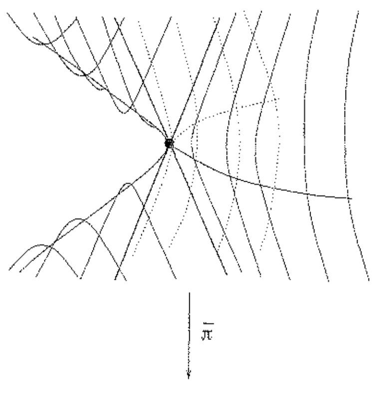
B. NORMAL FUNCTIONS AND LEFSCHETZ (1,1)

We now wish to vary the Abel-Jacobi map in families, over a smooth projective curve (i.e. compact Riemann surface) S. Let $\mathcal{X}$ be a smooth projective surface, and $\bar{\pi} : \mathcal{X} \to S$ a (projective) morphism which is

(a) smooth off a finite set $\Sigma = \{s_1, \ldots, s_e\} \subset S$, and

(b) locally of the form $(x_1, x_2) \mapsto x_1 x_2$ at singularities (of $\bar{\pi}$).

Write $X_s := \bar{\pi}^{-1}(s)$ $(s \in S)$ for the fibres. The singular fibres X_{s_i} $(i = 1, \ldots, e)$ then have only nodal, or "ODP" (ordinary double point) singularities



and writing $\mathcal{X}^*$ for their complement we have $\pi : \mathcal{X}^* \to S^* := S \backslash \Sigma$. Fixing a general $s_0 \in S^*$, the local monodromies $T_{s_i} \in$ *Aut* $(H^1(X_{s_0}, \mathbb{Z}) =: \mathbb{H}_{\mathbb{Z}, s_0})$ of the local system $\mathbb{H}_{\mathbb{Z}} := R^1 \pi_* \mathbb{Z}_{\mathcal{X}^*}$ about each s_i are then computed by the *Picard-Lefschetz formula*

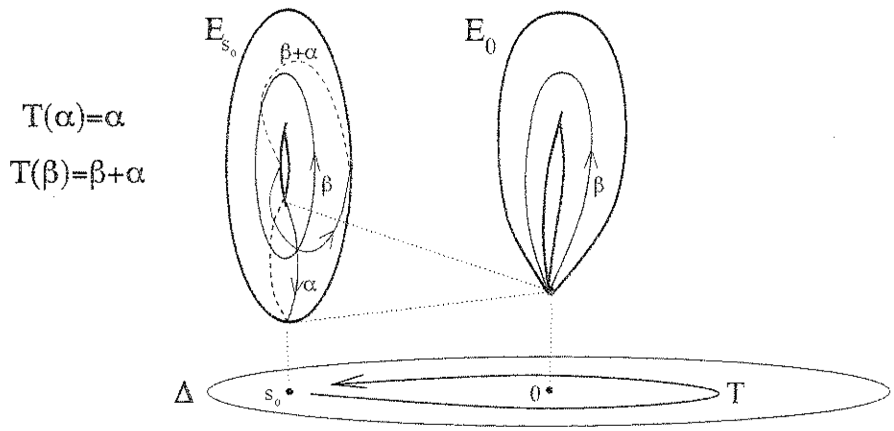
$$(1) \qquad (T_{s_i} - I)\gamma = \sum_j (\gamma \cdot \delta_j)\delta_j.$$

Here $\{\delta_j\}$ are the Poincaré duals[1] of the (quite possibly non-distinct) vanishing cycle classes $\in \ker\{H_1(X_{s_0}, \mathbb{Z}) \to H_1(X_{s_i}, \mathbb{Z})\}$ associated to each node on X_{s_i}, and "$\cdot$" the intersection form. A brief sketch of proof is provided in the appendix to this section.

Exercise. Show that $(T_{s_i} - I)^2 = 0$.

For a family of elliptic curves, (1) is the so-called *Dehn twist:*

[1] i.e. the images under $H_1(X_{S_0}, \mathbb{Z}) \overset{\cong}{\to} H^1(X_{s_0}, \mathbb{Z})$



(For the reader new to such pictures, the two crossing segments in the previous "local real" picture become the two touching "thimbles", i.e. a small neighborhood of the singularity in E_0, in this diagram.)

