B. Open and Singular Varieties

In this section we outline (in a couple of cases) Deligne's construction of functional MHS on the cohomology of arbitrary quasiprojective varieties / $\mathcal{O}$.

Suppose $U$ is smooth quasiprojective. By compactifying, resolving singularities contained in the complement (by blow-ups), followed by further blow-ups, one has

$$U = X \setminus D$$

where

- $X$ is smooth projective.
- $D \subset X$ is a normal crossing divisor (NCD):
  
  $$D = \bigcup_{j=1}^{n} D_j$$
  
  of smooth hypersurfaces intersecting transversally,

  i.e. $D$ is locally of the form $z_1 \cdots z_r = 0$ in holomorphic words about a generic point $p \in D$.

  Set
  
  $$\bigg\{ D^{[k]} := \bigcup_{\sum_{i=1}^{r} z_i = k} D_i, \quad \text{for } k \geq 1 \bigg\}$$

  $$(I_{\mathcal{O}}) \quad \text{(I won't worry about these signs.)}$$

Deligne's construction is independent of the realization of $U$ as $X \setminus D$.

Recall from residue theory that the cohomology of $\mathbb{P}^n \setminus \{ \text{smooth, compact} \}$ can be entirely computed by

(a) holomorphic forms w/ arbitrary poles — pole order gives Hodge filtration on $H^k(X_{\mathbb{C}})$

(b) $C^\infty$ forms w/ log poles — Hodge filtration on forms gives 

For generalizing, (b) seems like the more natural choice. More precisely, "recall" *

\[ \mathcal{L}_X^* \langle \log D \rangle \cong A_X^* \langle \log D \rangle \cong \mathcal{O}^* \]

\[ \Rightarrow H^k(U, \mathbb{C}) \cong H^k(X, \mathcal{L}_X^* \langle \log D \rangle). \]  

Set

\[ E^P_H(U, \mathbb{C}) := \text{Im} \left\{ H^k(X, \mathcal{L}_X^* \langle \log D \rangle) \to H^k(X, \mathcal{L}_X^* \langle \log D \rangle) \right\}. \]

Next, we define subcomplexes of \( \mathcal{L}_X^* \langle \log D \rangle \)

\[ W^*_X \mathcal{L}_X^* \langle \log D \rangle = \bigwedge^l \mathcal{L}_X^* \langle \log D \rangle \]  

\( l \) is the lowest power of \( \mathcal{O} \) and \( \frac{dz_i \wedge \cdots \wedge dz_k}{z_i \cdots z_k} \) (where \( i = 1, \ldots, n \)).

and set

\[ W^*_X H^k(U, \mathbb{C}) := \text{Im} \left\{ H^k(X, \mathcal{L}_X^* \langle \log D \rangle) \to H^k(X, \mathcal{L}_X^* \langle \log D \rangle) \right\} \]

**Lemma 1:** \( W^*_X \) is defined on \( H^k(U, \mathbb{C}) \).

**Idea:** \( W^*_X \) is, in fact, the Leray filtration associated to \( \pi : U \to X \). \( \square \)

It is fairly clear that sending

\[ \frac{dz_i \wedge \cdots \wedge dz_k}{z_i \cdots z_k} \to \left| \frac{dz_i \wedge \cdots \wedge dz_k}{z_i \cdots z_k} \right| \]

induces isomorphisms of complexes of sheaves on \( X \)

\[ (B.3) \quad \mathcal{O} \wedge \mathcal{L}_X^* \langle \log D \rangle \cong \left[ \frac{dz_i \wedge \cdots \wedge dz_k}{z_i \cdots z_k} \right]^{D_1} \]

Now set \( \mathcal{W}_X^* := \mathcal{W}_D^* \) (decreasing filtration) to produce

a spectral sequence

\[ E_1^{P, Q} := H^{P+Q}(X, (\mathcal{L}_X^* \mathcal{L}_X^* \langle \log D \rangle) \cong H^{P+Q}(D^P \cap, \mathbb{C}) \]

\[ = \left[ \frac{dz_i \wedge \cdots \wedge dz_k}{z_i \cdots z_k} \right]^{D_1-P} \cong \left[ \frac{dz_i \wedge \cdots \wedge dz_k}{z_i \cdots z_k} \right]^{P} \mathbb{C}[P] \]

We proved this for \( D \) smooth, but the proof is very similar here. (Exercise?)
Converging to
\[ E_{00}^{p,q} = G_{\xi}^{p} H^{p+q}(X, \Omega^q_X \langle \log D \rangle) = G_{\xi} H^{p+q}(U, C) \] (note \( P \leq 0 \))

Lemma 2: \( d_1: H^{2p+q}(D^{sp}, C)(P) \to H^{2p+q+2}(D^{sp+1}, C)(P1) \) is an alternating sum of Gysin maps. In particular, these are morphisms of HS.

Skeem: \( \text{Recent } d_1 \text{ is induced from } 0 \to G_{\xi}^{p+1} \to \frac{W^p}{W^{p+2}} \to G_{\xi}^p \to 0 \).

