C. Extensions of MHS

It is convenient at this point to officially drop our convention that a Hodge structures \( V \), the weight \( n \) and \((p,q)\)'s are non-negative. The preferred measure of complexity of \( V \) is then its

\[
\text{level}(V) := \text{difference between lowest & highest values of } p.
\]

When Hodge theorists talk about properties which emerge in "higher weight" they really mean "higher level". This notion extends to MHS, where one also has the

\[
\text{length}(V) := \text{difference between lowest & highest weights}.
\]

Now let \( C \) be an abelian category with enough injectives, \( X \in C \) and

\[
0 \to A \to B \to C \to 0
\]

a short exact sequence in \( C \). By Example I.F.11 & Prop. I.F.1, we have a long exact sequence

\[
(C.1) \quad 0 \to \text{Hom}_C(X, A) \to \text{Hom}_C(X, B) \to \text{Hom}_C(X, C) \to \text{Ext}_C^1(X, A) \to \text{Ext}_C^1(X, B) \to \text{Ext}_C^1(X, C) \to \ldots
\]

Taking an injective resolution \( A \to I^* \), we have

\[
\text{Ext}_C^1(X, A) := \frac{\ker f \text{Hom}_C(X, I^0) \to \text{Hom}_C(X, I^1)}{\text{im} f \text{Hom}_C(X, I^0) \to \text{Hom}_C(X, I^1)}
\]

\[
\cong \text{Hom}_C(X, K) \quad \text{(writing } K := \ker (I^1 \to I^2))
\]

\[
(C.2) \quad \cong \text{s.e.s. } A \to E \to X
\]

\[
\text{split s.e.s.}.
\]
Since in \[ 0 \to A \to I^0 \to K \to 0 \]
\[ 0 \to A \to E \to K \to 0 \]
\[ \ker(I^0 \otimes X \to K) \overset{\phi}{\to} \phi \text{ factors through } I^0, \]

one has a splitting \( X \to E \Leftrightarrow \phi \text{ factors through } I^0 \).

Now MHS does not have enough injectives, but there is a general theory due to Verber & Yomda \& H. that extends (C.1) \& (C.2) to our setting.

I'll give a more down-to-earth presentation that doesn't prove everything.

Let \( A, B \) be MHS:

**Definition 1**: \( H^g(A) := \text{Hom}_{\text{MHS}}(\mathbb{Z}(0), A) \sim A_2 \otimes \omega_0 A \otimes F^g A \).  

More generally, \( H^g(A) := \text{Hom}_{\text{MHS}}(\mathbb{Z}(-p), A) \sim A_2 \otimes \omega_p A \otimes F^p A \).

These are called the \{Hodge class in \( A \), Hodge class \} in \( F^p \).

**Definition 2**: (a) An extension of MHS is just an exact sequence of MHS \[ 0 \to A \to H \to B \to 0 \]

A section is a morphism \( s : B \to H \) s.t. \( \pi \circ s = \text{id}_B \); an extension with section is split.

A morphism of extensions is a diagram
\[
\begin{array}{ccc}
0 & \to & A & \to & E & \to & B & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A' & \to & E' & \to & B' & \to & 0
\end{array}
\]

A congruence of extensions is an isomorphism \( s, \beta \) s.t. \( s, \beta \) = identity.

(Split extensions are congruent to \( 0 \to A \to A \oplus B \to B \to 0 \).

* cf. the appendix A in [Peter-Sueling]*
(b) \( \text{Ext}^1_{\text{maps}}(B, A) := \frac{\text{extensions, congruence}}{\text{Bac-summation}} \) with the addition group structure given by Bac-summation (with split extension as "0")

\[
\begin{align*}
&\text{O} \rightarrow A \rightarrow E \rightarrow B \rightarrow \text{O} \\
&\text{O} \rightarrow A \oplus A \rightarrow E \oplus E' \rightarrow B \oplus B \rightarrow \text{O} \\
&\text{O} \rightarrow A \oplus A \rightarrow E'' \rightarrow B \rightarrow \text{O}
\end{align*}
\]

(This requires some work, but is the same as the proof for \( R \)-modules in MacLane's book "Homology".)

(c) \( J(A) := \frac{W_0 A_C}{P^0 W_0 A_C + (W_0 A A_2)} \) (generalized) Torsion of \( A \)

\[
J'(A) := \frac{W_{2p} A_C}{P^0 W_{2p} A_C + (W_{2p} A A_2)}
\]

(These are not in general algebraic, even for pure \( H^S \)'s, where they are compact complex tori. The problem is the independence of the polarizing form in higher level.)

Ex/ \( J'(A) \) is a Lie group \( \iff W_{2p} A = W_{2p-1} A \).

[Hint: draw the picture, and determine the condition under which \( W_{2p} A A_2 \) is a lattice in \( W_{2p} A C / (W_{2p} A P^0) A_C \).]

