IV. Mixed Hodge Structures

A. Examples + Definitions

Basic idea: we want to be able to extend Hodge-theoretic invariants to noncompact or singular algebraic varieties, in such a way as to be compatible with pullback, Res, Gysin, etc. To see why usual HS's won't do, consider the Residue exact sequence associated to a codimension-1 complex submanifold Y of a smooth projective variety X: writing $U = X \setminus Y$, we have

$$\cdots \longrightarrow \underbrace{H^{k-2}(Y)}_{(b)} \xrightarrow{Gy\cancel{/2\pi i}} \underbrace{H^{k}(X)}_{(a)} \xrightarrow{\iota_U} \underbrace{H^{k}(U)}_{(c)} \xrightarrow{2\pi i\,Res} \underbrace{H^{k-1}(Y)}_{(b)} \to \underbrace{H^{k+1}(X)}_{(a)} \to \cdots$$

(a) = put usual HS

(b) = put usual HS, but shifted up in weight by 2: $(p,q) \to (p+1, q+1)$

Notationally, one writes $H^{k-2}(Y)(-1)$, and the (-1) comes with a factor of $\frac{1}{2\pi i}$. This means we can remove the $\frac{1}{2\pi i}$ from Gy so that Gy itself is an "integral" morphism: $H^{k-2}(Y, \mathbb{Z}(-1)) \xrightarrow{Gy} H^{k}(X, \mathbb{Z})$.

(c) = ??? We must set this up so as to contain the coker of the first Gy, and so that its quotient by this is the ker of the second Gy. But even with the (-1) "twists"*

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* in the literature, these are called Tate twists — more on them in a bit

in (b), this means it "has a sub-HS of weight k and its quotient by this is a HS of weight $(k-1)$." This is not a HS. We need not just to formalize a notion of "mixed weight HS" to accommodate this sort of object, but to see how <u>intrinsically</u> we can put such a structure (in a functorial manner) on all smooth open varieties.*

<u>First question</u>: should we just consider sums of HS of different weights? Let me try to convince you that the answer is <u>no</u>: and the discrepancy captures <u>how Y sits in X</u>. (in particular, it contains <u>more</u> info than the HS's of X or Y alone)

<u>Example 1</u>:

$$\begin{tikzcd} X = \text{compact Riemann surface / smooth projective curve} \\ Y = \{p,q\} \arrow[u, hook, "\iota"] \end{tikzcd}$$

We have the short exact sequence of complexes of sheaves on X:

$$0 \to \Omega_X^\bullet \to \Omega_X^\bullet\langle\log Y\rangle \to \iota_*\Omega_Y^\bullet[-1] \to 0$$

which looks like

$$\begin{tikzcd} \Omega_X^1 \arrow[r, hook] & \Omega_X^1\langle\log Y\rangle \arrow[r, "2\pi i\,\mathrm{Res}"] & \iota_*\mathcal{O}_Y & \arrow[l, squiggly] \text{"F^1 subcomplex"} \\ \mathcal{O}_X \arrow[u, "d"] \arrow[r, equal] & \mathcal{O}_X \arrow[u, "d"] \arrow[r] & 0 \arrow[u] & \end{tikzcd}$$

The hypercohomology long exact sequence (noting $\mathbb{H}^0(\iota_*\Omega_Y^\bullet[-1]) = H^{-1}(\iota_*\mathcal{O}_Y) = 0$)

(A.1)
$$\begin{tikzcd} 0 \arrow[r] & H^1(X,\mathbb{C}) \arrow[r, hook] & H^1(U,\mathbb{C}) \arrow[r] & H^0(\{p,q\},\mathbb{C}) \overset{\cong\,\mathbb{C}^2}{} \arrow[r, "\mathrm{Gy}/2\pi i"'] & H^2(X,\mathbb{C}) \arrow[r] & 0 \\ & \Omega^1(X) \arrow[u, hook] \arrow[r, hook] & \Omega^1\langle\log Y\rangle(X) \arrow[u, hook] \arrow[r] & H^0(\{p,q\},\mathbb{C}) \arrow[u, equal] \arrow[r] & H^{1,1}(X) \arrow[u, equal] \arrow[r] & 0 \end{tikzcd}$$

inherits a sub-l.e.s. from the "F^1 subcomplex":

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* actually, one can do it on <u>all algebraic varieties</u>, but that would lead us too far afield — see [Peters + Steenbrink] book for that construction.

Since $\mathrm{Gy}(1_p,-1_q)=0$, we have $\xi_F \in \Omega^1\langle\log Y\rangle(X)$ with $\begin{cases}\mathrm{Res}_p\,\xi=+1\\ \mathrm{Res}_q\,\xi=-1\end{cases}$; ξ_F is defined up to a holomorphic form, and is the "Hodge F^1 lift" of $(1_p,-1_q)$.

