D. Homological vs. algebraic equivalence

One of the first great successes of Griffiths's theory of VHS, transversality, and normal functions, was the proof that these two equivalence relations on algebraic cycles are distinct for cycles of codimension ≥ 2. We will first prove a general result, then apply it to differences of lines on certain Calabi-Yau 3-folds.

Definition 1: For X a smooth projective variety, the codim. p Griffiths group is

$$\mathrm{Griff}^p(X) := \frac{Z^p(X)_{hom}}{Z^p(X)_{alg}} \left(= \frac{CH^p(X)_{hom}}{CH^p(X)_{alg}} \right).$$

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In particular, this group will be shown to be nontorsion by use of Griffiths's Abel-Jacobi map. I'll have to quote a couple of results from the theory of Lefschetz pencils.

We shall work with the following setting:

$\mathbb{X}$ = smooth projective $2m$-fold

$\{X_s\}_{s\in\mathbb{P}^1}$ = Lefschetz pencil of hyperplane sections of $\mathbb{X}$
($X_s = H_s \cdot \mathbb{X}$; singular fibres have one ODP each)

$B = X_0 \cap X_\infty$ base locus

$\mathcal{X}$ = blow-up of X along B

and the accompanying diagram as in §§B – C :

$$\begin{tikzcd} \mathcal{X}^* \arrow[r, hook, "\iota"] \arrow[d, "\pi \text{ smooth}"] & \mathcal{X} \arrow[r, two heads, "\beta"] \arrow[d, "\bar{\pi}"] & \overline{X} \\ \mathbb{P}^1 \setminus \Sigma = U \arrow[r, hook] & \mathbb{P}^1 & \end{tikzcd}$$

Let $X_{t (\in U)}$ denote a very general fibre (t not defined over the algebraic closure of the field of def. of $\mathcal{X}$)

$X_\Sigma = \amalg X_{s_i}$ singular fibres

Lemma 1 : Assume

(D.1) $\quad H^{2m-1}(X_t, \mathbb{C}) \neq H^{m,m-1}(X_t) \oplus H^{m-1,m}(X_t)$

(D.2) $\quad H^{2m-1}(\overline{X}) = \{0\}$.

Then $J^m_{alg}(X_t) = \{0\}$.

Proof : If $\rho : \pi_1(U) \to Aut(H^{2m-1}(X_t, \mathbb{Q}), Q)$ (intersection form), we quote the following (valid for Lefschetz pencils)*

FACTS : (i) $\Gamma := \mathrm{im}(\rho)$ acts irreducibly on $(H_{fix})^\perp \subseteq H^{2m-1}(X_t, \mathbb{Q})$

(ii) $H_{fix} = \iota^* H^{2m-1}(\overline{X})$. //

By (D.2), Γ acts irreducibly on $H^{2m-1}(X_t, \mathbb{Q})$.

Now recall that $J^m_{alg}(X_t)$ is the intermediate Jacobian associated to a sub-HS

(D.3) $\quad H_{alg}(X_t) \subset H^{m-1,m}(X_t) \oplus H^{m,m-1}(X_t)$.

Since any cycle $Z_t \in Z^m(X_t)_{alg}$ can be "spread out" to a family of cycles $Z_s \in Z^m(X_s)_{alg}$ $(s \in U)$, $H_{alg}(X_t, \mathbb{Q})$ is a Γ-submodule of $H^{2m-1}(X_t, \mathbb{Q})$. Since the latter is irreducible, it must be $\underline{H^{2m-1}(X_t, \mathbb{Q})}$ (impossible by (D.3) & (D.1)) or $\underline{\text{zero}}$. $\square$

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* the content here is that vanishing cycles span $H^{2m-1}(X_t)/H_{fix}$. Once you have that, (i) follows from the Picard-Lefschetz formula. (see Voisin?)

