D. Homological vs. algebraic equivalence

One of the first great successes of Griffiths's theory of VHS, transversality, and normal functions, was the proof that these two equivalence relations on algebraic cycles are distinct for cycles of codimension 2. We will first prove a general result, then apply it to differences of lines on certain Calabi-Yau 3-folds.

Definition 1: For $X$ a smooth projective variety, the Griffiths group is

$$G_{\text{Griff}}^p(X) := \frac{\mathbb{Z}^p(X)_{\text{hom}}}{\mathbb{Z}^p(X)_{\text{alg}}} = \frac{\mathbb{Q}^p(X)_{\text{hom}}}{\mathbb{Q}^p(X)_{\text{alg}}}$$

In particular, this group will be shown to be nontrivial by use of Griffiths's Abel-Jacobi map. I'll have to quote a couple of results from the theory of hyperelliptic pencils.

We shall work with the following setting:

- $\mathcal{X} = \text{smooth projective } 2m\text{-fold}$
- $\{X_s\}_{s \in \mathbb{P}^1} = \text{Lefschetz pencil of hyperplane sections of } \mathcal{X}$
  - $(X_s \subset H_s \cdot \mathcal{X}; \text{ singular fibers have one ODP each})$
- $B = X_0 \cap X_{\infty} \text{ base locus}$
- $\mathcal{X} = \text{blow-up of } X \text{ along } B$
and the accompanying diagram
\[ \begin{array}{ccc}
X^* & \overset{\pi}{\longrightarrow} & X \\
\downarrow \pi & & \downarrow \pi \\
\overline{\mathbb{P}^1} & \cong & \overline{\mathbb{P}^1}
\end{array} \]

\( \pi^* \mathcal{E} = U \hookrightarrow \overline{\mathbb{P}^1} \)

Let \( X_{\mathbb{C}(w)} \) denote a very general fibre (\( \tau \) not defined over the algebraic closure of the field of definition of \( X \)).

\( X_{\mathbb{C}} = \bigcup X_{\xi} \); singular fibres

**Lemma 1:** Assume

(D.1) \( H^{2m-1}(X_{\mathbb{C}}, \mathbb{C}) \neq H^{m,m-1}(X_{\mathbb{C}}) \oplus H^{m-1,m}(X_{\mathbb{C}}) \)

(D.2) \( H^{m-1}(\overline{\mathbb{P}^1}) = \{0\} \).

Then \( J_{\mathbb{C}}^m(X_{\mathbb{C}}) = \{0\} \).

**Proof:** If \( \rho : \pi_1(U) \to \text{Aut}(H^{2m-1}(X_{\mathbb{C}}, \mathbb{Q}), \mathbb{Q}) \), we prove the following (valid for any closed purely***

**FACTS:**
(i) \( \Gamma \) acts irreducibly on \( H^{2m-1}(X_{\mathbb{C}}, \mathbb{Q}) \).
(ii) \( H^{m-1}_{\text{fix}} = (\times H^{2m-1}(\overline{\mathbb{P}^1}) \).

By (D.2), \( \Gamma \) acts irreducibly on \( H^{2m-1}(X_{\mathbb{C}}, \mathbb{Q}) \).

Now recall that \( J_{\mathbb{C}}^m(X_{\mathbb{C}}) \) is the intermediate Jacobian associated to a sub-

(D.3) \( H_{\text{alg}}(X_{\mathbb{C}}) \subset H^{m-1,m}(X_{\mathbb{C}}) \oplus H^{m,m-1}(X_{\mathbb{C}}) \).

Since any cycle \( Z_{\mathbb{C}} \in \mathbb{Z}(X_{\mathbb{C}}) \) can be "spread out" to a family of cycles \( Z_s \in \mathbb{Z}(X_s) \) (\( s \in \mathbb{U} \)), \( H_s(X_s, \mathbb{Q}) \) is a \( \Gamma \)-submodule of \( H^{2m-1}(X_{\mathbb{C}}, \mathbb{Q}) \). Since the latter is irreducible, it must be \( H^{2m-1}(X_{\mathbb{C}}, \mathbb{Q}) \)`

impossible by (D.3) and (D.1) or zero.

\[ \Box \]

The content here is that vanishing cycles span \( H^{2m-1}(X_{\mathbb{C}})/H^* \).

Once you have that, (i) follows from the Picard-Lefschetz formula. (see Voisin?)
Now the Leray filtration on $H^{2m}(X^*, \mathbb{Q})$ satisfies (by Grothendieck) \[ Gr_0^U H^{2m}(X^*, \mathbb{Q}) = H^0(U, R^{2m} T^* \mathbb{Q}) \]
\[ Gr_1^U H^{2m}(X^*, \mathbb{Q}) = H^1(U, R^{2m-1} T^* \mathbb{Q}) \]
\[ Gr_2^U Z = \mathbb{Q} \] (since $U$ an affine curve).

