B. NORMAL FUNCTIONS AND LEFSCHETZ (1.1)

We now wish to vary the Abel-Jacobi map in families, over a smooth projective curve (i.e. compact Riemann surface) $S$. Let $\mathcal{X}$ be a smooth projective surface, and $\pi : \mathcal{X} \to S$ a (projective) morphism which is

(a) smooth off a finite set $\Sigma = \{s_1, \ldots, s_e\} \subset S$, and

(b) locally of the form $(x_1, x_2) \mapsto x_1x_2$ at singularities (of $\pi$).

Write $X_s := \pi^{-1}(s)$ ($s \in S$) for the fibres. The singular fibres $X_{s_i}$ ($i = 1, \ldots, e$) then have only nodal, or "ODP" (ordinary double point) singularities

and writing $\mathcal{X}^* = S^* := S \setminus \Sigma$. Fixing a general $s_0 \in S^*$, the local monodromies $T_{s_i}$ $\in Aut(H^1(X_{s_0}, \mathbb{Z})) =: \Pi Z_{s_0}$ of the local system $\Pi Z := R^1\pi_*\mathcal{O}_{\mathcal{X}}$ about each $s_i$ are then computed by the Picard-Lefschetz formula

$$ (T_{s_i} - I) \gamma = \sum_j (\gamma \cdot \delta_j) \delta_j. $$

Here $\{\delta_j\}$ are the Poincaré duals$^1$ of the (quite possibly non-distinct) vanishing cycle classes $\in \ker \{H_1(X_{s_0}, \mathbb{Z}) \to H_1(X_{s_i}, \mathbb{Z})\}$ associated to each node on $X_{s_i}$, and "\cdot" the intersection form. A brief sketch of proof is provided in the appendix to this section.

Exercise. Show that $(T_{s_i} - I)^2 = 0$.

For a family of elliptic curves, (1) is the so-called Dehn twist:

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$^1$ i.e. the images under $H_1(X_{s_0}, \mathbb{Z}) \xrightarrow{\pi_*} H^1(X_{s_0}, \mathbb{Z})$
\[ T(\alpha) = \alpha \]
\[ T(\beta) = \beta + \alpha \]

(For the reader new to such pictures, the two crossing segments in the previous "local real" picture become the two touching "thimbles", i.e. a small neighborhood of the singularity in \( E_0 \), in this diagram.)

For a family with singular fibers, the notion of relative differentials has to change. The relative canonical (or "dualizing") sheaf \( \omega_{X/S} := \Omega_X^2 \otimes \pi_* \theta_{S}^1 \cong \Omega_{X/S}^1 (\log \pi^{-1}(\Sigma)) \) is a sheaf on \( X \) which restricts to \( \Omega_{X_s}^1 \) on smooth fibers. Its restriction to singular fibers, denoted \( \omega_{X_s} \), identifies away from the ODP with the sheaf of holomorphic 1-forms; at the ODP, its sections are holomorphic 1-forms with log poles whose residues cancel.

**Abstract normal functions**

Now, in our setting, the bundle of Jacobians \( J := \bigcup_{s \in S} J^1(X_s) \) is a complex (algebraic) manifold. It admits a partial compactification to a fiber space of complex abelian Lie groups, by defining \( J^1(X_s) := \frac{H^1(\omega_{X_s})}{\text{im}(H^1(X_s, \mathcal{L}))} \) and \( \mathcal{J}_e := \bigcup_{s \in S} J^1(X_s) \). (How to topologize this is discussed for example in Clemens (1983).) The same notation will denote their sheaves of sections,

\[
0 \to \mathbb{H}_Z \to \mathcal{F}^\vee \to \mathcal{J} \to 0 \quad \text{(on } S^*)
\]

\[
0 \to \mathbb{H}_{Z,e} \to (\mathcal{F}_e)^\vee \to \mathcal{J}_e \to 0 \quad \text{(on } S)
\]

with \( \mathcal{F} := \pi_* \omega_{X/S}, \mathcal{F}_e := \pi_* \omega_{X/S}, \mathbb{H}_Z = R^1 \pi_* \mathbb{Z}, \mathbb{H}_{Z,e} = R^1 \pi_* \mathbb{Z} \).

Note that the stalk of \( \mathbb{H}_{Z,e} \) at a "singular" point \( s_i \) is the part of cohomology invariant under monodromy, i.e. \( \ker(T_{s_i} - I) \subset H^1(X_{s_i}, \mathbb{Z}) \).

