B. Open and singular varieties

In this section we outline (in a couple of cases) Deligne's construction of functorial MHS on the cohomology of arbitrary quasi-projective varieties $/\mathbb{C}$.

Suppose U is smooth quasiprojective. By compactifying, resolving singularities contained in the complement (by Hironaka), followed by further blowups, one has

$$\begin{array}{l} U = X \setminus D \\ \downarrow \jmath \end{array} \quad \text{where}$$

- X is smooth projective
- $D \overset{?}{\subset} X$ is a normal crossing divisor (NCD):

$$D = \bigcup_{j=1}^{M} D_j \text{ of smooth hypersurfaces intersecting transversally,}$$

i.e. D is locally of the form $z_1 \cdots z_r = 0$ in holo. coords. about a general point $p \in D_{i_1} \cap \cdots \cap D_{i_r} =: D_I$.

Set
$$\begin{cases} D^{[0]} := X, \text{ and for } k \geq 1 \\ D^{[k]} := \coprod_{|I|=k} D_I \xrightarrow[\sum (\pm) \imath_I]{\imath^{[k]}} X \end{cases}.$$

(I won't worry about these signs.)

Deligne's construction is independent of the realization of U as $X \setminus D$.

Recall from residue theory that the cohomology of $\mathbb{P}^n \setminus \{F=0\}$ (smooth hypersurf.) can be entirely computed by

(a) holomorphic forms w/ arbitrary poles — pole order gives Hodge filtration on $H^*(\{F=0\})$

(b) C^∞ forms w/ log poles — Hodge filtration on forms gives " " " ".

For generalizing, (b) seems like the more natural choice. More precisely, "recall"*

$$\Omega^\bullet_X\langle\log D\rangle \xhookrightarrow{\simeq} A^\bullet_X\langle\log D\rangle \xhookrightarrow{\simeq} j_* A^\bullet_U$$

$$\implies H^k(U,\mathbb{C}) \cong \mathbb{H}^k(X,\Omega^\bullet_X\langle\log D\rangle).$$ Set

(B.1) $$\boxed{F^p H^k(U,\mathbb{C}) := \mathrm{Im}\left\{\mathbb{H}^k(X,\Omega^{\bullet\geq p}_X\langle\log D\rangle) \to \mathbb{H}^k(X,\Omega^\bullet_X\langle\log D\rangle)\right\}.}$$

Next, we define subcomplexes of $\Omega^\bullet_X\langle\log D\rangle$

$$\tilde{W}_\ell\Omega^\bullet_X\langle\log D\rangle = \Lambda^\ell\Omega^1_X\langle\log D\rangle \wedge \Omega^{\bullet-\ell}_X$$ ("worst" poles of form $\frac{dz_{i_1}}{z_{i_1}}\wedge\cdots\wedge\frac{dz_{i_\ell}}{z_{i_\ell}}\wedge$ smooth)

and set

(B.2) $$\boxed{W_\ell H^k(U,\mathbb{C}) := \mathrm{Im}\left\{\mathbb{H}^k(X,\tilde{W}_{\ell-k}\Omega^\bullet_X\langle\log D\rangle) \to \mathbb{H}^k(X,\Omega^\bullet_X\langle\log D\rangle)\right\}}$$

$$\quad = \tilde{W}_{\ell-k}H^k(U,\mathbb{C})$$

<u>Lemma 1</u>: $W_\bullet$ is defined on $H^k(U,\underline{\underline{\mathbb{Q}}})$.

<u>Idea</u>: $W_\bullet$ is, in fact, the Leray filtration associated to $j: U \hookrightarrow X$. □

It is fairly clear that sending

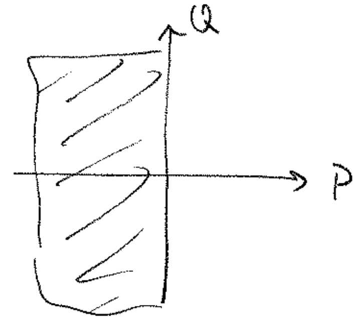
$$\frac{dz_I}{z_I}\wedge\alpha \longmapsto \alpha|_{D_I}$$

induces isomorphisms of complexes of sheaves on X

(B.3) $$Gr^{\tilde{W}}_\ell\Omega^\bullet_X\langle\log D\rangle \xrightarrow[\mathrm{Res}^\ell]{\cong} \imath^{(\ell)}_*\Omega^{\bullet-\ell}_{D^{(\ell)}}.$$ [Pf.: see Voisin p.209]

