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C. THE HODGE CONJECTURE

Let's begin by reviewing the ground covered thus far: in the classical algebraic geometry of curves, Abel's theorem and Jacobi inversion articulate the relationship (involving rational integrals) between configurations of points with integer multiplicities, or zero-cycles, and an abelian variety known as the Jacobian of the curve: the latter algebraically parametrizes the cycles of degree 0 modulo the subgroup arising as divisors of meromorphic functions.

Given a family $\mathcal{X}$ of algebraic curves over a complete base curve S, with smooth fibers over S^* (S minus a finite point set Σ over which fibers have double point singularities), Poincaré defined *normal functions* as holomorphic sections of the corresponding family of Jacobians over S which behave "normally" (or "logarithmically") in some sense near the boundary. His main result, which says essentially that they parametrize 1-dimensional cycles on $\mathcal{X}$, was then used by Lefschetz (in the context where $\mathcal{X}$ is a pencil of hyperplane sections of a projective algebraic surface) to prove his famous $(1,1)$ theorem for algebraic surfaces.

Lefschetz's result later became the basis for the Hodge conjecture, which says that certain *topological-analytic* invariants of an *algebraic* variety must come from *algebraic* subvarieties:

Conjecture 1. *For a smooth projective complex algebraic variety X, with $Hg^m(X)_{\mathbb{Q}}$ the classes in $H^{2m}_{sing}(X^{an}_{\mathbb{C}},\mathbb{Q})$ of type (m,m), and $CH^m(X)$ the "Chow group" of codimension-m algebraic cycles modulo rational equivalence, the fundamental class map $CH^m(X)\otimes\mathbb{Q}\to Hg^m(X)_{\mathbb{Q}}$ is surjective.*

You can think of this as a "metaphor for transforming transcendental computations into algebraic ones". For example, if you're handed period matrices (with parameter) of families of algebraic varieties and asked to find some kind of morphism between the corresponding Hodge structures (i.e. a Hodge class in one variation of Hodge structure tensor the dual of the other), that might be too hard . . . *unless* you can find an algebraic correspondence between the two families (a cycle in their fiber product) inducing such a morphism.

The Chow group is defined below. Hodge's original formulation (c. 1950) was that the fundamental class map was *integrally* surjective; this was shown to be false by Atiyah and Hirzebruch.

In this section we shall describe the attempts to directly generalize Lefschetz's success to higher-codimension cycles which led to Griffiths's Abel-Jacobi map (from the codimension m cycle group of a variety X to its m^{th} "intermediate" Jacobian), horizontality and variations of mixed Hodge structure, and S. Zucker's "Theorem on Normal Functions". The upshot will be that the breakdown (beyond codimension 1) of the relationship between cycles and (intermediate) Jacobians, and the failure of the Jacobians to be algebraic, meant that the same game played in 1 parameter does not work outside very special cases.

Hodge structures: a quick review

As we know, a $\mathbb{Z}$-Hodge structure (HS) of weight m comprises a finitely generated abelian group $H_{\mathbb{Z}}$ together with a descending filtration $F^{\bullet}$ on $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $F^p H_{\mathbb{C}} \oplus \overline{F^{m-p+1} H_{\mathbb{C}}} = H_{\mathbb{C}}$, the *Hodge filtration*; we denote the lot by H. Examples include the m^{th} (singular/Betti + de Rham) cohomology groups of smooth projective varieties $/\mathbb{C}$, with $F^p H^m_{dR}(X, \mathbb{C})$ being that part of the de Rham cohomology represented by C^{∞} forms on X^{an} with *at least* p holomorphic differentials wedged together in each monomial term. (These are forms of *Hodge type* $(p, m-p) + (p+1, m-p-1) + \cdots$; note that $H^{p,m-p}_{\mathbb{C}} := F^p H_{\mathbb{C}} \cap \overline{F^{m-p} H_{\mathbb{C}}}$.) To accommodate H^m of non-smooth or incomplete varieties, the notion of a ($\mathbb{Z}$-)mixed Hodge structure (MHS) V is required: in addition to $F^{\bullet}$ on $V_{\mathbb{C}}$, introduce a decreasing *weight filtration* $W_{\bullet}$ on $V_{\mathbb{Q}}$ such that the $\left(Gr_i^W V_{\mathbb{Q}}, \left(Gr_i^W (V_{\mathbb{C}}, F^{\bullet})\right)\right)$ are weight i $\mathbb{Q}$-HS. Mixed Hodge structures have Hodge and Jacobian groups $Hg^p(V) := \ker\{V_{\mathbb{Z}} \oplus F^p W_{2p} V_{\mathbb{C}} \to V_{\mathbb{C}}\}$ (for for $V_{\mathbb{Z}}$ torsion-free becomes $V_{\mathbb{Z}} \cap F^p W_{2p} V_{\mathbb{C}}$) and $J^p(V) := \frac{W_{2p} V_{\mathbb{C}}}{F^p W_{2p} V_{\mathbb{C}} + W_{2p} V_{\mathbb{Q}} \cap V_{\mathbb{Z}}}$, with special cases $Hg^m(X) := Hg^m(H^{2m} X))$ and $J^m(X) := J^m(H^{2m-1}(X))$. Jacobians of HS yield complex tori, and subtori correspond bijectively to sub-HS.