**Theorem 1**: There is a canonical 1-functor isomorphism of groups

\[
\text{Ext}^1_{\text{maps}}(B, A) \cong J(\text{Hom}(B, A))
\]

Viewed as \( \text{MRS} : \{ W, \text{Hom}(B, A) = \{ \phi \in \text{Hom} | \phi(W_0)(\text{Hom}) \} \}

Proof: Let \( 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0 \in \text{Ext}^1 \), and choose a section \( \sigma : B \rightarrow E \) strictly preserving \( W_0 \).

Any 2 differ by \( \text{Hom}(B, A)_Z \cap W_0 \), and we define...
Next, choose a section \( \sigma_F : B_0 \to E_0 \) satisfying \( \text{pr}_0 \circ F^0 \circ W_0 = F^0 \).

Use Deligne's \( \mathbb{Z}^{p,q} \). Any \( 2 \) differ by \( \text{Hom}_c(B, A) \circ W_0 \circ F^0 \), and we define

\[
A_0 \otimes B_0 \xrightarrow{f} E_0
\]

\[
(a, b) \mapsto a(a) + \sigma_F(b).
\]

Curvilinear

\[
f(\sigma_F)^* \circ f(\sigma_0) \in \text{Aut}_c(A_0 \otimes B_0) \circ W_0
\]

\[
= \left( \begin{array}{cc}
1_A & \phi \\
0 & 1_B
\end{array} \right),
\]

where \( \phi = \partial \circ (\sigma_0 - \sigma_F) + W_0 \text{Hom}_c(B, A) \).

Clearly the extension class

\[
[\phi] \in \frac{W_0 \text{Hom}(B, A) \circ}{F^0 (\text{Hom}) + (\text{Hom}) \otimes} = J(\text{Hom}(B, A))
\]

is well-defined and if the extension is split gives zero.

To show that all classes in \( J(\ldots) \) occur: let \( \phi \in \text{Hom}(B, A) \circ W_0 \).

Then \( g_\phi := \left( \begin{array}{cc}
1_A & \phi \\
0 & 1_B
\end{array} \right) \in W_0 \text{Aut}_c(A_0 \otimes B_0) \), and we define a

MHS \( E \) by

\[
F^*_\phi(A_0 \otimes B_0) := g_\phi \left( F^*(A_0) \otimes F^*(B_0) \right)
\]

\[
= F^*(A_0) + (\mathbb{Z}_n \otimes \phi) F^*(B_0).
\]

Example 1: \( \text{Ext}^1_{\text{mhs}}(\mathbb{Z}(0), \mathbb{Z}(n)) \equiv J(\text{Hom}(\mathbb{Z}(0), \mathbb{Z}(n))) \equiv J(\mathbb{Z}(n)) = \mathbb{Q} \mathbb{Z} \equiv \mathbb{C}^* \mathbb{Z}
\]

\[
\mathbb{Q}(n) = \mathbb{C}/(\mathbb{Z})^n \otimes \mathbb{C}.
\]
Example 2: \( \text{Ext}^1_{\text{mhs}}(\mathbb{Z}(0), A) \cong J(A) \)
\( \text{Ext}^1_{\text{mhs}}(\mathbb{Z}(p), A) \cong J^p(A) \)

Example 3: \( \text{Ext}^1_{\text{mhs}}(\mathbb{Z}(1), \mathfrak{H}(C)) \cong \text{Ext}^1_{\text{mhs}}(\mathbb{Z}(0), \mathfrak{H}(C)(1)) \) \[ \cong \frac{\mathfrak{H}(C, C)}{\mathfrak{H}^1 + \mathfrak{H}^2} \cong J(C) \text{ (Tamagawa) or } C \text{.} \]

Theorem 2: All \( \text{Ext}^1_{\text{mhs}} \) for \( i \geq 2 \) vanish.

Proof: This is a formal consequence of the surjectivity of \( J(\mathfrak{g}) \to J(C) \) when \( \mathfrak{g} \to C \) (clear from Thm. 1), using the aforementioned general theory.

A new result in Carlson's article on \( \text{Ext}_{\text{mhs}} \) is a Torrelli theorem for open curves. That is, the extensions in Deligne's MHS capture "where the points lie" (up to \( \epsilon \)) on \( C \) (compactification of open curve). The content of this, beyond Torrelli for curves, is really just AbrL theorem, which we shall prove in \( \& \text{ Thm. 1.} \)

Appendix: At top of p. 229, if \( \exists \text{ lift } \phi: X \to I^\circ \) of \( \phi \),
you don't need to use injectivity of \( I^\circ \) to get a splitting \( X \to E \).
It's more trivial than that: just recall \( E := \ker \left\{ I^0 \otimes X \to K \right\} \) and map \( X \) to \( E \) by \( x \mapsto (x, -\phi(x)) \). That's it.