For a family with singular fibers, the notion of relative differentials has to change. The relative canonical (or "dualizing") sheaf $\omega_{\mathcal{X}/S} := \Omega^2_{\mathcal{X}} \otimes \bar{\pi}^*\theta^1_S \cong \Omega^1_{\mathcal{X}/S}(\log \bar{\pi}^{-1}(\Sigma))$ is a sheaf on $\mathcal{X}$ which restricts to $\Omega^1_{X_s}$ on smooth fibers. Its restriction to singular fibers, denoted $\omega_{X_{s_i}}$, identifies away from the ODP with the sheaf of holomorphic 1-forms; at the ODP, its sections are holomorphic 1-forms with log poles whose residues cancel.

Abstract normal functions

Now, in our setting, the bundle of Jacobians $\mathcal{J} := \bigcup_{s\in S^*} J^1(X_s)$ is a complex (algebraic) manifold. It admits a partial compactification to a fiber space of complex abelian Lie groups, by defining $J^1(X_{s_i}) := \frac{H^0(\omega_{X_{s_i}})}{\mathrm{im}\{H^1(X_{s_i},\mathbb{Z})\}}$ and $\mathcal{J}_e := \cup_{s\in S} J^1(X_s)$. (How to topologize this is discussed for example in Clemens (1983).) The same notation will denote their sheaves of sections,

$$0 \to \mathbb{H}_{\mathbb{Z}} \to \mathcal{F}^\vee \to \mathcal{J} \to 0 \qquad (\text{on } S^*) \tag{2}$$

$$0 \to \mathbb{H}_{\mathbb{Z},e} \to (\mathcal{F}_e)^\vee \to \mathcal{J}_e \to 0 \qquad (\text{on } S) \tag{3}$$

with $\mathcal{F} := \pi_*\omega_{\mathcal{X}/S}$, $\mathcal{F}_e := \bar{\pi}_*\omega_{\mathcal{X}/S}$, $\mathbb{H}_{\mathbb{Z}} = R^1\pi_*\mathbb{Z}$, $\mathbb{H}_{\mathbb{Z},e} = R^1\bar{\pi}_*\mathbb{Z}$.

Note that the stalk of $\mathbb{H}_{\mathbb{Z},e}$ at a "singular" point s_i is the part of cohomology invariant under monodromy, i.e. $\ker(T_{s_i} - I) \subset H^1(X_{s_0}, \mathbb{Z})$.

Definition 1. A *normal function* (NF) is a holomorphic section (over S^*) of $\mathcal{J}$. An *extended (or Poincaré) normal function* (ENF) is a holomorphic section (over S) of $\mathcal{J}_e$. A NF is *extendable* if it lies in $\mathrm{im}\{H^0(S,\mathcal{J}_e) \to H^0(S^*,\mathcal{J})\}$.

Next consider the long-exact cohomology sequence (sections over S^*)

$$0 \to H^0(\mathbb{H}_{\mathbb{Z}}) \to H^0(\mathcal{F}^\vee) \to H^0(\mathcal{J}) \to H^1(\mathbb{H}_{\mathbb{Z}}) \to H^1(\mathcal{F}^\vee); \tag{4}$$

the *topological invariant* of a normal function $\nu \in H^0(\mathcal{J})$ is its image $[\nu] \in H^1(S^*, \mathbb{H}_{\mathbb{Z}})$.

Exercise. Show that the restriction of $[\nu]$ to $H^1(\Delta_i^*, \mathbb{H}_{\mathbb{Z}})$ (Δ_i a punctured disk about s_i) computes the local monodromy $(T_{s_i} - I)\tilde{\nu}$ (where $\tilde{\nu}$ is a multivalued local lift of ν to $\mathcal{F}^\vee$), modulo the monodromy of topological cycles.

We say that ν is locally liftable if all these restrictions vanish, i.e. if $(T_{s_i} - I)\tilde{\nu} \in \mathrm{im}\{(T_{s_i} - I)\mathbb{H}_{\mathbb{Z},s_0}\}$. Together with the assumption that as a (multivalued, singular) "section" of $\mathcal{F}_e^\vee$, $\tilde{\nu}_e$ has at worst logarithmic divergence at s_i, this is equivalent to extendability.