**Definition 1.** A normal function (NF) is a holomorphic section (over \( S^* \)) of \( \mathcal{J} \). An extended (or Poincaré) normal function (ENF) is a holomorphic section (over \( S \)) of \( \mathcal{J}_e \). A NF is extendable if it lies in \( \text{im}\{ H^0(S, \mathcal{J}_e) \to H^0(S^*, \mathcal{J}) \} \).
Next consider the long-exact cohomology sequence (sections over $S^*$)

\[(4) \quad 0 \to H^0(\mathbb{H}_\mathbb{Z}) \to H^0(\mathcal{F}^\vee) \to H^0(\mathcal{J}) \to H^1(\mathbb{H}_\mathbb{Z}) \to H^1(\mathcal{F}^\vee);\]

the topological invariant of a normal function $\nu \in H^0(\mathcal{J})$ is its image $[\nu] \in H^1(S^*, \mathbb{H}_\mathbb{Z})$.

**Exercise.** Show that the restriction of $[\nu]$ to $H^1(\Delta^*_s, \mathbb{H}_\mathbb{Z})$ ($\Delta_s$ a punctured disk about $s_i$) computes the local monodromy $(T_{s_i} - I)\hat{\nu}$ (where $\hat{\nu}$ is a multivalued local lift of $\nu$ to $\mathcal{F}^\vee$), modulo the monodromy of topological cycles.

We say that $\nu$ is locally liftable if all these restrictions vanish, i.e. if $(T_{s_i} - I)\hat{\nu} \in \text{im}\{(T_{s_i} - I)\mathbb{H}_{2,,s_0}\}$. Together with the assumption that as a (multivalued, singular) "section" of $\mathcal{F}^\vee$, $\hat{\nu}_c$ has at worst logarithmic divergence at $s_i$, this is equivalent to extendability.

**Remark.** The "at worst log divergence" aspect is the reason for the term "normal".

**INTERLUDE ON ALGEBRAIC CYCLES**

In our discussion of Abel's theorem, we used the notation $\text{Div}(M)$ for divisors on a compact Riemann surface $M$, with $\text{Div}^0(M)$ for divisors of degree zero, $P\text{Div}(M)$ for divisors of meromorphic functions, and $\text{Pic}^0(M)$ for their quotient. Replacing $M$ by a projective algebraic $n$-manifold $Y$, we introduce the more general notation:

- $\mathbb{Z}^p(Y)$ for codimension $p$ algebraic cycles, i.e. the free abelian group generated by subvarieties of codimension $p$; and
- $\mathbb{Z}^p(Y)_{\text{hom}}$ for algebraic cycles which, viewed as topological ones (of real dimension $2n - 2p$), are trivial in homology.

For now we shall just take $p = 1$ (and $n = \dim Y \leq 2$ for the most part), so we are really still dealing with divisors. A divisor $Z = \sum q_i V_i$ is said to be rationally equivalent to zero, $Z \equiv 0$, if it is the divisor of a rational (equivalently meromorphic) function; and I will just remind you that such functions on surfaces (unlike those on curves) have points where they are not well-defined: so the zeroes and poles can intersect. It is clear that $f^{-1}(\mathbb{R}_{>0})$ bounds on the divisor $(f) = (f)_0 - (f)_\infty$, so that topological cycle class is well-defined modulo rational equivalence. Defining

- $CH^1(Y) := \frac{\mathbb{Z}^1(Y)_{\text{rat}}}{\text{rat}}$ (Chow group),

we therefore have a map $[\cdot] : CH^1(Y) \to H_{2n-2}(Y, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$, with kernel

- $CH^1(Y)_{\text{hom}} := \frac{\mathbb{Z}^1(Y)_{\text{hom}}}{\text{rat}}$.  