Now the Leray filtration on $H^{2m}(\mathcal{X}^*)$ satisfies (by Leray s.s. / Deligne) 260

$$\mathrm{Gr}^0_{\mathcal{L}} H^{2m}(\mathcal{X}^*, \mathbb{Q}) = H^0(U, R^{2m}\pi_* \mathbb{Q})$$
$$\mathrm{Gr}^1_{\mathcal{L}} H^{2m}(\mathcal{X}^*, \mathbb{Q}) = H^1(U, R^{2m-1}\pi_* \mathbb{Q})$$
$$\mathrm{Gr}^{\geq 2}_{\mathcal{L}} \quad " \quad = \{0\} \quad \text{(since } U \text{ affine curve).}$$

By definition, $H^{2m}(\overline{X})_{prim}$ is the part of $H^{2m}(\overline{X})$ which, under pullback to $H^{2m}(\mathcal{X}^*)$, has vanishing $\mathrm{Gr}^0_{\mathcal{L}}$-image (because this computes pullback to $H^{2m}(X_s)$'s). So we get a composition

$$\begin{tikzcd} H^{2m}(\overline{X}, \mathbb{Q})_{prim} \arrow[r, "(\beta\circ r)^*"] \arrow[rr, bend right, "\mathcal{Y}"'] & \mathcal{L}^1 H^{2m}(\mathcal{X}^*, \mathbb{Q}) \equiv H^1(U, R^{2m-1}\pi_* \mathbb{Q}) \end{tikzcd}$$

Lemma 2: $\mathcal{Y}$ is injective.

Proof: β^* is injective, but r^* is not: one has the exact sequence

$$\to H^{2m}_{X_\Sigma}(\mathcal{X}) \xrightarrow[(\iota_\Sigma)_*]{} H^{2m}(\mathcal{X}) \xrightarrow[r^*]{} H^{2m}(\mathcal{X}^*) \to$$
$$H^{2m}_{X_\Sigma}(\mathcal{X}) \cong H_{2m}(X_\Sigma)$$

So $\beta^*(\ker \mathcal{Y}) \subset \mathrm{im}((\iota_\Sigma)_*)$

$\Rightarrow \ker \mathcal{Y}$ is ~~supported~~ on the X_{s_i},

~~and~~ since they have only ODP singularities,

FACT: $\mathrm{im}\, H_{2m}(X_{s_i}) = \mathrm{im}\, H_{2m}(X_t)$ in $(H_{2m}(\mathcal{X}) \cong H^{2m}(\mathcal{X})$.

But being in

$$\mathrm{im}\{H_{2m}(X_t)\} = \mathrm{im}\{H^{2m-2}(X_t) \xrightarrow{Gy} H^{2m}(\overline{X})\} = \mathrm{im}\{\cup [H_t]\}$$

is impossible for a primitive class (by definition). Hence $\ker \mathcal{Y} = 0$. $\square$

Theorem 1: Under the assumptions (0.1-2), if $Z \in Z^m(\overline{X})$ and $Z_t := Z \cap X_t \underset{alg}{\equiv} 0$ (t general), then $Z \underset{hom}{\equiv} 0$. [Note: $Z_t \underset{alg}{\equiv} 0 \Rightarrow Z_t \underset{hom}{\equiv} 0 \Rightarrow Z \in Z^m(\overline{X})_{prim}$.]

Proof: $Z_t \underset{alg}{\equiv} 0 \implies AJ(Z_t) \in J^m_{alg}(X_t)$

$$\underset{\text{Lemma 1}}{\implies} AJ(Z_t) = 0$$

$$\implies \nu_Z = 0 \text{ in } H^0(U, \mathbb{Q})$$

$$\implies [\nu_Z] = 0 \text{ in } H^1(U, R^{2m-1}\pi_* \mathbb{Q})$$

$\overset{\|}{\nu([Z])}$ ← (this is by the analogue of Prop. B.2(ii) mentioned in §C for higher (co)dimension)

$$\underset{\text{Lemma 2}}{\implies} [Z] = 0 \text{ in } H^{2m}(\mathfrak{X}, \mathbb{Q}).$$

□

Write $cl^m_{\mathfrak{X}} = [\cdot] : CH^m(\mathfrak{X}) \to H^{2m}(\mathfrak{X}, \mathbb{Q})$. The contrapositive of Theorem 1 is

Corollary 1: Assuming (D.1-2), if $\mathrm{im}(cl^m_{\mathfrak{X}}) \cap H^{2m}(\mathfrak{X}, \mathbb{Q})_{pr} \neq \{0\}$ and $\{X_t\}$ is a Lefschetz pencil, then $\mathrm{Griff}^m(X_t) \otimes \mathbb{Q} \neq \{0\}$ (t very general).

Theorem 2: Very general hyperplane sections of

$$X := \left\{ \sum_{i=0}^{5} z_i^5 = 0 \right\} \subset \mathbb{P}^5$$

have nontorsion Griff^2.