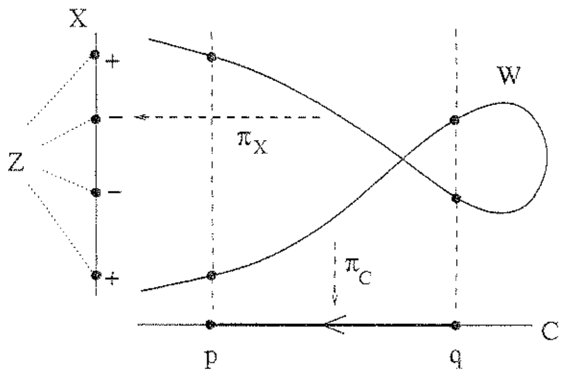
A *polarization* of a HS H is a morphism Q of HS (defined over $\mathbb{Z}$; complexification respects $F^{\bullet}$) from $H \times H$ to the trivial HS $\mathbb{Z}(-m)$ of weight $2m$ (and type (m, m)), such that viewed as a pairing Q is nondegenerate and satisfies a positivity constraint generalizing that in §1.1 (the *second Hodge-Riemann bilinear relation*). A consequence of this definition is that under Q, F^p is the annihilator of F^{m-p+1} (the *first Hodge-Riemann bilinear relation* in abstract form). If X is a smooth projective variety of dimension d, $[\Omega]$ the class of a hyperplane section, write (for $k \leq d$, say) $H^m(X, \mathbb{Q})_{prim} := \ker\{H^m(X, \mathbb{Q}) \overset{\cup \Omega^{d-k+1}}{\longrightarrow} H^{2d-m+2}(X, \mathbb{Q})\}$. This Hodge structure is then polarized by $Q(\xi, \eta) := (-1)^{\binom{m}{2}} \int_X \xi \wedge \eta \wedge \Omega^{d-k}$, $[\Omega]$ the class of a hyperplane section (obviously since this is a $\mathbb{Q}$-HS, the polarization is only defined $/\mathbb{Q}$).

Griffiths's AJ map

Let X be a smooth projective $(2m-1)$-fold; we shall consider some equivalence relations on algebraic cycles of codimension m on X. Writing (as in §V.B) $Z^m(X)$ for the free abelian group on irreducible (complex-)codimension m subvarieties of X, two cycles $Z_1, Z_2 \in Z^m(X)$ are homologically equivalent if their difference bounds a C^{∞} chain $\Gamma \in C^{top}_{2m-1}(X^{an}; \mathbb{Z})$ (of real dimension $2m-1$). Algebraic equivalence is generated by (the projection to X of) differences of the form $W \cdot (X \times \{p_1\}) - W \cdot (X \times \{p_2\})$ where C is an algebraic curve,

$W \in Z^m(X \times C)$, and $p_1, p_2 \in C(\mathbb{C})$ (or $C(K)$ if we are working over a subfield $K \subset \mathbb{C}$). Rational equivalence is obtained by taking C to be rational (i.e. $C \cong \mathbb{P}^1$), and for $m = 1$ is generated by divisors of meromorphic functions. We write $Z^m(X)_{rat}$ for cycles $\equiv_{rat} 0$, etc; note that $CH^m(X) := \frac{Z^m(X)}{Z^m(X)_{rat}} \supset CH^m(X)_{hom} := \frac{Z^m(X)_{hom}}{Z^m(X)_{rat}} \supset CH^m(X)_{alg} := \frac{Z^m(X)_{alg}}{Z^m(X)_{rat}}$ are proper inclusions in general.

Now let $W \subset X \times C$ be an irreducible subvariety of codimension m, with π_X, π_C the projections from a desingularization of W to X resp. C. If we put $Z_i := \pi_{X_*}\pi_C^*\{p_i\}$, then $Z_1 \equiv_{alg} Z_2 \implies Z_1 \equiv_{hom} Z_2$, which can be seen explicitly by setting $\Gamma := \pi_{X_*}\pi_C^*(\overrightarrow{q.p})$ (so that $Z_1 - Z_2 = \partial\Gamma$).