This is a residue s.e.s. and we know the \( d_1 \) in the \( \text{s.e.s. of } G_y \) (by definition). E.g., if \( D \) is smooth, for \( P = -1 \) this is just \( 0 \to G_{\xi}^0 \to G_{\xi}^0 \langle \log D \rangle \to G_{\xi}^0 \to 0 \).

Lemma 3: \( E_r \) degenerates at \( E_2 \).

Idea: Tracing through all of this with \( \Omega^\infty_x \langle \log D \rangle \), one finds that 
\[ \tau^a (c_6 \langle B.1 \rangle) \text{ induces } F^{a+p} \text{ on RHS of (B.4)} \; \text{that is, } \mathbb{F} \] of 
\[ E_1^{p,q} = H^{2p+q}(D^{1-p}, C)(P) \] which is a pure HS of weight \( Q \).

Showing inductively that the \( d_r \) are morphisms of (MHS (with Lemma 2 as "base case") from \( E_r^{p,q} \to E_r^{p+r, Q-r+1} \) wt. \( Q \) wt. \( Q-r+1 \) \) one concludes (by strictness) that \( d_r = 0 \) for \( r > 1 \).

Remark 1: There is a spectral sequence for the "Hodge" filtration \( F^r \) too, and it degenerates at \( E_2 \). This is the \( \Omega^\infty \langle \log D \rangle \) analogue of what we proved by hand in Cor II D.1. (It follows for the special case \( \Omega^\infty \langle \log (F_{13}) \rangle \) from that Corollary.)

\# For the sign see Vaisin.
Theorem 1. \((\mathcal{H}^\bullet(U, \mathcal{O}), W, F^\bullet)\) is a MHS with

![Diagram]

(Compare Example A.1)

"Proof": By the lemma,

\[
\ker \left\{ \mathcal{H}^{k-m}(D) [m] \to \mathcal{H}^{k-m+m}(D) \right\} \cong \text{im} \left\{ \mathcal{I}^{1+k}(x) \right\}
\]

and \(F^\bullet\) induces the Hodge filtration on the RHS.

This is the object that sits in the residue sequence in \(\mathcal{A}\),

for \(D\) smooth. For \(D\) generically a NCD, one has

\[
\mathcal{H}^k(X) \to \mathcal{H}^k(U) \to \mathcal{H}^k(D) \to \mathcal{H}^{k+1}(X) \to \mathcal{H}^{k+2}(X) \to \ldots
\]

where \(\mathcal{H}^{k+1}(X)\) is computed via a spectral sequence as above, but without \(D^{[0]}(x)\). This is only \(\mathcal{H}^{k+1}(D)\) if \(D\) is smooth!!

Indeed, cohomology with support on \(D\) is really homology of \(D\).

In fact, the cohomology of \(D\) sits in a very

\[
\text{if } \dim X = n, \text{ then } \mathcal{H}^{k+1}(X) \cong H_{2n-k-1}(D) \text{ (for our compact NCD).}
\]
different (well, essentially dual) sequence:

\[ \cdots \to H^i(X, D) \to H^i(X) \to H^i(D) \to H^{i+1}(X, D) \to \cdots \]

The MMJ look like

\[ \delta = \text{alternating sum of pullback maps (as implied differentials)} \]

\[ d = \text{exterior derivative} \]

\[ K^{a, b} = (A^b(D^{[a+1]})), \text{ with} \]

\[ F^p K^{a, b} = F^p A^b(D^{[a+1]}) \quad \text{and} \]

\[ W^* \text{ the filtration by subcomplexes as shown} \]

We have \( H^k(\text{Tot}^* W^*, *) = H^k(D, \mathcal{O}) \), and set

\[ W_{k+1} H^k(D) := \text{im} \left\{ H^k(\text{Tot}^* W^* K^*, *) \to H^k(D) \right\} \]

The resulting spectral sequence \( E^2_{r, s} = K^{a, b}, d_0 = \delta \)

\[ E^{n, 0}_{r, s} = H^r(D^{[s+n]}) \quad d_1 = [\delta] \]

degenerates at \( E^2 \), and we have
\( E_{20}^{a,b} = E_{2}^{a,b} = \frac{\ker \left\{ H^b(D^{[a+1]}) \rightarrow H^b(D^{[a+2]}) \right\}}{\text{im} \left\{ H^b(D^{[a+1]}) \rightarrow H^b(D^{[a+2]}) \right\}} \)

\[ \cong \left( \mathcal{G}_{\omega} \mathcal{H}^{a+b}(D) \right) \]

\[ \cong \mathcal{G}_{\omega^b} \mathcal{H}^{a+b}(D) \]

**Theorem 2:** \( F^* \) and \( W_* \) induce a MHS on \( H^\bullet(D) \).

**Proof:** This is essentially just because

(a) \( F^* \) restricts to \( D^{[a+1]} \) to induce the Hodge filtration on \( E_2^{a,b} \)

and

(b) \( W_* \) is dual to a filtration on the derived

complex of topological chains computing homology of \( D \)

(see hence is defined /\( \alpha \)).

One can also do \( H^k(X, D) \) in this way. (Set \( D^{[0]} = X \)

again and set \( K^{a,b} = A^b(D^{[a+1]}) \).) For a more complicated singular

variety you have to go beyond the simplicial approach to "abelian

hyper-resolution", cf. [Peter-Seminar].