{if} \quad n > 1, \]

except for the case of cubic surfaces \( (d = n = 3) \).

This says that locally** the Hodge field (or \( \overline{\Phi} ) \) recovers the isomorphism

class of \( X_t \) (as a complex manifold).

Proof:\n
\[ G \equiv 0 \in R^d \iff G \in J^r \]
\[ \iff G = \sum A_{ij} \frac{2}{r} \frac{\partial F}{\partial z_j} \]
\[ \iff \{ F = \frac{x}{n-1} + G = 0 \} \text{ is tangent to the action of } \text{PGL}_n(\mathbb{C}) \]

\[ \iff \text{by Theorem 6} \]

\[ \text{Use 1-parameter subgroup given by } \exp(t [A_{ij}]). \]

Claim (i): \( R^d \times R^{(n-p)d-n-1} \to R^{(n-p+1)d-n-1} \)

has no left kernel

Claim (ii): \( R^d \to R^{(n-p+1)d-n-1} \otimes (R^{(n-p)d_n})^* \)

which produces \( \gamma \)'s of hypersurfaces

* not just in \( D \) but in any ball inside \( \mathbb{D} = \mathbb{P}H^0(\mathbb{P}^n, O(d)) \).
Claim (iii): \((R^{(n-p)d-n-1})^\vee \otimes R^{(n-p)k} \rightarrow (R^d)^\vee\)

Proof of the Lemma (due to M. Green):
\[ W = C \langle \frac{1}{2} z_0, \ldots, \frac{1}{2} z_n \rangle \]

From the map of sheaves on \(\mathbb{P}^n\):
\[ W \otimes O \rightarrow O(1-d) \]

one builds a long-exact sequence (of sheaves on \(\mathbb{P}^n\)) called a Koszul complex.

**Proof:**
\[ 0 \rightarrow W \otimes O(n(d-1)) \rightarrow W \otimes O(n-d) \rightarrow \cdots \rightarrow W \otimes O(1-d) \]

In computing hypercohomology, \(H^*(\mathbb{P}^n, E^*) = H^*(\text{Tot}^{**} \{ C^w(\mathbb{P}^n, E^w) \}) \) by spectral sequence, there are 2 ways to proceed:

(a) \[ d_0 = d_1 \]

and (b) \[ d_0 = d_1 \]

(a) gives \(E_1^{**} = 0\) by exactness of \(C^*\).

(b) gives

\[ E_1^{**} = H^i(\mathbb{P}^n, E^i) = 0 \]

for \(1 \leq i \leq n-1\) by Bott vanishing.
Since they must give the same answer, and in (a) $d_1, d_2, d_3, \ldots$ are evidently the only nonzero differentials, we must have in particular that

\[
\begin{align*}
\text{(B.7)} & \quad E^{0,n}_{n+1} \xrightarrow{d_{n+1}} E^{n+1,0}_{n+1} \\
\text{when} & \\
\text{(LHS (B.7))} & = \left( \ker E^{0,n}_{1} \xrightarrow{d_1} E^{1,n}_{1} \right)^{\vee} \\
& = \left( \ker \left\{ H^n(I^n, C^\bullet) \to H^n(I^n, C^\bullet) \right\} \right)^{\vee} \\
& = \left( \ker \left\{ H^n(I^n, K^n_{\text{top}} \otimes O(2-2)) \to H^n(I^n, W \otimes K^n_{\text{top}} \otimes O(x-c+1-2)) \right\} \\
& \quad \otimes O(x-1) \right)^{\vee} \\
& = \text{coker} \left\{ H^0(I^n, W \otimes O(x-e-x+1)) \to H^0(I^n, O(x-2)) \right\}^{\vee} \\
& = R^{e-x}, \quad \text{and} \\
\text{(RHS (B.7))} & = \text{coker} \left\{ H^0(I^n, W \otimes O(x-e-x+1)) \to H^0(I^n, O(x-2)) \right\}^{\vee} \\
& = R^x.
\end{align*}
\]

(The compatibility of the duality with multiplicative comes from the fact that cup products of sections of $O(a) \otimes O(b)$ are computed by multiplying polynomials, and the compatibility of some duality with cup products.)