Now set $\hat{W}^\ell := \tilde{W}_{-\ell}$ (<u>decreasing</u> filtration) to produce a spectral sequence

(B.4) $$E_1^{P,Q} := \mathbb{H}^{P+Q}(X, \underbrace{Gr^P_{\hat{W}}\Omega^\bullet_X\langle\log D\rangle}_{=\imath^{[-P]}_*\Omega^{\bullet+P}_{D^{[-P]}}\,\simeq\,\imath^{[-P]}_*\mathbb{C}[P]}) \cong H^{2P+Q}(D^{[-P]},\mathbb{C})$$



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* we proved this for D <u>smooth</u> but the proof is very similar here. (Exercise?)

Converging to

$$(B.5)\qquad E_\infty^{P,Q} = \underbrace{Gr^{P}_{\hat{W}}}_{(\text{or } Gr^{\tilde{W}}_{-P})} H^{P+Q}(X, \Omega^\bullet_X\langle \log D\rangle) = Gr^W_Q H^{P+Q}(U, \mathbb{C})$$

add $P+Q$

(note $P \leq 0$)

<u>Lemma 2</u>: $d_1 : H^{2P+Q}(D^{[-P]}, \mathbb{C})(P) \to H^{2P+Q+2}(D^{[-P-1]}, \mathbb{C})(P+1)$ is an alternating* sum of Gysin maps. In particular, these are morphisms of HS.

<u>Sketch</u>: Recall d_1 is induced from $0 \to Gr^{P+1}_{\hat{W}} \to \frac{\hat{W}^P}{\hat{W}^{P+2}} \to Gr^P_{\hat{W}} \to 0$. This is a residue s.e.s. and we know the δ in the l.e.s. is Gy (by definition). e.g. if D is smooth, for $P=-1$ this is just $0 \to \Omega^\bullet_X \to \Omega^\bullet_X\langle \log D\rangle \to \imath_*\Omega^{\bullet-1}_D \to 0$. $\square$

<u>Lemma 3</u>: $E^{\bullet,\bullet}_\bullet$ degenerates at E_2.

<u>Idea</u>: Tracing through all of this with $\Omega^{\bullet \geq a}_X\langle \log D\rangle$, one finds that $\mathcal{F}^a$ (cf. (B.1)) induces F^{a+P} on RHS of (B.4); that is, F^a of $E_1^{P,Q} = H^{2P+Q}(D^{[-P]}, \mathbb{C})(P)$ — which is a pure HS of weight Q.

Showing inductively that the d_r are morphisms of (M)HS (with lemma 2 as "base case") from $\underset{\text{wt. } Q}{E_r^{P,Q}} \to \underset{\text{wt. } Q-r+1}{E_r^{P+r,\, Q-r+1}}$, one concludes [by strictness] that $d_r = 0$ for $r > 1$. $\square$

<u>Remark 1</u>: There is a spectral sequence for the "Hodge" filtration $F^\bullet$ too, and it degenerates at E_1. This is the $\Omega^\bullet\langle \log\rangle$ analogue of what we proved by hand in Cor II.D.1. (It followed for the special case $\Omega^\bullet_{\mathbb{P}^n}\langle \log\{F=0\}\rangle$ from that Corollary.)

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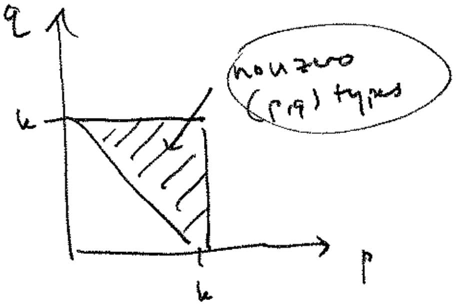
* for the sign see Voisin.

<u>Theorem 1</u>: $\left(H^k(U,\mathbb{Z}),\ W_\cdot\ ,\ F^\cdot\right)$ is a MHS with

"picture"



(compare Example A.1).