Remark. The "at worst log divergence" aspect is the reason for the term "normal".

Interlude on algebraic cycles

In our discussion of Abel's theorem, we used the notation $Div(M)$ for divisors on a compact Riemann surface M, with $Div^0(M)$ for divisors of degree zero, $PDiv(M)$ for divisors of meromorphic functions, and $Pic^0(M)$ for their quotient. Replacing M by a projective algebraic n-manifold Y, we introduce the more general notation:

- $Z^p(Y)$ for codimension p *algebraic cycles*, i.e. the free abelian group generated by subvarieties of codimension p; and
- $Z^p(Y)_{hom}$ for algebraic cycles which, viewed as topological ones (of real dimension $2n-2p$), are trivial in homology.

For now we shall just take $p = 1$ (and $n = \dim Y \leq 2$ for the most part), so we are really still dealing with divisors. A divisor $Z = \sum q_i V_i$ is said to be rationally equivalent to zero, $Z \underset{\text{rat}}{\equiv} 0$, if it is the divisor of a rational (equivalently meromorphic) function; and I will just remind you that such functions on surfaces (unlike those on curves) have points where they are not well-defined: so the zeroes and poles can intersect. It is clear that $f^{-1}(\mathbb{R}_{\geq 0})$ bounds on the divisor $(f) = (f)_0 - (f)_\infty$, so that topological cycle class is well-defined modulo rational equivalence. Defining

- $CH^1(Y) := \frac{Z^1(Y)}{\underset{\text{rat}}{\equiv}}$ (Chow group),

we therefore have a map $[\cdot] : CH^1(Y) \to H_{2n-2}(Y, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$, with kernel

- $CH^1(Y)_{hom} := \frac{Z^1(Y)_{hom}}{\underset{\text{rat}}{\equiv}}$.

(Of course, this identifies with $Pic^0(Y)$ as defined above when Y is a curve.) The most important observation is this: one can also view $[\cdot]$ as being defined by integration over Z and duality, $CH^1(Y) \to \{H^{2n-2}(Y,\mathbb{C})\}^\vee \cong H^2(Y,\mathbb{C})$. Since pulling back to the components V_i of Z kills $(n, n-2)$ forms and $(n-2, n)$ forms, but not $(n-1, n-1)$ forms, the image lies in $H^{1,1}(Y,\mathbb{C})$. Hence, the cycle class map is

$$[\cdot] : CH^1(Y) \to H^2(Y,\mathbb{Z}) \cap H^{1,1}(Y,\mathbb{C}) \cong Hg^1(Y).$$

Normal functions of geometric origin

Let $\mathfrak{Z} \in Z^1(\mathcal{X})_{prim}$ be a divisor properly intersecting fibres of $\bar{\pi}$ (i.e. no component of $\mathfrak{Z}$ lies in a fibre) and avoiding its singularities, and which is *primitive* in the sense that each $Z_s := \mathfrak{Z} \cdot X_s$ ($s \in S^*$) is of degree 0. In fact, the intersection conditions can be done away with, by moving the divisor in a rational equivalence. The main point is that $\mathfrak{Z}$ is "fiberwise homologous to zero on smooth fibres", but not in general homologous to zero on $\mathcal{X}$. When this is true, $s \mapsto AJ(Z_s)$ defines a section $\nu_{\mathfrak{Z}}$ of $\mathcal{J}$, and it can be shown that a multiple $N\nu_{\mathfrak{Z}} = \nu_{N\mathfrak{Z}}$ of $\nu_{\mathfrak{Z}}$ is always extendable. One says that $\nu_{\mathfrak{Z}}$ itself is *admissible.*

Now assume $\bar{\pi}$ has a section $\sigma : S \to \mathcal{X}$ (also avoiding singularities) and consider the analogue of (4) for $\mathcal{J}_e$

$$0 \to \frac{H^0(\mathcal{F}_e^\vee)}{H^0(\mathbb{H}_{\mathbb{Z},e})} \to H^0(\mathcal{J}_e) \to \ker\left\{H^1(\mathbb{H}_{\mathbb{Z},e}) \to H^1(\mathcal{F}_e^\vee)\right\} \to 0.$$