Proof: We need $Z \in Z^2(\mathfrak{X})_{pr}$ with $[Z] \neq 0$. Let $\zeta = e^{2\pi i/5}$ and

$$P_1 = \{ z_3 = -\zeta z_0, \ z_4 = -\zeta z_1, \ z_5 = -\zeta z_2 \} \subset \mathbb{P}^5$$

$$P_2 = \{ z_3 = -\zeta^2 z_0, \ z_4 = -\zeta^2 z_1, \ z_5 = -\zeta^2 z_2 \} \subset \mathbb{P}^5.$$

Both lie in X, and $P_1 \cap P_2 = \emptyset$. Set $L_i := H_t \cap P_i$ (2 lines).

We have $h^4(X_t) = 1$ and $H^4(X_t, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ for H' a second hyperplane.

$$\gamma \longmapsto \gamma \cdot H'$$

Since H' meets L_1 & L_2 in one point each, we have

$\underset{(\text{on } X_t)}{[L_i] \neq 0}$ $(i=1,2)$ and $\underset{(\text{on } X_t)}{[L_1 - L_2] = 0}$ $\implies$ $\underset{(\text{on } \mathfrak{X})}{[P_i] \neq 0}$ $(i=1,2)$

and $[P_1 - P_2] \in H^4(\mathfrak{X})_{prim}$. If $[P_1] = [P_2]$ (i.e. $[P_1 - P_2] = 0$) then P_1 & P_2 would have to have the same intersection pairing with any other cohomology class represented by a cycle. But setting

$$P_0 := \{ Z_3 = -\zeta Z_1 ,\ Z_4 = -\zeta Z_0 ,\ Z_5 = -\zeta^3 Z_2 \},$$

we find $\begin{cases} [P_0]\cdot[P_1] = 1 \\ [P_0]\cdot[P_2] = 0 \end{cases}$. So $[P_1 - P_2] \neq 0 \in H^4(\mathfrak{X})_{prim}$.

Finally, we just need to verify (D.1–2). We have $H^3(\mathfrak{X}) = 0$ simply because $\mathfrak{X}$ is a smooth projective hypersurface of dimension 4 (by the Lefschetz hyperplane theorem). The X_t are quintic 3-folds in $H_t \cong \mathbb{P}^4$, hence have

$$h^{3,0} = 1 \implies H^3(X_t) \neq H^{2,1} \oplus H^{1,2}.$$

So we're now done by Corollary 1. $\square$

Tracing through the proofs of Theorems 1 & 2, we find that for sufficiently general X_t, $\underline{AJ_{X_t}(L_1 - L_2) \text{ is nontorsion in}}$ $\underline{J^2(X_t)}$ $\left(= J^2(X_t) / \cancel{J^2_{alg}(X_t)}^{\,0} \right)$, proving $L_1 - L_2$ is nontorsion in $\mathrm{Griff}^2(X_t)$.

Appendix :

Here are some results + conjectures which complete the panorama of cycles & AJ maps. We have (X smooth proj.)

$$cl_X^m = [\cdot] : CH^m(X) \to Hg^m(X)$$

$$AJ_X^m : \ker(cl_X^m) = CH^m(X)_{hom} \longrightarrow J^m(X)$$

Hodge Conj : $cl_X^m \otimes \mathbb{Q}$ surjective.

Do we have an analogue of Abel's theorem? NO —

Mumford (1968) (as generalized by Roitman) : If $\Omega^j(X) \neq \{0\}$ for some $j \in [2, m) \cap \mathbb{Z}$, then $\ker(AJ)$ is huge (the technical term is nonprorepresentable).

For arithmetic cycles, one still hopes that AJ is enough to detect everything:

Bloch-Beilinson Conj: If X is defined $/\bar{\mathbb{Q}}$, then on $CH^m_{hom}(X_{\bar{\mathbb{Q}}}) \otimes \mathbb{Q}$ (the cycles with defining eqns. having coeffs. in $\bar{\mathbb{Q}}$), AJ is still injective.

How about an analogue of Jacobi inversion? Nope —

Clemens - Griffiths (70's/80's) : The image of $\mathrm{Griff}^{m}(X)$ ($m \geq 2$) in $J^m(X)/J^m_{Hg}(X)$ is in general countable and not finitely generated.

This is actually good, because whenever you have a countable image (of some cycle class map) in transcendental AG, the image points have arithmetic meaning (because, being rigid, they don't change as we spread the cycle to one defined $/\bar{\mathbb{Q}}$).