<u>"Proof"</u>: By the lemmas,

$$Gr^W_{m+k} H^k(U) \cong \frac{\ker\left\{H^{k-m}(D^{[m]})(-m) \xrightarrow{\Sigma(\pm G_\gamma)} H^{k-m+2}(D^{[m-1]})(-m+1)\right\}}{\operatorname{im}\left\{\Sigma(\pm G_\gamma)\right\}}$$

and $F^\cdot$ (on forms) induces the Hodge filtration on the RHS. $\square$

<u>This</u> is the object that sits in the residue sequence in §A, for <u>D smooth</u>. For D generically a NCD, one has

$$\longrightarrow H^k(X) \to H^k(U) \xrightarrow{\text{Res}} H^{k+1}_{|D|}(X) \to H^{k+1}(X) \to$$

where $H^{k+1}_{|D|}(X)$ is computed via a spectral sequence as above, but without $D^{[0]}(=X)$. <u>This is only $H^{k-1}(D)$ if D is smooth !!</u>

Indeed, cohomology with support on D is really <u>homology</u>* of D.

In fact, the <u>cohomology</u> of D sits in a very

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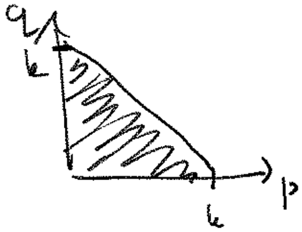
* if $\dim_{\mathbb{C}} X = n$, then $H^{k+1}_{|D|}(X) \cong H_{2n-k-1}(D)$ (for our [compact] NCD).

different (well, essentially dual!) sequence:

$$\cdots \to H^k(X,D) \to H^k(X) \xrightarrow{\iota^*} H^k(D) \to H^{k+1}(X,D) \to \cdots$$

The MHS here looks like



and is constructed (roughly) by setting up a double complex of forms

δ = alternating sum of pullback maps (or simplicial differential)

d = exterior derivative

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
A^k(D^{[1]}) & \xrightarrow{\delta} & A^k(D^{[2]}) & \xrightarrow{\delta} & A^k(D^{[3]}) & \to & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
A^{k-1}(D^{[1]}) & \xrightarrow{\delta} & A^{k-1}(D^{[2]}) & \xrightarrow{\delta} & A^{k-1}(D^{[3]}) & \to & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
A^{k-2}(D^{[1]}) & \xrightarrow{\delta} & A^{k-2}(D^{[2]}) & \xrightarrow{\delta} & A^{k-2}(D^{[3]}) & \to & \\
\uparrow & & \uparrow & & \uparrow & & \\
 & & \tilde{W}^1 & & \tilde{W}^2 & & \text{etc.}
\end{array}$$

i.e. $K^{a,b} := \begin{cases} 0 \text{ for } a<0 \\ A^b(D^{[a+1]}) \end{cases}$, with

$F^p K^{a,b} := F^p A^b(D^{[a+1]})$ and

$\tilde{W}^\bullet$ the filtration by subcomplexes as shown.

We have $H^k(\mathrm{Tot}^\bullet K^{\bullet,\bullet}) \overset{\text{Theorem}}{=} H^k(D,\mathbb{C})$, and set

$$W_{k-\ell} H^k(D) := \mathrm{im}\left\{ H^k(\mathrm{Tot}^\bullet \tilde{W}^\ell K^{\bullet,\bullet}) \to H^k(D) \right\}.$$

The resulting spectral sequence $\begin{cases} E_0^{a,b} = K^{a,b}, \ d_0 = d \\ E_1^{a,b} = H^b(D^{[a+1]}), \ d_1 = [\delta] \end{cases}$

degenerates at E_2, and we have

$$E_\infty^{a,b} = E_2^{a,b} = \frac{\ker\{H^b(D^{[a+1]}) \xrightarrow{[\delta]} H^b(D^{[a+2]})\}}{\operatorname{Im}\{H^b(D^{[a-1]}) \xrightarrow{[\delta]} H^b(D^{[a]})\}}$$

$$\cong Gr^a_{\tilde{W}} H^{a+b}(D)$$

$$\cong Gr^W_b H^{a+b}(D) .$$

<u>Theorem 2</u>: $F^\bullet$ and $W_\bullet$ induce a MHS on $H^k(D)$.

<u>Pf</u>: This is essentially just because

(a) $F^\bullet$ "restricts" to $D^{[a+1]}$ to induce the Hodge filtration on $E_2^{a,b}$

and

(b) $W_\bullet$ is dual to a filtration on the dual complex of topological chains computing homology of D (and hence is defined $/\mathbb{Q}$). $\square$

One can also do $H^k(X, D)$ in this way. (Set $D^{[0]} = X$ again and set $K^{a,b} = A^b(D^{[a]})$.) For a more complicated singular variety you have to go beyond the simplicial approach to "cubical hyper-resolutions", cf. [Peters-Steenbrink].