Let ω be a d-closed form of Hodge type $(j, 2m-j-1)$ on X, for j at least m. Consider $\int_\Gamma \omega = \int_p^q \kappa$, where $\kappa := \pi_{C_*}\pi_X^*\omega$ is a d-closed 1-current of type $(j-m+1, m-j)$ as integration along the $(m-1)$-dimensional fibres of π_C eats up $(m-1, m-1)$. So $\kappa = 0$ unless $j = m$, and by a standard regularity theorem in that case κ is holomorphic. In particular, if C is rational, we have $\int_\Gamma \omega = 0$. This is essentially the reasoning behind the following result:

Proposition 2. *The Abel-Jacobi map*

$$(1) \qquad CH^m(X)_{hom} \xrightarrow{AJ} \frac{\left(F^m H^{2m-1}(X,\mathbb{C})\right)^\vee}{\int_{H_{2m-1}(X,\mathbb{Z})}(\cdot)} \cong J^m(X)$$

induced by $Z = \partial\Gamma \mapsto \int_\Gamma(\cdot)$*, is well-defined and restricts to*

$$(2) \qquad CH^m(X)_{alg} \xrightarrow{AJ_{alg}} \frac{F^m H^{2m-1}_{hdg}(X,\mathbb{C})}{\int_{H_{2m-1}(X,\mathbb{Z})}(\cdot)} \cong J^m(H^{2m-1}_{hdg}(X)) =: J^m_h(X)$$

where $H^{2m-1}_{hdg}(X)$ *is the largest sub-HS of* $H^{2m-1}(X)$ *contained (after* $\otimes\mathbb{C}$*) in* $H^{m-1,m}(X,\mathbb{C}) \oplus H^{m,m-1}(X,\mathbb{C})$*. While* $J^m(X)$ *is in general only a complex torus,* $J^m_h(X)$ *is an abelian variety and defined (along with the point* $AJ_{alg}(Z)$*) over the field of definition of* X*.*

Remark 3. (i) As for Jacobians of curves, to see that $J^m_h(X)$ is an abelian variety, one uses the Kodaira embedding theorem: by the Hodge-Riemann bilinear relations, the polarization of $H^{2m-1}(X)$ induces a Kähler metric $h(u,v) = -iQ(u,\bar{v})$ on $J_h^m(X)$ with rational Kähler class.

(ii) The mapping (1) is neither surjective nor injective in general, and (2) is not injective in general; however, (2) is conjectured to be surjective, and regardless of this $J^m_{alg}(X) := \mathrm{im}(AJ_{alg}) \subseteq J_h^m(X)$ is in fact a sub-abelian-variety. This is because $W \subset X \times C$ induces a map on cohomology $H^1(C) \to H^{2m-1}(X)$ which descends to a map of Jacobians $J(C) \to J^m(X)$; taken over all C and W, the sum of images of such maps is $J^m_{alg}(X)$.

(iii) A point in $J^m(X)$ is naturally the invariant of an extension of MHS

$$0 \to (H =)H^{2m-1}(X, \mathbb{Z}(m)) \to E \to \mathbb{Z}(0) \to 0$$

(where the "twist" $\mathbb{Z}(m)$ reduces weight by $2m$, to (-1)). The invariant is evaluated by taking two lifts $\nu_F \in F^0 W_0 E_{\mathbb{C}}$, $\nu_{\mathbb{Z}} \in W_0 E_{\mathbb{Z}}$ of $1 \in \mathbb{Z}(0)$, so that $\nu_F - \nu_{\mathbb{Z}} \in W_0 H_{\mathbb{C}}$ is well-defined modulo the span of $F^0 W_0 H_{\mathbb{C}}$ and $W_0 H_{\mathbb{Z}}$ hence is in $J^0(H) \cong J^m(X)$. The resulting isomorphism $J^m(X) \cong Ext^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2m-1}(X, \mathbb{Z}(m)))$ is part of Carlson's extension-class approach to AJ maps described in §IV.C.

Horizontality

Generalizing the setting of §V.B, let $\mathcal{X}$ be a smooth projective $2m$-fold fibred over a curve S with singular fibres $\{X_{s_i}\}$ each of either

(i) *NCD*(=normal crossing divisor) *type*: locally $(x_1, \ldots, x_{2m}) \stackrel{\pi}{\longmapsto} \prod_{j=1}^k x_j$; or

(ii) *ODP*(=ordinary double point) *type*: locally $(\underline{x}) \longmapsto \sum_{j=1}^{2m} x_j^2$.