We have the "topological version" of $(\dagger)$, by replacing $\mathbb{C}$ with $\mathbb{Z}$ (dual to the Tube exact sequence on homology). There is then an "integral lift" of $(1_p,-1_q)$ to $\xi_{\mathbb{Z}} \in H^1(U,\mathbb{Z}) \cong H_1(X,Y;\mathbb{Z})$ by taking any path Γ from q to p; it is defined up to an element of $H^1(X,\mathbb{Z})$ (or topological 1-cycles). [To be consistent with $2\pi i\,\mathrm{Res}$ one should actually take $\mathbb{Z}(1)$ coeffs. — so "$2\pi i\,\Gamma$".]

Now Gy sends ξ_F and $\xi_{\mathbb{Z}}$ to $(1,-1)$ so $\xi_{\mathbb{Z}}-\xi_F$ to zero. This gives a class in $H^1(X,\mathbb{C})$ well-defined modulo $H^1(X,\mathbb{Z}(1))+F^1H^1(X,\mathbb{C})$. If $H^1(U,\mathbb{C})$ were going to be of the form $\underbrace{H^1(X,\mathbb{C})}_{\text{wt. }1}\oplus\underbrace{\ker(\mathrm{Gy})(-1)}_{\text{wt. }2}$ as generalized HS, then one should be able to take lifts to be the same. Our class

$$\begin{tikzcd}
{[\xi_{\mathbb{Z}}-\xi_F]} \arrow[d, mapsto] & \in & \dfrac{H^1(X,\mathbb{C})}{H^1_{\mathbb{Z}}+F^1} \arrow[d, "\cong"'] & \cong H^{0,1}\cong\Omega^1(X)^\vee \\
{\left[\int_\Gamma(\cdot)\right]} & \in & \dfrac{\Omega^1(X)^\vee}{H_1(X,\mathbb{Z})} & \text{lattice in}
\end{tikzcd}$$

Ex

gives an obstruction to this, clearly nonzero in general*.

(Later, it will be called the Abel-Jacobi class $AJ_X([p]-[q])$; and its vanishing will be shown to be equivalent to $[p]-[q]$ being the divisor of a meromorphic function on X.) □

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* for X an elliptic curve, for example, just vary p or q a bit.

All of the above is to motivate the

Definition 1: A *mixed Hodge structure* (MHS) consists of

- $V_{\mathbb{Z}}$ = $\mathbb{Z}$-module (fin. gen.)
- $W_\bullet$ = *increasing* filtration on $V (= V_{\mathbb{Q}})$
- $F^\bullet$ = *decreasing* filtration on $V_{\mathbb{C}}$

such that $(Gr_i^W V, Gr_i^W F^\bullet)$ is a weight i HS $(\forall i)$. □

(I won't worry about polarizations for now.)

The example above suggested that in some way a MHS "V" $= (V_{\mathbb{Z}}, W_\bullet, F^\bullet)$ is not a $\oplus$ of pure HS in different weights (even though, as an abelian group, $V = \oplus Gr_i^W V$), i.e. that we had a nontrivial "extension" of (M)HS. For this even to be a definable concept, we first need the idea of morphisms.

Definition 2: A *morphism* of MHS $\Theta : V \to \tilde{V}$ is a morphism $\Theta_{\mathbb{Z}} : V_{\mathbb{Z}} \to \tilde{V}_{\mathbb{Z}}$ of $\mathbb{Z}$-mod. which satisfies $\begin{cases} \Theta(W_i) \subset \tilde{W}_i & (\forall i) \\ \Theta_{\mathbb{C}}(F^p) \subset \tilde{F}^p & (\forall p) \end{cases}$ □

So in our example, it was *not* in general possible to define a morphism from ker(Gy) with the trivial (M)HS of $\left\{ \begin{array}{l} \text{weight } 2, \text{ rank } 1 \\ \text{type } (1,1) \end{array} \right.$, to $H^1(U)$ with the MHS given by