By a deep result in Hodge theory (due to Griffiths, Deligne, and Schmid) called the Theorem of the Fixed Part, $H^0(\mathbb{H}_{\mathbb{Z},e})$ — regarded as a constant sub-local-system of $\mathbb{H}_{\mathbb{Z}}$ — underlies an actual constant sub-VHS $\mathcal{H}_{fix} \subset \mathcal{H}^1_{\mathcal{X}^*/S^*}$, called the *fixed part.* The Jacobian of the fixed part $J^1(\mathcal{X}/S)_{fix} \hookrightarrow J^1(X_s)$ ($\forall s \in S$) gives a constant sub-bundle of $\mathcal{J}_e$, and the left hand term of the above sequence is but constant sections of this.[2] Next, if $H^2(\mathcal{X})_{prim}$ denotes the classes restricting to 0 on a general fibre X_{s_0} of $\bar{\pi}$, then $V_{\mathbb{Z}} := \frac{H^2(\mathcal{X},\mathbb{Z})_{prim}}{\mathbb{Z}\langle[X_{s_0}]\rangle} \cong H^1(\mathbb{H}_{\mathbb{Z},e})$ while $V^{0,2} = H^1(\mathcal{F}_e^\vee)$. So the right hand term becomes $\ker\{V_{\mathbb{Z}} \to V_{\mathbb{C}}/F^1V_{\mathbb{C}}\} \cong V_{\mathbb{Z}} \cap V^{1,1} =: Hg^1(V)$. The upshot is that with some work, the above short-exact sequence becomes

$$(5) \qquad 0 \to J^1(\mathcal{X}/S)_{fix} \longrightarrow \text{ENF} \xrightarrow{[\cdot]} \frac{Hg^1(\mathcal{X})_{prim}}{\mathbb{Z}\langle[X_{s_0}]\rangle} \to 0\ ,$$

where the primitive Hodge classes $Hg^1(\mathcal{X})_{prim}$ are the Q-orthogonal complement of X_{s_0} in $Hg^1(\mathcal{X})$.

Proposition 2. *Let ν be an ENF.*

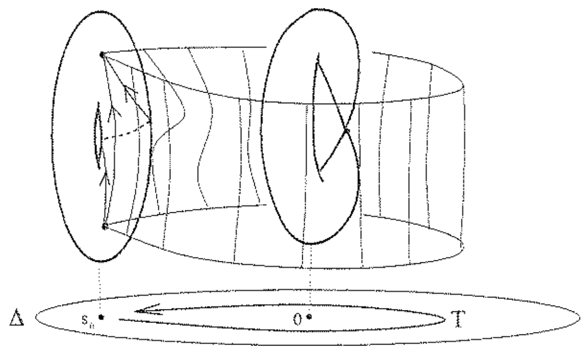
(i) If $[\nu] = 0$ then ν is a constant section of $\mathcal{J}_{fix} := \bigcup_{s\in S} J^1(\mathcal{X}/S)_{fix} \subset \mathcal{J}_e$;

[2] for curves, this fixed part business could be dealt with explicitly via curvature or degree arguments (which are beyond our scope here)

(ii) If $(\nu =)\nu_{\mathfrak{Z}}$ *is of geometric origin, then* $[\nu_{\mathfrak{Z}}] = \overline{[\mathfrak{Z}]}$ *(*$[\mathfrak{Z}]$ *= fundamental class);*

(iii) [Poincaré Existence Theorem] *Every ENF is of geometric origin.*

We note that (i) follows immediately from (5). To see (iii), apply "Jacobi inversion with parameters" and $q_i(s) = \sigma(s)$ $(\forall i)$ over S^* (really, over the generic point of S), and then take Zariski closure.[3] Finally, when ν is geometric, the monodromies of a lift $\tilde{\nu}$ (to $\mathcal{F}_e^\vee$) around each loop in S (which determine $[\nu]$) are just the corresponding monodromies of a bounding 1-chain Γ_s $(\partial\Gamma_s = Z_s)$



which identify with the Leray $(1,1)$ component of $[\mathfrak{Z}]$ in $H^2(\mathcal{X})$; this gives the gist of (ii).