An immediate consequence is that all $T_{s_i} \in Aut\,(H^{2m-1}(X_{s_0}, \mathbb{Z}))$ are *unipotent*: $(T_{s_i} - I)^n = 0$ for $n \geq 2m$ in case (i) or $n \geq 2$ in case (ii). (If all fibers are of NCD type, then we say the family $\{X_s\}$ of $(2m-1)$-folds is *semistable.*)

The Jacobian bundle of interest is $\mathcal{J} := \bigcup_{s \in S^*} J^m(X_s)\,(\supset \mathcal{J}_{alg})$. Writing

$$\left\{\mathcal{F}^{(m)} := \mathbb{R}^{2m-1}\pi_* \Omega^{\bullet \geq m}_{\mathcal{X}^*/S^*}\right\} \subset \left\{\mathcal{H} := \mathbb{R}^{2m-1}\pi_*\Omega^{\bullet}_{\mathcal{X}^*/S^*}\right\} \supset \left\{\mathbb{H}_{\mathbb{Z}} := R^{2m-1}\pi_*\mathbb{Z}_{\mathcal{X}^*}\right\},$$

and noting $\mathcal{F}^{\vee} \cong \frac{\mathcal{H}}{\mathcal{F}}$ via $Q : \mathcal{H}^{2m-1} \times \mathcal{H}^{2m-1} \to \mathcal{O}_{S^*}$, the sequences (V.B.2) and (V.B.4), as well as the definitions of NF and topological invariant $[\cdot]$, all carry over. A normal function of geometric origin, likewise, comes from $\mathfrak{Z} \in Z^m(\mathcal{X})_{prim}$ with $Z_{s_0} := \mathfrak{Z} \cdot X_{s_0} \equiv_{hom} 0$ (on X_{s_0}), but now has an additional feature known as *horizontality*, which we now explain.

Working locally over an analytic ball $(s_0 \in)\, U \subset S^*$, let $\tilde{\omega} \in \Gamma(\mathcal{X}_U, F^{m+1}A^{2m-1}_{\mathcal{X}})$ be a "lift" of $\omega(s) \in \Gamma(U, \mathcal{F}^{m+1})$, and $\Gamma_s \in C^{top}_{2m-1}(X_s; \mathbb{Z})$ be a continuous family of chains with $\partial\Gamma_s = Z_s$.

Let P^ϵ be a path from s_0 to $s_0+\epsilon$; then $\hat{\Gamma}^\epsilon := \bigcup_{s\in P^\epsilon}\Gamma_s$ has boundary $\Gamma_{s_0+\epsilon}-\Gamma_{s_0}+\bigcup_{s\in P^\epsilon} Z_s$, and

$$\begin{aligned}(3)\qquad \left(\tfrac{\partial}{\partial s}\int_{\Gamma_s}\omega(s)\right)_{s=s_0} &= \lim_{\epsilon\to 0}\tfrac{1}{\epsilon}\int_{\Gamma_{s_0+\epsilon}-\Gamma_{s_0}}\tilde{\omega} = \\ &\lim_{\epsilon\to 0}\tfrac{1}{\epsilon}\left(\int_{\partial\hat{\Gamma}^\epsilon}\tilde{\omega}-\int_{s_0}^{s_0+\epsilon}\int_{Z_s}\omega(s)\right) = \\ &\int_{\Gamma_{s_0}}\left\langle\widetilde{d/ds}, d\tilde{\omega}\right\rangle - \int_{Z_{s_0}}\omega(s_0)\end{aligned}$$

where $\pi_*\widetilde{d/ds} = d/ds$ (with $\widetilde{d/ds}$ tangent to $\hat{\Gamma}^\epsilon$, $\hat{Z}^\epsilon$).

The Gauss-Manin connection $\nabla : \mathcal{H}\to\mathcal{H}\otimes\Omega^1_{S^*}$ differentiates the periods of cohomology classes (against topological cycles) in families, satisfies Griffiths transversality $\nabla(\mathcal{F}^m)\subset\mathcal{F}^{m-1}\otimes\Omega^1_{S^*}$, and is computed by $\nabla\omega = [\langle\widetilde{d/ds}, d\tilde{\omega}\rangle]\otimes dt$. Moreover, the pullback of any form of type F^m to Z_{s_0} (which is of dimension $m-1$) is zero, so that $\int_{Z_{s_0}}\omega(s_0)=0$ and $\int_{\Gamma_{s_0}}\nabla\omega$ is well-defined. If $\tilde{\Gamma}\in\Gamma(U,\mathcal{H})$ is any lift of $AJ(\Gamma_s)\in\Gamma(U,\mathcal{J})$, we therefore have

$$\begin{aligned}Q\left(\nabla_{d/ds}\tilde{\Gamma},\omega\right) &= \frac{d}{ds}Q(\tilde{\Gamma},\omega)-Q(\tilde{\Gamma},\nabla_{d/ds}\omega)\\ &= \frac{d}{ds}\int_{\Gamma_s}\omega-\int_{\Gamma_s}\nabla_{d/ds}\omega\end{aligned}$$

which is zero by (3) and the remarks just made. We have shown that $\nabla_{d/dt}\tilde{\Gamma}$ kills $\mathcal{F}^{m+1}$, and so $\nabla_{d/dt}\tilde{\Gamma}$ is a local section of $\mathcal{F}^{m-1}$.