(Of course, this identifies with \( \text{Pic}^0(Y) \) as defined above when \( Y \) is a curve.) The most important observation is this: one can also view \([\cdot]\) as being defined by integration over \( Z \) and duality, \( CH^1(Y) \to \{H^{2n-2}(Y, \mathbb{C})\}^\vee \cong H^2(Y, \mathbb{C}) \). Since pulling back to the components \( V_i \) of \( Z \) kills \((n, n - 2)\) forms and \((n - 2, n)\) forms, but not \((n - 1, n - 1)\) forms, the image lies in \( H^{1,1}(Y, \mathbb{C}) \). Hence, the cycle class map is

\[
[\cdot] : CH^1(Y) \to H^2(Y, \mathbb{Z}) \cap H^{1,1}(Y, \mathbb{C}) \cong Hg^1(Y).
\]

**NORMAL FUNCTIONS OF GEOMETRIC ORIGIN**

Let \( 3 \in \mathbb{Z}^1(\mathcal{X}) \) be a divisor properly intersecting fibres of \( \pi \) (i.e. no component of \( 3 \) lies in a fibre) and avoiding its singularities, and which is *primitive* in the sense that each \( Z_s := 3 \cdot X_s \) (\( s \in S^* \)) is of degree 0. In fact, the intersection conditions can be done away with, by moving the divisor in a rational equivalence. The main point is that \( 3 \) is "fiberwise homologous to zero on smooth fibres", but not in general homologous to zero on \( \mathcal{X} \). When this is true, \( s \mapsto AJ(Z_s) \) defines a section \( \nu_3 \) of \( J \), and it can be shown that a multiple \( N\nu_3 = \nu_3 \) of \( \nu_3 \) is always extendable. One says that \( \nu_3 \) itself is *admissible*.

Now assume \( \pi \) has a section \( \sigma : S \to \mathcal{X} \) (also avoiding singularities) and consider the analogue of (4) for \( J_\sigma \)

\[
0 \to \frac{H^0(F_\sigma)}{H^0(\mathbb{H}_{Z, \sigma})} \to H^0(J_\sigma) \to \ker \{ H^1(\mathbb{H}_{Z, s}) \to H^1(F_\sigma) \} \to 0.
\]

By a deep result in Hodge theory (due to Griffiths, Deligne, and Schmid) called the Theorem of the Fixed Part, \( H^0(\mathbb{H}_{Z, s}) \) — regarded as a constant sub-local-system of \( \mathbb{H}_Z \) — underlies an actual constant sub-VHS \( \mathcal{H}_{\text{fix}} \subset \mathcal{H}_{\mathcal{X} / S} \), called the *fixed part*. The Jacobian of the fixed part \( J^1(X/S)_{\text{fix}} \to J^1(X_S) \) (\( \forall s \in S \)) gives a constant sub-bundle of \( J_\sigma \), and the left hand term of the above sequence is but constant sections of this.\(^2\)

Next, if \( H^2(\mathcal{X})_{\text{prim}} \) denotes the classes restricting to 0 on a general fibre \( X_{s_0} \) of \( \pi \), then \( V_\sigma := \frac{H^2(X, \mathbb{Z})_{\text{prim}}}{\mathbb{Z}(X_{s_0})} \cong H^1(\mathbb{H}_{Z, s}) \) while \( V^{0,2} = H^1(F_\sigma) \). So the right hand term becomes \( \ker \{ V_\sigma \to V_\sigma / F^1V_\sigma \} \cong V_\sigma \cap V^{1,1} \). The upshot is that with some work, the above short-exact sequence becomes

\[
0 \to J^1(X/S)_{\text{fix}} \longrightarrow \text{ENF} \longrightarrow \frac{H^0(\mathcal{X})_{\text{prim}}}{\mathbb{Z}(X_{s_0})} \to 0,
\]

where the primitive Hodge classes \( Hg^1(\mathcal{X})_{\text{prim}} \) are the \( Q \)-orthogonal complement of \( X_{s_0} \) in \( Hg^1(\mathcal{X}) \).

**Proposition 2.** Let \( \nu \) be an ENF.

(i) If \( [\nu] = 0 \) then \( \nu \) is a constant section of \( J_{\text{fix}} := \bigcup_{s \in S} J^1(X/S)_{\text{fix}} \subset J_\sigma \);

\(^2\)for curves, this fixed part business could be dealt with explicitly via curvature or degree arguments (which are beyond our scope here)
(ii) If \((\nu =) \nu_3\) is of geometric origin, then \([\nu_3] = [3] / [3] = \text{fundamental class}\);

(iii) \([\text{Poincaré Existence Theorem}]\) Every ENF is of geometric origin.