A normal function is said to be *motivated over* K ($K \subset \mathbb{C}$ a subfield) if it is of geometric origin as above, and if the coefficients of the defining equations of $\mathfrak{Z}$, $\mathcal{X}$, $\bar{\pi}$, and S belong to K.

The Lefschetz (1,1) Theorem

Now take $X \subset \mathbb{P}^N$ to be a smooth projective surface of degree d, and $\{X_s := X \cdot H_s\}_{s\in\mathbb{P}^1}$ a *Lefschetz pencil* of hyperplane sections: the singular fibres have exactly one (nodal) singularity. Let $\beta : \mathcal{X} \twoheadrightarrow X$ denote the blow-up at the base locus $B := \bigcap_{s\in\mathbb{P}^1} X_s$ of the pencil, and $\bar{\pi} : \mathcal{X} \to \mathbb{P}^1 =: S$ the resulting fibration. We are now in the situation considered above, with $\sigma(S)$ replaced by d sections $E_1 \amalg \cdots \amalg E_d = \beta^{-1}(B)$, and fibres of genus $g = \binom{d-1}{2}$; and with the added bonus that there is no torsion in any $H^1(\Delta_i^*, \mathbb{H}_{\mathbb{Z}})$, so that admissible $\Longrightarrow$ extendable.[4] Hence, given $Z \in Z^1(X)_{prim}$ $(\deg(Z \cdot X_{s_0}) = 0)$: $\beta^* Z$ is primitive, $\upsilon_Z := \nu_{\beta^* Z}$ is an ENF, and $[\upsilon_Z] = \beta^*[Z]$ under $\beta^* : Hg^1(X)_{prim} \hookrightarrow \frac{Hg^1(\mathcal{X})_{prim}}{\mathbb{Z}\langle[X_{s_0}]\rangle}$.

If on the other hand we start with a Hodge class $\xi \in Hg^1(X)_{prim}$, $\beta^*\xi$ is (by (5) + Poincaré existence) the class of a geometric ENF $\nu_{\mathfrak{Z}}$; and $[\mathfrak{Z}] \equiv [\nu_{\mathfrak{Z}}] \equiv \beta^*\xi \mod \mathbb{Z}\langle[X_{s_0}]\rangle \Longrightarrow$

[3]Here the $q_i(s)$ are as in Jacobi Inversion (Theorem A.2) (but varying with respect to a parameter). If at a generic point $\nu(\eta)$ is a special divisor then additional argument is needed.
[4]since removing the one node will not disconnect the singular fibre, we can always draw a chain Γ_{s_i} bounding on Z_{s_i} to avoid the node, and compute $AJ(Z_{s_i})$ directly.

$\xi \equiv \beta_* \beta^* \xi \equiv [\beta_* \mathfrak{Z} =: Z]$ in $\frac{Hg^1(X)}{\mathbb{Z}\langle [X_{s_0}] \rangle} \implies \xi = [Z']$ for some $Z' \in Z^1(X)_{(prim)}$. This is the gist of Lefschetz's original proof (1924) of

Theorem 3. *Let X be a (smooth projective algebraic) surface. The fundamental class map $CH^1(X) \stackrel{[\cdot]}{\to} Hg^1(X)$ is (integrally) surjective.*

This continues to hold in higher dimension, as can be seen from an inductive treatment with ENF's or (more easily) from the "modern" treatment of Theorem 3 using the exponential exact sheaf sequence

$$0 \to \mathbb{Z}_X \longrightarrow \mathcal{O}_X \stackrel{e^{2\pi i(\cdot)}}{\longrightarrow} \mathcal{O}_X^* \to 0.$$

One simply puts the induced long-exact sequence in the form

$$0 \to \frac{H^1(X,\mathcal{O})}{H^1(X,\mathbb{Z})} \to H^1(X,\mathcal{O}^*) \to \ker\left\{H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O})\right\} \to 0,$$

and interprets it as

$$(6) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & J^1(X) & \longrightarrow & \left\{\begin{array}{c}\text{holomorphic}\\ \text{line bundles}\end{array}\right\} & \longrightarrow & Hg^1(X) & \longrightarrow & 0 \\ & & & & \vdots\downarrow & \nearrow & & & \\ & & & & CH^1(X) & & & & \end{array}$$

where the dotted arrow takes the divisor of a meromorphic section of a given bundle. Existence of this meromorphic section is the hard part of this approach, and is proved in Appendix A (last page).

