D. Limit MHS and boundary components

Let \( V = (V_2, Q, F^*) \) be a weight \( n \) MHS over \( \Delta^* \), with \( T = \text{monodromy generator} \)

\[
\Phi : \Delta^* \to \langle T \rangle \quad \text{the period map.}
\]

We can associate to this a MHS \( V_{\text{in}} \) which describes how \( V \) "degenerates" at \( s \to 0 \). Let \( s \in \Delta^* \). The monodromy theorem is \( (T^m - 1)^{m+1} = 0 \) for \( m \leq n \).

Assume unipotent monodromy \( (m = 1) \Rightarrow \) can define the unipotent endomorphism.

\[
N := \log (T) = \log (1 - (1 - T)) = \sum_{k \geq 1} \frac{(y - 1)^k}{k} \quad (T - I)^k \in \text{End} (V_2, s_0).
\]

Remark: \( T \) preserves \( Q \Rightarrow Q(N(\cdot, \cdot)) = -Q(\cdot, N(\cdot)) \).

Consider a basis \( \{ Y_i \} \subset V_{2, s_0} \) (view as multi-valued section of \( V \)), and define the single-valued (!) sections of \( V \)

\[
\tilde{Y}_i := e^{-\langle s | N \rangle} Y_i \in \Gamma (\Delta^*, V).
\]

These give a trivial ("untwisted") local system

\[
\tilde{W}_2 := \mathbb{Z} \langle \{ \tilde{Y}_i \} \rangle
\]

and we define the pruned extension of \( V \) to \( \Delta^* \) by

\[\text{†} \text{ carefully, this means that the period matrix entries will blow up like powers of } \log(s). \]
\[ V_e := W_{\mathbb{Z}} \otimes O_A. \]

(The point is that any holomorphic vector bundle over \( \Delta^* \) extends to \( \Delta \), but this is a special choice of extension.) Pretending \( V_{\mathbb{Z}} \) is our "true \( \mathbb{Z} \)-structure" and writing \( \{ \mathbb{Z}^\text{+} \}\) defines a new "period map"\(^{1+}\)

\[ \widetilde{\mathcal{F}} := e N \mathcal{F} : h \to D. \]

\(^{1+}\)strictly speaking, it really isn't one — it doesn't correspond in general to a VHS.

**Theorem 1 (Schmidt):** This descends to \( \mathcal{F} : \Delta^* \to \tilde{D} \) and extends across the origin.

**Remark 2:** \( \widetilde{\mathcal{F}}(0) \), not the "naive" (or "Baily-Borel") limit (in \( \overline{N}(0) \in \Delta \)), is equivalent to considering the limiting flag \( F_{\infty} \subset \mathcal{F}_{\infty} \subset \mathcal{F}_{0} \) against \( \mathcal{F}_{z_0} \) and is the "right" object. The point is that the "re-normalization" prevents periods from blowing up meaning that the limit will carry more information.

**Example 1:** Recall Example III.C.1 \((n=1)\)

\[ W_{z_0} = \langle \beta, \lambda \rangle, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{so} \quad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

\[ F_s = \mathbb{C} \langle \omega_s \rangle, \quad \omega_s = \beta + (g(s) - \lambda(s)) \lambda \]

\[ \tilde{\omega} = \lambda, \quad \tilde{\beta} = \beta - \lambda(s) \lambda \Rightarrow \omega_s = \tilde{\beta} + g(s) \tilde{\omega}, \quad \text{with} \]

\[ \lim_{s \to 0} \tilde{\beta} + g(s) \tilde{\omega}. \]

So \( \widetilde{\mathcal{F}}(0) = g(s) + \mathbb{C} \tilde{\omega} = \tilde{D} \) (and this may not be in \( h \)).

\[ \text{Ex/} \quad \text{The reparameterisation by } g(s), \text{ in getting rid of } g, \text{ makes } \widetilde{\mathcal{F}}(0) = 0 \in \mathbb{C} \tilde{\omega}. \]

* This is a consequence of a theorem of Seindorf called the \( \alpha \)-point orbit theorem.*
Remark 3: Schmid showed that the nilpotent orbit
\[ \Phi_{\text{nilp}} = e^{-\lambda(0)N} \Phi(0) : \Delta^* \to \mathfrak{g} \]
"strongly approximates" the original period map. (This is the closest thing to a "constant" PVHS when the local system has monodromy.)

In particular, this satisfies transversality because
\[ 2\pi i \text{Res}_{\infty}(\nabla) = N \quad \Rightarrow \quad N(F^* \Phi) \subset F^* \Phi. \]

Example 1 (contd): The PVHS corresponding to \( \Phi_{\text{nilp}} \) is given by
\[ w_{\text{nilp}} = \beta + (g(0) - \lambda(0)) \ll \]

Remark 4: Kato & Usui have constructed a theory of boundary components for period domains. Given a strongly convex, finitely generated rational polyhedral cone \( \Sigma = \sum_{j=1}^{\infty} N_j \subset \mathbb{R}^m \),
the \( \{ N_j \} \) commuting nilpotents, and \( F^* \in D^* \),
\[ e^{\lambda(0)N_j} \text{ is a } \Sigma^* \text{-nilpotent orbit} \iff \begin{cases} e^{\sum_j N_j y_j} F^* \in D \text{ for } y_j >> 0 \\ N_j F^* \subset F^{*-1} \end{cases} \]

The boundary component \( D_0 \) is then the set of \( \Sigma^* \)-nilpotent orbits.

Now back to our PVHS \( V \) over \( \Delta^* \). Associated to the nilpotent \( \mathbf{N} \in \text{End}(W_{\infty}, \mathbf{s}_0) \) is a unique filtration
\[ W_{-1} = \mathbf{0} \subset W_0 \subset W_1 \subset \ldots \subset W_{n-1} = W_{\infty}, \mathbf{s}_0 \]
such that
(i) \( N(W_k) \subset W_{k-1} \quad \text{and} \)
(ii) \( N^k : \mathfrak{g}^W_{n+k} \to \mathfrak{g}^W_{n+k} \) is an isomorphism.
(This is just linear algebra. The elegant way to do it is by extending \( N \) to an \( SL_2 \)-representation.) By (i), \( W_0 \)
is preserved under monodromy hence extends to a filtration of \( V_\mathbb{Q} \) by sub-local-systems, and also to a filtration \( W \) of \( V_\mathbb{E} \).

**Theorem 2 (Schmid)**: \((W_{\mathbb{Q}}, W, \Gamma_{\mathbb{Q}})\) defines a MHS on \( V_{\mathbb{E}} \), called the limiting mixed Hodge structure (LMHS).

**Example 1 (cont’d)**: Write \( \omega = \omega \circ, B = \beta \circ, A = \alpha \circ \). Then

\[
V_{\text{lim}} = \mathbb{Z} \langle B, A \rangle, \quad \{ W_0 V_{\text{lim}} = \mathbb{Q} \langle A \rangle \}, \quad F^1 = \omega = B + g(0) A.
\]

[note \( N(B) = A \)]

The extension class is given by \( g(0) \in C/\mathbb{Z} \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(1), \mathbb{Z}(0)) \).

Notice that this captures the constant term of the log-period of the VHS.

**Example 2**: For a VHS of weights and Hodge type \( h^{3,0} h^{2,1} h^{1,2} h^{0,3} \), the possible LMHS “look like”:

\[
T = I \quad (T-I)^2 = 0 \quad (T-I)^4 = 0
\]

The crazy Example III. 2 (p. 201) has of Hodge-Tate type with

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

In this is a consequence of a bigger theorem called the \( SL_2 \)-orbit theorem.