We note that (i) follows immediately from (iii). To see (iii), apply “Jacobi inversion with parameters” and \(q_{i}(s) = \sigma(s) (\forall i)\) over \(S^*\) (really, over the generic point of \(S\)), and then take Zariski closure.\(^3\) Finally, when \(\nu\) is geometric, the monodromies of a lift \(\nu\) (to \(\mathcal{F}_{\nu}^*\)) around each loop in \(S\) (which determine \([\nu]\)) are just the corresponding monodromies of a bounding 1-chain \(\Gamma_s (\partial \Gamma_s = Z_s)\)

which identify with the Leray \((1,1)\) component of \([3]\) in \(H^2(\mathcal{X})\); this gives the gist of (ii).

A normal function is said to be \textit{motivated over} \(K (K \subset \mathbb{C} \text{ a subfield})\) if it is of geometric origin as above, and if the coefficients of the defining equations of \(3, \mathcal{X}, \pi, \) and \(S\) belong to \(K\).

\textbf{The Lefschetz \((1,1)\) Theorem}

Now take \(X \subset \mathbb{P}^N\) to be a smooth projective surface of degree \(d\), and \(\{X_s := X \cdot H_s\}_{s \in \mathbb{P}^1}\) a \textit{Lefschetz pencil} of hyperplane sections: the singular fibres have exactly one (nodal) singularity. Let \(\beta : \mathcal{X} \to X\) denote the blow-up at the base locus \(B := \bigcap_{s \in \mathbb{P}^1} X_s\) of the pencil, and \(\pi : \mathcal{X} \to \mathbb{P}^1 =: S\) the resulting fibration. We are now in the situation considered above, with \(\sigma(S)\) replaced by \(d\) sections \(E_1, \ldots, E_d = \beta^{-1}(B)\), and fibres of genus \(g = (d-1)\); and with the added bonus that there is no torsion in any \(H^1(\Delta_s^*, \mathcal{E}_s)\), so that admissible \(\Rightarrow\) extendable.\(^4\) Hence, given \(Z \in Z^1(\mathcal{X})_{\text{prim}}\) \((\deg(Z \cdot X_{s_0}) = 0)\): \(\beta^*Z\) is primitive, \(\nu_Z := \nu_{\beta^*Z}\) is an ENF, and \([\nu_Z] = \beta^*[Z]\) under \(\beta^* : H^1(\mathcal{X})_{\text{prim}} \to \frac{H^1(\mathcal{X})_{\text{prim}}}{\mathcal{Z}([X_{s_0}])}\).

If on the other hand we start with a Hodge class \(\xi \in H^1(\mathcal{X})_{\text{prim}}\), \(\beta^*\xi\) is (by (5) + Poincaré existence) the class of a geometric ENF \(\nu_3\); and \([3] = [\nu_3] \equiv \beta^*\xi \mod \mathcal{Z}([X_{s_0}]) \Rightarrow\)

\(^3\)Here the \(q_i(s)\) are as in Jacobi Inversion (Theorem A.2) (but varying with respect to a parameter). If at a generic point \(\nu(\eta)\) is a special divisor then additional argument is needed.

\(^4\)since removing the one node will not disconnect the singular fibre, we can always draw a chain \(\Gamma_s\) bounding on \(Z_s\), to avoid the node, and compute \(AJ(Z_s)\) directly.
\[ \xi = \beta \cdot \beta^* \xi = [\beta, \beta^* : Z] \quad \text{in} \quad \frac{H^1(X)}{Z([X_{\text{prim}}])} \quad \implies \quad \xi = [Z'] \quad \text{for some} \quad Z' \in Z^1(X)_{\text{(prim)}}. \] This is the gist of Lefschetz's original proof (1924) of

**Theorem 3.** Let \( X \) be a (smooth projective algebraic) surface. The fundamental class map \( CH^1(X) \rightarrow H^1(X) \) is (integrally) surjective.

This continues to hold in higher dimension, as can be seen from an inductive treatment with ENF's or (more easily) from the "modern" treatment of Theorem 3 using the exponential exact sheaf sequence

\[ 0 \rightarrow \mathbb{Z}_X \rightarrow \mathcal{O}_X \xrightarrow{\exp(2\pi i \cdot \cdot)} \mathcal{O}_X^* \rightarrow 0. \]

One simply puts the induced long-exact sequence in the form

\[ 0 \rightarrow \frac{H^1(X, \mathcal{O})}{H^1(X, \mathbb{Z})} \rightarrow H^1(X, \mathcal{O}^*) \rightarrow \ker \{ H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}) \} \rightarrow 0, \]

and interprets it as

\[ (6) \quad 0 \rightarrow J^1(X) \rightarrow \left\{ \text{holomorphic line bundles} \right\} \rightarrow H^1(X) \rightarrow 0 \]

\[ \xrightarrow{CH^1(X)} \]

where the dotted arrow takes the divisor of a meromorphic section of a given bundle. Existence of this meromorphic section is the hard part of this approach, and is proved in Appendix A (last page).