Definition 4. A NF $\nu\in H^0(S^*,\mathcal{J})$ is *horizontal* if for any local lift $\tilde{\nu}\in\Gamma(U,\mathcal{H})$, $\nabla\tilde{\nu}\in\Gamma(U,\mathcal{F}^{m-1}\otimes\Omega^1_U)$. Equivalently, if we set $\mathcal{H}_{hor} := \ker\left(\mathcal{H}\overset{\nabla}{\to}\frac{\mathcal{H}}{\mathcal{F}^{m-1}}\otimes\Omega^1_{S^*}\right)\supset\mathcal{F}^m =: \mathcal{F}$, $(\mathcal{F}^\vee)_{hor} := \frac{\mathcal{H}_{hor}}{\mathcal{F}}$, and $\mathcal{J}_{hor} := \frac{(\mathcal{F}^\vee)_{hor}}{\mathbb{H}_\mathbb{Z}}$, then $\mathrm{NF}_{hor} := H^0(S,\mathcal{J}_{hor})$.

Much as an AJ image was encoded in a MHS in Remark 3(ii), we may encode horizontal normal functions in terms of variations of MHS. A VMHS $\mathcal{V}/S^*$ consists of a $\mathbb{Z}$-local system $\mathbb{V}$ with an increasing filtration of $\mathbb{V}_\mathbb{Q} := \mathbb{V}_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{Q}$ by sub- local systems $W_i\mathbb{V}_\mathbb{Q}$, a decreasing filtration of $\mathcal{V}_{(\mathcal{O})} := \mathbb{V}_\mathbb{Q}\otimes_\mathbb{Q}\mathcal{O}_{S^*}$ by holomorphic vector bundles $\mathcal{F}^j (=\mathcal{F}^j\mathcal{V})$, and a connection $\nabla : \mathcal{V}\to\mathcal{V}\otimes\Omega^1_{S^*}$ such that (i) $\nabla(\mathbb{V})=0$, (ii) the fibres $(\mathbb{V}_s, W_\bullet, V_s, F^\bullet_s)$ yield $\mathbb{Z}$-MHS, and (iii) [transversality] $\nabla(\mathcal{F}^j)\subset\mathcal{F}^{j-1}\otimes\Omega^1_{S^*}$. (Of course, a VHS is just a VMHS with one nontrivial $Gr^W_i\mathbb{V}_\mathbb{Q}$, and $((\mathbb{H}_\mathbb{Z},\mathcal{H},\mathcal{F}^\bullet),\nabla)$ in the geometric setting above gives one.) A horizontal normal function corresponds to an extension of VMHS

$$(4)\qquad 0\to\overbrace{\mathcal{H}(m)}^{\substack{\text{wt. -1}\\ \text{VHS}}}\to\mathcal{E}\to\mathbb{Z}(0)_{S^*}\to 0$$

"varying" the setup of Remark 3(iii), with the transversality of the lift of $\nu_F(s)$ (together with flatness of $\nu_\mathbb{Z}(s)$) reflecting horizontality.

An important result on VHS over a smooth quasi-projective base (mentioned already in §V.B for families of curves) is that the global sections $H^0(S^*, \mathbb{V})$ (resp. $H^0(S^*, \mathbb{V}_{\mathbb{R}})$, $H^0(S^*, \mathbb{V}_{\mathbb{C}})$) span the $\mathbb{Q}$-local system (resp. its $\otimes\mathbb{R}$, $\otimes\mathbb{C}$) of a (necessarily constant) sub-VMHS $\subset \mathcal{V}$, called the *fixed part* $\mathcal{V}_{fix}$ (with constant Jacobian bundle $\mathcal{J}_{fix}$).

