D. Limit MHS and boundary components

Let $V = (\mathbb{V}_{\mathbb{Z}}, Q, \mathcal{F}^\bullet)$ be a weight n PVHS$/\Delta^*$,

with T = monodromy operator

$$\Phi : \underset{\text{coord. } s}{\Delta^*} \to \langle T\rangle \backslash D \quad \text{the period map.}$$

We can associate to this a MHS V_{lim} which describes how V "degenerates"† as $s \xrightarrow{\in \Delta^*} 0$. Let $s_0 \in \Delta^*$. The monodromy theorem $\Rightarrow$ $(T^\mu - \mathbb{1})^{m+1} = 0$ for $m \leq n$.

Assume unipotent monodromy ($\mu = 1$) $\Rightarrow$ can define the nilpotent (why?) endomorphism

$$N := \log(T) := \log(\mathbb{1} - (\mathbb{1} - T)) := \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (T - \mathbb{1})^k \in \mathrm{End}(\mathbb{V}_{\mathbb{Q}, s_0}).$$

Remark 1: T preserves Q $\Rightarrow$ $Q(N(\cdot), \cdot) = -Q(\cdot, N(\cdot))$. //

Consider a basis $\{\gamma_i\} \subset \mathbb{V}_{\mathbb{Z}, s_0}$ (view as multivalued section of $\mathbb{V}$), and define the single-valued(!) sections of V

$$\tilde{\gamma}_i := e^{-\ell(s)N} \gamma_i \in \Gamma(\Delta^*, \mathcal{V}).$$

[calculation: $T e^{-\ell(s)N} \gamma_i = e^{-\ell(s+1)N} T \gamma_i = e^{-\ell(s)N} e^{-N} T \gamma_i = e^{-\ell(s)N} \cancel{T^{-1} T} \gamma_i$.]

These give a trivial ("untwisted") local system

$$\tilde{\mathbb{V}}_{\mathbb{Z}} := \mathbb{Z}\langle\{\tilde{\gamma}_i\}\rangle$$

and we define the privileged extension of V to Δ by

---

† concretely, this means that the period matrix entries will blow up like powers of $\log(s)$.

$$\mathcal{V}_e := \widetilde{\mathbb{V}}_{\mathbb{Z}} \otimes \mathcal{O}_\Delta .$$

(The point is that any holomorphic vector bundle over Δ^* extends to Δ, but this is a special choice of extension.) Pretending $\widetilde{\mathbb{V}}_{\mathbb{Z}}$ is our "new $\mathbb{Z}$-structure" and writing $[F^\bullet_s]_{\tilde{\gamma}}$ defines a new "period map"(†)

$$\widetilde{\Psi} := e^{\ell(s)N} \underbrace{\widetilde{\Phi}}_{\text{lift to } \mathfrak{h} \to \check{D} \text{ of } \Phi} : \mathfrak{h} \to \check{D} .$$

(†) strictly speaking, it really isn't one — doesn't correspond in general to a VHS.

<u>Theorem 1 (Schmid*)</u>: This descends to $\Psi : \Delta^* \to \check{D}$ and extends across the origin.

<u>Remark 2</u>: $\Psi(0)$, <u>not</u> the "naive" (or "Borel-Borel") limit $\lim_{s \to 0} \Phi(s) \in \overline{D}$, (i.e. $D \cup \partial D \subset \check{D}$) is equivalent to considering the limiting flag $F^\bullet_{e,0} \subset V_{e,0}$ ($=: F^\bullet_{\lim}$) against $\widetilde{\mathbb{V}}_{\mathbb{Z},0}$, and is the "right" object. The point is that this "renormalization" prevents periods from blowing up meaning that the limit will carry more information.

There is actually recent work by Bloch–Kreimer relating this to renormalization in physics (QFT).

<u>Example 1</u>: Recall Example III.C.1 ($n=1$)

$$\mathbb{V}_{\mathbb{Z},s_0} = \langle \beta, \alpha \rangle , \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \quad \text{so } N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

$$F^1_s = \mathbb{C}\langle \omega_s \rangle , \quad \omega_s = \beta + (\underset{\mathcal{O}(\Delta)}{g(s)} - \ell(s))\alpha$$

$$\tilde{\alpha} = \alpha , \quad \tilde{\beta} = \beta - \ell(s)\alpha \Rightarrow \underline{\omega_s = \tilde{\beta} + g(s)\tilde{\alpha}} , \quad \text{with}$$

limit $\tilde{\beta} + g(0)\tilde{\alpha}$.

So $\Psi(0) = g(0) \in \mathbb{P}^1 = \check{D}$ (and this may not lie in $\mathfrak{h}$).

EX/ The reparametrization by $q(s)$, in getting rid of g, makes $\Psi(0) = 0 \in \mathbb{P}^1$.

---

* this is a consequence of a bigger thm. of Schmid called the <u>nilpotent orbit theorem</u>.

Remark 3: Schmid showed that the nilpotent orbit

$$\Phi_{nilp} := e^{-\lambda(s)N}\,\Psi(0) : \Delta^* \to {}_{\langle T\rangle}\backslash D$$

"strongly approximates" the original period map. (This is the closest thing to a "constant" PVHS when the local system has monodromy.)