We note that for \( \mathcal{X} \rightarrow \mathbb{P}^1 \) a Lefschetz pencil of \( X \), in (5) \( J^1(\mathcal{X}/\mathbb{P}^1)_{\text{fix}} = J^1(X) := \frac{H^1(X, \mathcal{O})}{\mathbb{P}^1 H^1(X, \mathbb{Z}) + H^1(X, \mathbb{Z})} \), which is zero if \( X \) is a complete intersection; in that case ENF is finitely generated and \( H^1(X)_{\text{prim}} \xrightarrow{\text{gen}} \text{ENF} \).

**Example 4.** For \( X \) a cubic surface \( \subset \mathbb{P}^3 \), divisors with support on the 27 lines already surject onto \( H^1(X) = H^2(X, \mathbb{Z}) \cong \mathbb{Z}^7 \). Differences of these lines generate all primitive classes, hence all of \( \text{im}(\beta^*) (\cong \mathbb{Z}^6) \) in ENF (\( \cong \mathbb{Z}^8 \)). Note that \( J_e \) is essentially an elliptic surface and ENF comprises the (holomorphic) sections passing through the \( \mathbb{C}^* \)'s over points of \( \Sigma \). There are no torsion sections.

**APPENDIX: PICARD-LEFSCHETZ FORMULA**

Given a family of curves over a disk \( \Delta \), with singular fiber (singularities of ODP type only) at the center \( 0 \in \Delta \), the idea is that a topological 1-cycle meeting a vanishing cycle gets twisted by a copy of the vanishing cycle as \( t \) goes around 0. This is a completely local phenomenon.
All I’m going to do now is give the basic computation which exhibits this twist. A much more detailed, much more careful version is in Vol. II of Voisin.

We consider just a local degeneration $xy = t$, and only look at what happens inside a ball $|x|^2 + |y|^2 \leq 1$. Here $x$ and $y$ denote complex coordinates, and $t$ is considered to vary over the unit disk with the singular fiber over $t = 0$. The “fibres” $X_t := \{xy = t\} \cap \{ |x|^2 + |y|^2 \leq 1\}$ for $t \neq 0$ look like tubes which are pinched to a point at their center as $t \to 0$. Consider the “piece of 1-cycle”

$$(x(s), y(s)) := \left( \epsilon^{s(1 + 2^{1/2})^{1/2}}, \epsilon^{1-s(1 + 2^{1/2})^{1/2}} \right), \quad s \in [0, 1]$$

in $X_t$, which runs along the tube, from end to end. Now we let $t(\theta) = \epsilon e^{2\pi i \theta}$ go around $0$ while holding the endpoints of this segment (on the sphere $|x|^2 + |y|^2 = 1$) as close as possible to fixed, we want to see that the segment gets twisted. For our endpoints, we need $(x(\theta, 0), y(\theta, 0)) = \left( \sqrt{1 + \epsilon^2}, \frac{\epsilon^{\frac{\pi i \theta}{\sqrt{1 + \epsilon^2}}} }{\sqrt{1 + \epsilon^2}} \right)$ and $(x(\theta, 1), y(\theta, 1)) = \left( \frac{\epsilon^{\frac{\pi i \theta}{\sqrt{1 + \epsilon^2}}}}{\sqrt{1 + \epsilon^2}}, \sqrt{1 + \epsilon^2} \right)$ since these are the ones that limit to $(1, 0)$ and $(0, 1)$ as $\epsilon \to 0$. Now fixing $\epsilon$, a short computation gives us

$$(x(\theta, s), y(\theta, s)) = \left( \epsilon^{s(1 + 2^{1/2})^{1/2}} e^{2\pi i \theta s}, \epsilon^{1-s(1 + 2^{1/2})^{1/2}} e^{2\pi i \theta s} \right), \quad s \in [0, 1]$$

on $X_{t(\theta)}$, which you will notice is not the same for $\theta = 0$ and $\theta = 1$ (even though $t$ is). Voila: the twist.