Infinitesimal invariant

Given $\nu \in \mathrm{NF}_{hor}$, the "$\nabla\tilde{\nu}$" for various local liftings patch together after going modulo $\nabla\mathcal{F}^m \subset \mathcal{F}^{m-1} \otimes \Omega^1_{S^*}$. If $\nabla\tilde{\nu} = \nabla f$ for $f \in \Gamma(U, \mathcal{F}^m)$, then the alternate lift $\tilde{\nu} - f$ is flat, i.e. equals $\sum_i c_i \gamma_i$ where $\{\gamma_i\} \subset \Gamma(U, \mathbb{V}_{\mathbb{Z}})$ is a basis and the c_i are complex constants. Since the composition ($s \in S^*$) $H^{2m-1}(X_s, \mathbb{R}) \hookrightarrow H^{2m-1}(X_s, \mathbb{C}) \twoheadrightarrow \frac{H^{2m-1}(X_s, \mathbb{C})}{F^m}$ is an isomorphism, we may take the $c_i \in \mathbb{R}$, and then they are unique in $\mathbb{R}/\mathbb{Z}$. This implies that $[\nu]$ lies in the torsion group $\ker\left(H^1(\mathbb{H}_{\mathbb{Z}}) \to H^1(\mathbb{H}_{\mathbb{R}})\right)$, so that a multiple $N\nu$ lifts to $H^0(S^*, \mathbb{H}_{\mathbb{R}}) \subset \mathcal{H}_{fix}$. This motivates the definition of an infinitesimal invariant

$$\delta\nu \in \mathbb{H}^1\left(S^*, \mathcal{F}^m \xrightarrow{\nabla} \mathcal{F}^{m-1} \otimes \Omega^1_{S^*}\right) \underset{\text{affine}}{\overset{\text{if } S^*}{=\!=\!=}} H^0\left(S, \frac{\mathcal{F}^{m-1} \otimes \Omega^1}{\mathcal{F}^m}\right) \tag{5}$$

as the image of $\nu \in H^0\left(S^*, \frac{\mathcal{H}_{hor}}{\mathcal{F}}\right)$ under the connecting homomorphism induced by

$$0 \to \mathrm{Cone}\left(\mathcal{F}^m \xrightarrow{\nabla} \mathcal{F}^{m-1} \otimes \Omega^1\right)[-1] \to \mathrm{Cone}\left(\mathcal{H} \xrightarrow{\nabla} \mathcal{H} \otimes \Omega^1\right)[-1] \to \frac{\mathcal{H}_{hor}}{\mathcal{F}} \to 0. \tag{6}$$

Proposition 5. *If $\delta\nu = 0$, then up to torsion, $[\nu] = 0$ and ν is a (constant) section of $\mathcal{J}_{fix}$.*

An interesting application to the differential equations satisfied by normal functions is essentially due to Manin. For simplicity let $S = \mathbb{P}^1$, and suppose $\mathcal{H}$ is generated by $\omega \in H^0(S^*, \mathcal{F}^{2m-1})$ as a D-module, with monic *Picard-Fuchs operator* $F(\nabla_{\delta_s := s\frac{d}{ds}}) \in \mathbb{C}(\mathbb{P}^1)^*[\nabla_{\delta_s}]$ killing ω. Then its periods satisfy the homogeneous P-F equation $F(\delta_s)\int_{\gamma_i} \omega = 0$, and one can look at the multivalued holomorphic function $Q(\tilde{\nu}, \omega)$ (where Q is the polarization, and $\tilde{\nu}$ is a multivalued lift of ν to $\mathcal{H}_{hor}/\mathcal{F}$), which in the geometric case is just $\int_{\Gamma_s} \omega(s)$. The resulting equation

$$(2\pi i)^m F(\delta_s) Q(\tilde{\nu}, \omega) =: G(s) \tag{7}$$

is called the *inhomogeneous Picard-Fuchs equation* of ν.

Proposition 6. *(i)* [del Angel + Müller-Stach] *$G \in \mathbb{C}(\mathbb{P}^1)^*$ is a rational function holomorphic on S^*.*

(ii) [Manin, Griffiths] $G \equiv 0 \iff \delta\nu = 0$.

Returning to the setting described above (cf. "HORIZONTALITY"), since we are assuming unipotent local monodromies, there are *canonical extensions* $\mathcal{H}_e, \mathcal{F}^\bullet_e$ of $\mathcal{H}, \mathcal{F}^\bullet$ across the

s_i as holomorphic vector bundles resp. subbundles (as in §IV.D); e.g. if all fibres are of NCD type then $\mathcal{F}^p_e \cong \mathbb{R}^{2m-1}\bar{\pi}_* \Omega^{\bullet \geq p}_{\mathcal{X}/S}(\log(\mathcal{X}\backslash\mathcal{X}^*))$. Writing[1] $\mathbb{H}_{\mathbb{Z},e} := R^{2m-1}\bar{\pi}_*\mathbb{Z}_{\mathcal{X}}$ and $\mathcal{H}_{e,hor} := \ker\left\{\mathcal{H}_e \overset{\nabla}{\to} \frac{\mathcal{H}_e}{\mathcal{F}^{m-1}_e} \otimes \Omega^1_S(\log\Sigma)\right\}$, we have short exact sequences

$$(8) \qquad 0 \to \mathbb{H}_{\mathbb{Z},e} \to \frac{\mathcal{H}_{e(,hor)}}{\mathcal{F}^m_e} \to \mathcal{J}_{e(,hor)} \to 0$$

and set $\mathrm{ENF}_{(hor)} := H^0(S, \mathcal{J}_{e(,hor)})$.