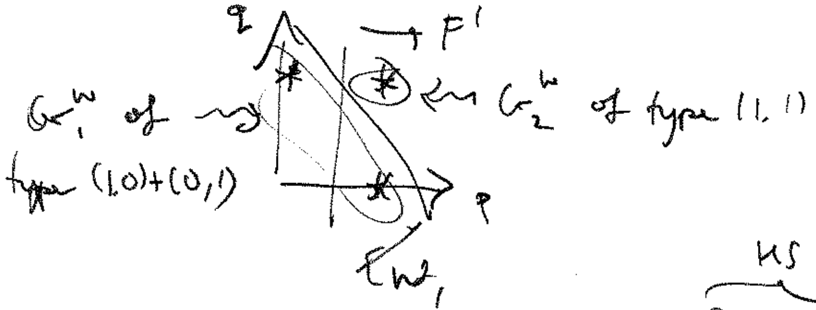
$$F^1 H^1(U)_{\mathbb{C}} = \Gamma(\Omega^1\langle \log Y \rangle), \quad F^2 = \{0\}, \quad F^0 = \text{everything}$$

$$W_1 H^1(U) = H^1(X), \quad W_0 = \{0\}, \quad W_2 = \text{everything}.$$

We say that the extension

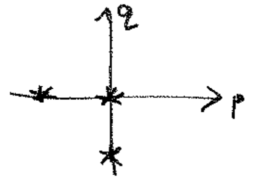
$$0 \to H^1(X) \to H^1(U) \to \ker(\cup_y) \to 0$$

<u>does not split</u> (when $[\xi_I - \xi_*] \not\equiv 0$). We can also draw a "picture" of $H^1(U)$:



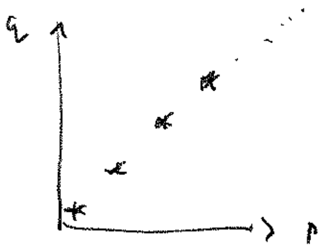
Where a "$*$" at (p,q) means that $V^{p,q} := \overbrace{\left(\mathrm{Gr}^W_{p+q} V\right)}^{\text{HS of wt. } p+q}{}^{(p,q)} \neq \{0\}$.
(These are not subobjects of $V_{\mathbb{C}}$ in the mixed case.)

<u>Remark 1</u>: It is useful to denote a trivial HS of rank 1 & weight $2n$ (type (n,n)) by $\mathbb{Z}(-n)$. Given a MHS V, $V \otimes \mathbb{Z}(n) =: V(n)$ has $F^p W_k(V(n)) = F^{p-n} W_{k-2n} V$. So e.g. $H^1(U)(+1)$ would "look like"



Also, we can write with this notation $0 \to H^1(X) \to H^1(U) \to \mathbb{Z}(-1) \to 0$.

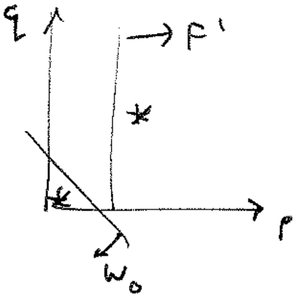
<u>Definition 3</u>: A MHS V is <u>Hodge-Tate</u> if $\mathrm{Gr}^W_{2k+1} V = \{0\}$ $(\forall k)$ and $\mathrm{Gr}^W_{2n} V \cong \mathbb{Q}(-n)^{\oplus m_n}$ for some integers. This "looks like"



Example 2: $V_{\mathbb{Z}} = \mathbb{Z}\langle \gamma_0, \gamma_2 \rangle$

$$\underbrace{W_2V = V} \supset \underbrace{W_1V = \mathbb{Q}\langle\gamma_0\rangle = W_0V} \supset \underbrace{W_{-1}V = \{0\}}$$

$$F^1V_{\mathbb{C}} = \mathbb{C}\langle \gamma_2 + \underset{\boxed{\frac{\log t}{2\pi i}}}{\ell(t)}\gamma_0 \rangle$$



gives an extension

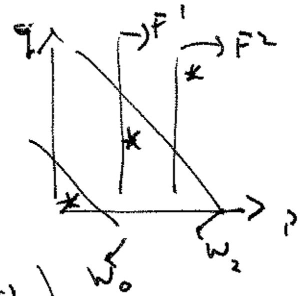
$$0 \to \mathbb{Z}(0) \to V \to \underset{\bar{\gamma}_2}{\underset{\downarrow}{\mathbb{Z}(-1)}} \to 0$$

and (ignoring $2\pi i$'s) $\bar{\gamma}_2$ lifts to $\begin{cases} \xi_F = \gamma_2 + \ell(t)\gamma_0 \in F^1V_{\mathbb{C}} \\ \xi_{\mathbb{Z}} = \gamma_2 \in V_{\mathbb{Z}} \end{cases}$

$$\Longrightarrow \quad [\xi_F - \xi_{\mathbb{Z}}] = \underset{\underset{t}{\downarrow}}{[\ell(t)]} \in \underset{\underset{\mathbb{C}^*}{\cong\downarrow \exp(2\pi i(\cdot))}}{\mathbb{C}/\mathbb{Z}(0)} \left(\cong \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), \mathbb{Z}(0)), \text{ as we shall see} \right).$$

We can represent this MHS by its period matrix $\begin{pmatrix} 1 & 0 \\ \ell(t) & 1 \end{pmatrix}$. $\square$