We note that for $\mathcal{X} \to \mathbb{P}^1$ a Lefschetz pencil of X, in (5) $J^1(\mathcal{X}/\mathbb{P}^1)_{fix} = J^1(X) := \frac{H^1(X,\mathbb{C})}{F^1H^1(X,\mathbb{C})+H^1(X,\mathbb{Z})}$, which is zero if X is a complete intersection; in that case ENF is finitely generated and $Hg^1(\mathcal{X})_{prim} \stackrel{\beta^*}{\hookrightarrow} ENF$.

Example 4. For X a cubic surface $\subset \mathbb{P}^3$, divisors with support on the 27 lines already surject onto $Hg^1(X) = H^2(X,\mathbb{Z}) \cong \mathbb{Z}^7$. Differences of these lines generate all primitive classes, hence all of $\mathrm{im}(\beta^*)(\cong \mathbb{Z}^6)$ in ENF$(\cong \mathbb{Z}^8)$. Note that $\mathcal{J}_e$ is essentially an elliptic surface and ENF comprises the (holomorphic) sections passing through the $\mathbb{C}^*$'s over points of Σ. There are no torsion sections.

Appendix: Picard-Lefschetz formula

Given a family of curves over a disk Δ, with singular fiber (singularities of ODP type only) at the center $0 \in \Delta$, the idea is that a topological 1-cycle meeting a vanishing cycle gets twisted by a copy of the vanishing cycle as t goes around 0. This is a completely local phenomenon.

All I'm going to do now is give the basic computation which exhibits this twist. A much more detailed, much more careful version is in Vol. II of Voisin.

We consider just a local degeneration $xy = t$, and only look at what happens inside a ball $|x|^2+|y|^2 \leq 1$. Here x and y denote complex coordinates, and t is considered to vary over the unit disk with the singular fiber over $t = 0$. The "fibres" $X_s := \{xy = t\} \cap \{|x|^2 + |y|^2 \leq 1\}$ for $t \neq 0$ look like tubes which are pinched to a point at their center as $t \to 0$. Consider the "piece of 1-cycle"

$$(x(s), y(s)) := \left(\epsilon^s(1+\epsilon^2)^{\frac{1}{2}-s},\ \epsilon^{1-s}(1+\epsilon^2)^{s-\frac{1}{2}}\right), \quad s \in [0,1]$$

in X_ϵ, which runs *along* the tube, from end to end. Now we let $t(\theta) = \epsilon e^{2\pi i\theta}$ go around 0 while holding the endpoints of this segment (on the sphere $|x|^2 + |y|^2 = 1$) as close as possible to fixed, we want to see that the segment gets twisted. For our endpoints, we need $(x(\theta,0), y(\theta,0)) = \left(\sqrt{1+\epsilon^2}, \frac{\epsilon e^{2\pi i\theta}}{\sqrt{1+\epsilon^2}}\right)$ and $(x(\theta,1), y(\theta,1)) = \left(\frac{\epsilon e^{2\pi i\theta}}{\sqrt{1+\epsilon^2}}, \sqrt{1+\epsilon^2}\right)$ since these are the ones that limit to $(1,0)$ and $(0,1)$ as $\epsilon \to 0$. Now fixing ϵ, a short computation gives us

$$(x(\theta,s), y(\theta,s) = \left(\epsilon^2(1+\epsilon^2)^{\frac{1}{2}-s}e^{2\pi i\theta s},\ \epsilon^{1-s}(1+\epsilon^2)^{s-\frac{1}{2}}e^{2\pi i\theta s}\right), \quad s \in [0,1]$$

on $X_{t(\theta)}$, which you will notice is not the same for $\theta = 0$ and $\theta = 1$ (even though t is). Voila: the twist.