Theorem 7. *(i)* $\mathfrak{Z} \in Z^m(\mathcal{X})_{prim} \implies N\nu_{\mathfrak{Z}} \in ENF_{hor}$ *for some* $N \in \mathbb{N}$*; and*

(ii) $\nu \in ENF_{hor}$ *with* $[\nu]$ *torsion* $\implies \delta\nu = 0$.

Remark 8. (ii) derives from a recent result of M. Saito. For $\nu \in \mathrm{ENF}_{hor}$, $\delta\nu$ lies in the subspace $\mathbb{H}^1\left(S, \mathcal{F}^m \overset{\nabla}{\to} \mathcal{F}^{m-1}_e \otimes \Omega^1_S(\log\Sigma)\right)$, the restriction of $\mathbb{H}^1\left(S^*, \mathcal{F}^m \overset{\nabla}{\to} \mathcal{F}^{m-1} \otimes \Omega^1_{S^*}\right) \to H^1(S^*, \mathbb{H}_{\mathbb{C}})$ to which is injective.

No dice

Putting together Theorem 7(ii) and Proposition 6(ii), we see that a horizontal ENF with trivial topological invariant lies in $H^0(S, \mathcal{J}_{fix}) =: J^m(\mathcal{X}/S)_{fix}$ (constant sections). In fact, the long-exact sequence associated to (8) yields

$$0 \to J^m(\mathcal{X}/S)_{fix} \to \mathrm{ENF}_{hor} \overset{[\cdot]}{\to} \frac{Hg^m(\mathcal{X})_{prim}}{\mathrm{im}\,\{Hg^{m-1}(X_{s_0})\}} \to 0,$$

with $[\nu_{\mathfrak{Z}}] = \overline{[\mathfrak{Z}]}$ (if $\nu_{\mathfrak{Z}} \in \mathrm{ENF}$) as before. If $\mathcal{X} \overset{\bar{\pi}}{\to} \mathbb{P}^1 = S$ is a Lefschetz pencil on a $2m$-fold X, this becomes

$$(9) \qquad \begin{array}{ccccc}
J^m(X) & \hookrightarrow & \mathrm{ENF}_{hor} & \underset{(*)}{\overset{[\cdot]}{\twoheadrightarrow}} & Hg^m(X)_{prim} \oplus \ker\left\{\begin{array}{c} Hg^{m-1}(B) \\ \to Hg^m(X) \end{array}\right\} \\
\uparrow AJ & & \uparrow \nu_{(\cdot)} \ \ \nwarrow v_{(\cdot)} & & \uparrow (\mathrm{id.},0) \\
 & & CH^m(\mathcal{X})_{prim} & & \\
 & & \uparrow \beta^* & & \\
\ker([\cdot]) & \hookrightarrow & CH^m(X)_{prim} & \underset{(**)}{\overset{[\cdot]}{\longrightarrow}} & Hg^m(X)_{prim}
\end{array}$$

where surjectivity of (*) is Zucker's "Theorem on Normal Functions" (which followed on work of Griffiths and Bloch establishing the surjectivity for *sufficiently ample* Lefschetz pencils). What we are after ($\otimes\mathbb{Q}$) is surjectivity of the fundamental class map (**). This would clearly follow from surjectivity of $\nu_{(\cdot)}$, i.e. a Poincaré existence theorem, as in §V.B. By Remark 3(ii) this cannot work in most cases; however we do have

[1]Warning: while $\mathcal{H}_e$ has no jumps in rank, the stalk of $\mathbb{H}_{\mathbb{Z},e}$ at $s_i \in \Sigma$ is of strictly smaller rank than at $s \in S^*$.

Theorem 9. *The Hodge Conjecture HC(m,m) is true for X if $J^m(X_{s_0}) = J^m(X_{s_0})_{alg}$ for a general member of the pencil.*

Example 10. As $J^2 = J^2_{alg}$ is true for cubic threefolds by the work of Griffiths and Clemens, HC$(2,2)$ holds for cubic fourfolds in $\mathbb{P}^5$.