In particular, this satisfies transversality because $2\pi i\,\mathrm{Res}_0(\nabla) = N$ (Gauss-Manin connection on original PVHS)

$\Rightarrow N(F^{\bullet}_{\lim}) \subset F^{\bullet-1}_{\lim}$. //

Example 1 (cont'd): The VHS corresponding to Φ_{nilp} is given by

$$\omega_s^{nilp} := \beta + (g(0) - \lambda(s))\,\alpha.$$ //

Remark 4: Kato & Usui have constructed a theory of boundary components for period domains. Given a strongly convex, finitely generated rational polyhedral cone $\sigma = \sum_j \mathbb{R}_{\geq 0} N_j \subset \mathfrak{g}_{\mathbb{R}}$, with the $\{N_j\}$ commuting nilpotents, and $F^{\bullet} \in \check{D}$,

(!! nilpotent cone)

$$e^{\sigma_{\mathbb{C}}} F^{\bullet} \text{ is a } \sigma\text{-nilpotent orbit} \iff \begin{cases} e^{\sum i y_j N_j} F^{\bullet} \in D \text{ for } y_j \gg 0 \\ \text{AND} \\ N_j F^p \subset F^{p-1} \quad \forall j \end{cases}.$$

The boundary component D_σ is then the set of σ-nilpotent orbits. //

Now back to a weight n VHS $\mathbb{V}$ over Δ^*. Associated to the nilpotent $N \in \mathrm{End}(\mathbb{V}_{\mathbb{Q},s_0})$ is a unique filtration

$$W_{-1} = \{0\} \subset W_0 \subset W_1 \subset \cdots \subset W_{2n} = \mathbb{V}_{\mathbb{Q},s_0}$$

such that

(i) $N(W_k) \subset W_{k-2}$ and

(ii) $N^\ell : Gr^W_{n+\ell} \longrightarrow Gr^W_{n-\ell}$ is an isomorphism.

(this is just linear algebra. The elegant way to do it is by extending N to an sl_2-representation.) By (i), $W_\bullet$ is preserved under monodromy hence extends to a filtration of $\mathbb{V}_{\mathbb{Q}}$ by sub-local-systems, and also to a filtration $W_\bullet$ of $V_{e,0}$.

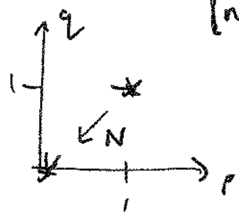
<u>Theorem 2 (Schmid)*</u>: $(\mathbb{V}_{\mathbb{Z},0}, W_\bullet, F^\bullet_{e,0})$ defines a MHS on $V_{e,0}$, called the <u>limiting mixed Hodge structure</u> (LMHS). (Call this V_{lim}.)

<u>Example 1 (cont'd)</u>: Write $\omega := \omega_0$, $B = \tilde{\beta}_0$, $A = \tilde{\alpha}_0$. Then

$$V_{lim,\mathbb{Z}} = \mathbb{Z}\langle B, A\rangle, \quad \begin{cases} W_0 V_{lim} = \mathbb{Q}\langle A\rangle \\ Gr^W_2 V_{lim} = \mathbb{Q}\langle \overline{B}\rangle \end{cases}, \quad F^1_{lim} = \omega_0 = B + g(0)A.$$

[note $N(B) = A$]

<u>Picture</u>:

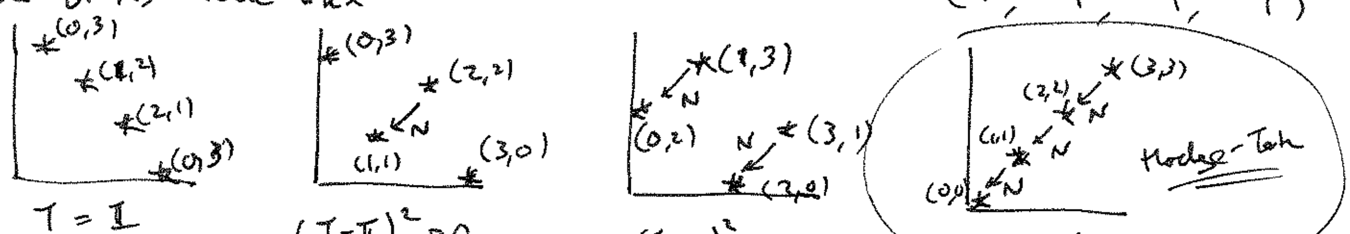


The extension class is given by

$$g(0) \in \mathbb{C}/\mathbb{Z} \cong \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), \mathbb{Z}(0)).$$

Notice that this captures the constant term of the (holomorphic part of the) log-period of the VHS.

<u>Example 2</u>: For a VHS of weight 3 and Hodge #'s $h^{3,0}\ h^{2,1}\ h^{1,2}\ h^{0,3}$ $(1, 1, 1, 1)$, the possible LMHS "look like"



$T = \mathbb{I}$ (pure HS) $\qquad (T-\mathbb{I})^2 = 0 \qquad (T-\mathbb{I})^2 = 0 \qquad (T-\mathbb{I})^4 = 0$

The crazy Example III.C.2 (p. 201) has LMHS of Hodge-Tate type with period matrix

$$\begin{pmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 25/6 & 11/2 & 1 & \\ \frac{25i}{\pi^3}\zeta(3) & 25/6 & 0 & 1 \end{pmatrix}.$$

Columns express $I^{3,3}, I^{2,2}, I^{1,1}, I^{0,0}$ generators in <u>integral</u> basis

gives highly nontrivial $\mathrm{Ext}(\mathbb{Z}(-3), \mathbb{Z}(0))$ element.

---

* this is a consequence of a bigger theorem called the <u>Sl_2-orbit Theorem</u>