Example 3: $V_{\mathbb{Z}} = \mathbb{Z}\langle \underbrace{\overbrace{\gamma_0}^{W_0}, \gamma_2}_{W_2}, \gamma_4 \rangle$



$$\left.\begin{array}{l} F^2V_{\mathbb{C}} = \mathbb{C}\langle 1^{\text{st}} \text{ column} \rangle \\ F^1V_{\mathbb{C}} = \mathbb{C}\langle 1^{\text{st}}\ 2 \text{ columns} \rangle \end{array}\right\} \text{ of } \begin{pmatrix} 1 & 0 & 0 \\ \ell(t) & 1 & 0 \\ \frac{Li_2(t)}{(2\pi i)^2} & -\ell(1-t) & 1 \end{pmatrix}$$

where $Li_2(t) := -\int_0^t \log(1-t)\frac{dt}{t}$ is the dilogarithm.

What is special about this particular MHS? Well, the definition of a variation of MHS is that we must have

$\mathbb{V}_{\mathbb{Z}}$ = $\mathbb{Z}$-local system

$W_i\mathbb{V}$ = (increasing filtration by) sub-$\mathbb{Q}$-local systems of $\mathbb{V}$ $\Big\}$ (assoc. holo. vector bundle)

$F^p\mathcal{V}$ = (decreasing filtration by) holomorphic subbundles of $\mathcal{V}$

such that

and → $\nabla(\mathcal{F}^p) \subset \Omega^1 \otimes \mathcal{F}^{p-1}$ $(\forall p)$

→ $Gr_i^W\mathcal{V}$ = VHS* of weight i $(\forall i)$.

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* if these are polarized then the VMHS is called graded-polarized.

But in Example 3, $\nabla\Big(\underbrace{1^{st}\text{ Column}}_{\text{gen. }F^2}\Big) = \begin{pmatrix} 0 \\ 1/2\pi i t \\ \frac{-\log(1-t)}{(2\pi i)^2 t} \end{pmatrix} = \frac{1}{2\pi i t}\begin{pmatrix} 0 \\ 1 \\ -\ell(1-t) \end{pmatrix} \in F^1$!!

$\Rightarrow$ varying $t \in \mathbb{P}^1 \backslash \{0,1,\infty\}$ gives a VMHS. There are generalizations of this for higher (& multiple) polylogarithms. $\square$

In the next two sections I'll discuss MHS on cohomology of open spaces and extensions of MHS. I conclude this section with some generalities:

<u>Proposition 1</u>: All the operations $\oplus$, $\otimes$, $^\vee$, ker, coker apply to pairs/morphisms of MHS to yield a MHS, and MHS form an abelian category. Moreover, morphisms of MHS are strict with respect to the 2 filtrations.

<u>Sketch</u>: For a MHS V, Deligne proved* that there is an unique (+ functorial) bigrading $V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}(V)$ s.t.

- $F^r V_{\mathbb{C}} = \bigoplus_{\substack{p,q \\ p \geq r}} I^{p,q}(V)$
- $W_k V_{\mathbb{C}} = \bigoplus_{\substack{p,q \\ p+q \leq k}} I^{p,q}(V)$
- $\overline{I^{p,q}(V)} \equiv I^{q,p}(V) \mod \bigoplus_{\substack{p'<p \\ q'<q}} I^{p',q'}(V)$.

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* the reference is actually a paper of Cattani, Kaplan, & Schmid; but a partial proof of the $I^{p,q}$ result which suffices to prove Prop. 1 can be found in [Peters-Steenbrink], p. 64.

This has the very ugly formula

$$(A.2) \qquad \boxed{I^{p,q} = F^p \cap W_{p+q} \cap \left(F^q \cap W_{p+q} + \sum_{j \geq 2} \overline{F^{q-j+1}} \cap W_{p+q-j} \right)}$$

Under a morphism of MHS $\Theta : V \to \tilde{V}$, $\Theta(I^{p,q}(V)) \subseteq I^{p,q}(\tilde{V})$ is clear from the formula, and this implies the statement of the Proposition exactly as for pure HS.

$\square$

Of course, $I^{p,n-p}(V)$ maps to $V^{p,n-p}$ in $Gr^W_n V_{\mathbb{C}}$. For Hodge-Tate MHS the nonzero $I^{p,q}$'s are (by (A.2))

$$I^{p,p}(V) = (F^p \cap W_{2p})\, V \, .$$

In Example 3 this gives $I^{2,2}(V) = \mathbb{C}\langle 1^{st} \text{ column}\rangle$, $I^{1,1}(V) = \mathbb{C}\langle 2^{nd} \text{ column}\rangle$, $I^{0,0} = \mathbb{C}\langle 3^{rd} \text{ column}\rangle$.

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