TAKE TWO

The Lefschetz paradigm, of taking a 1-parameter family of slices of a primitive Hodge class to get a normal function and constructing a cycle by Jacobi inversion, appears to have led us (for the most part) to a dead end in higher codimension. A beautiful new idea of Griffiths and Green, replaces the Lefschetz pencil by a complete linear system (of higher degree sections of X) so that $\dim(S) \gg 1$, and proposes to recover algebraic cycles *dual* to the given Hodge class from features of the (admissible) normal function in codimension ≥ 2 on S. I'll give only the briefest account of this here; probably the easiest place to read about it in more detail is in my paper with Greg Pearlstein.

The story begins with the Deligne cycle-class map, which replaces the fundamental and AJ classes of an algebraic cycle by one object. Writing $\mathbb{Z}(m) := (2\pi i)^m\mathbb{Z}$, define the Deligne cohomology of X (smooth projective of any dimension) by $H^*_{\mathcal{D}}(X^{an}, \mathbb{Z}(m)) :=$

$$H^*\left(\mathrm{Cone}\left\{C^\bullet_{top}(X^{an}; \mathbb{Z}(m)) \oplus F^m\mathcal{D}^\bullet(X^{an}) \to D^\bullet(X^{an})\right\}[-1]\right),$$

and $c_{\mathcal{D}} : CH^m(X) \to H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))$ by $Z \mapsto (2\pi i)^m(Z_{top}, \delta_Z, 0)$. One easily derives the exact sequence

$$0 \to J^m(X) \to H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m)) \to Hg^m(X) \to 0,$$

which invites comparison to the top row of (9).

Let X be a smooth projective variety of dimension $2m$ and $L \to X$ be a very ample line bundle.[2] Let $S = |L|$ and

$$\mathcal{X} = \{(x,s) \in X \times S \mid s(x) = 0\} \tag{10}$$

be the incidence variety associated to the pair (X, L). Let $\bar{\pi} : \mathcal{X} \to S$ denote projection on the second factor, and let $\hat{X} \subset S$ denote the dual variety of X (i.e. the points $s \in S$ such that $X_s = \bar{\pi}^{-1}(s)$ is singular). Let $\mathcal{H}(m)$ be the variation of Hodge structure of weight -1 over $S^* = S\backslash\hat{X}$ attached to the local system $\mathbb{H}(m) := R^{2m-1}\pi^{sm}_*\mathbb{Z}(m)$.

For a pair (X, L) as above, an integral Hodge class ζ of type (m,m) on X is primitive with respect to π^{sm} if and only if it is primitive in the usual sense of being annihilated by cup product with $c_1(L)$. Let $Hg^m(X)_{prim}$ denote the group of all such primitive Hodge

[2] you can think of L as $\mathcal{O}_{\mathbb{P}^N}(1)|_X$ for some embedding of X into a $\mathbb{P}^N$

classes, and note that $Hg^m(X)_{prim}$ is unchanged upon replacing L by $L^{\otimes k}$ for $k > 0$. Given $\zeta \in Hg^m(X)_{prim}$, a simple chase of the diagram

(11)
$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & Hg^m(X)_{prim} & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & J^m(X) & \longrightarrow & H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m)) & \longrightarrow & Hg^m(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^m(X_s) & \longrightarrow & H^{2m}_{\mathcal{D}}(X_s, \mathbb{Z}(m)) & \longrightarrow & Hg^m(X_s) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

yields a well-defined class in $J^m(X)$ for each $s \in S^*$, and hence a normal function ν_ζ. Viewing ν_ζ as an extension of VMHS

$$0 \to \mathcal{H}(m) \to \mathcal{E} \to \mathbb{Z}(0)_{S^*} \to 0,$$

we define its singularity $\sigma_p(\nu_\zeta)$ at $p \in \hat{X}$ to be the image of 1 under the composition

$$H^0(S^*, \mathbb{Z}(0)) \to H^1(S^*, \mathbb{H}(m)) \to \varinjlim_{U \ni p} H^1(U \cap S^*, \mathbb{H}(m)),$$

where the limit is taken over all analytic open neughborhoods of p in S.

With this terminology in place, we can at last state the beautiful

Theorem 11. (Griffiths-Green, 2007; Brosnan-Fang-Nie-Pearlstein, 2009; de Cataldo-Migliorini, 2009) *The Hodge Conjecture (HC) on a $2m$-dimensional smooth projective variety X is equivalent to the following statement for* each *primitive Hodge (m, m) class ζ and very ample line bundle $L \to X$: there exists $k \gg 0$ such that the natural normal function ν_ζ over $|L^k| \setminus \hat{X}$ (the complement of the dual variety in the linear system) has a nontorsion singularity at some point of $\hat{X}$.*

This is a highly nontrivial result with an equally nontrivial drawback: so far it has not led to a single new proof of the Hodge Conjecture.