By definition, $H^{2m}(X)_{prim}$ is the part of $H^{2m}(X)$ which, under pullback to $H^{2m}(X^*)$, has vanishing $Gr_2^U$ image (become this comes pullback to $H^{2m}(X^*)$'s). So we get a composition
\[
\begin{align*}
Gr_{prim}^{U} H^{2m}(X^*, \mathbb{Q}) & \xrightarrow{(\text{Born})^*} \mathbb{Z} \cdot H^{2m}(X^*, \mathbb{Q}) \\
\mathbb{Z} \cdot H^{2m}(X^*, \mathbb{Q}) & \xrightarrow{r^*} H^{2m}(X^*) \\
H^{2m}(X^*) & \xrightarrow{m} H^{2m}(X) \\
H^{2m}(X) & \text{ injection} \\
\end{align*}
\]

Lemma 2: $\text{ker } m$ is injective.

Proof: $p^*$ is injective, but $r^*$ is not: one has the exact sequence
\[
\begin{align*}
\cdots & \xrightarrow{\epsilon_*} H^{2m}(X) \xrightarrow{\text{incl}} H^{2m}(X) \\
& \xrightarrow{r^*} H^{2m}(X^*) \\
& \xrightarrow{m} H^{2m}(X) \\
\end{align*}
\]

So $p^*(\text{ker } m) \subset \text{image}(\epsilon_*)$
\[ \Rightarrow \text{ker } m \text{ is supported on the } X_{s_i}, \]
and since they have only odd singularities,

Fact: $\text{image } H^{2m}(X_{s_i}) = \text{image } H^{2m}(X_e)$ in $(H^{2m}(X) \cong H^{2m}(X))$.

But being in
\[ \text{image } \{ H^{2m}(X_e) \} \subset \text{image } \{ H^{2m-2}(X_e) \} \overset{\cong}{\rightarrow} H^{2m}(X) \]
is impossible for a primitive class (by definition). Hence $\text{ker } m = 0$.  

Theorem 1: Under the assumptions (0.1-2), if $Z \in Z^{2m}(X)$ and $Z_e := Z \cap X_e \equiv 0$ (generally) then $Z \equiv 0$. [Note: $Z_e \equiv 0 \Rightarrow Z_e \equiv 0 \Rightarrow Z \equiv 0$]
Proof. \( E_+ \equiv 0 \Rightarrow AJ(\mathcal{E}_+^0) \in J_{\text{alg}}^1(\mathcal{X}_+^0) \)

\[ AJ(\mathcal{E}_+^0) = 0 \]

Lemma 1:

\[ \nabla \mathcal{E}_+^0 \in H^0(U, \mathbb{C}) \]

\[ [\nabla \mathcal{E}_+^0] = 0 \quad \text{in} \quad H^1(U, \mathbb{R}^{2m-1} \cdot \mathbb{C}_\tau) \]

\[ \mathcal{M}(\mathcal{E}_+^0) \quad \text{(this is by the analogue of Prop. B.2(ii))} \]

\[ \mathcal{M}(\mathcal{E}_+^0) \quad \text{measured in } \mathbb{C} \text{ for } \text{higher (co)dimension} \]

Lemma 2:

\[ [\mathcal{E}_+^0] = 0 \quad \text{in} \quad H^{2m}(\mathcal{X}, \mathbb{C}). \]

Write \( c_{\mathcal{X}}^{2m} = [\cdot] : \text{Ch}^{2m}(\mathcal{X}) \to H^{2m}(\mathcal{X}, \mathbb{C}). \) The contrapositive of

Theorem 1 is

Corollary 1: Assuming \((D.1-2)\), if \( m(\text{ch}_{\mathbb{R}}^{2m}) \cap H^{2m}(\mathcal{X}, \mathbb{C}) \neq \{0\} \)

and \( \{\mathcal{X}_+\} \) is a Lefschetz pencil, then \( \text{Griff}^{m}(\mathcal{X}_+)^0 \mathbb{C} \neq \{0\} \) (4 very

Theorem 2: Very general hyperplane sections of

\[ \mathcal{X} : = \{ \sum_{i=0}^{5} \mathcal{E}_i^5 = 0 \} \subset \mathbb{P}^5 \]

have nonvanishing \( \text{Griff}^{\mathcal{E}_+} \).

Proof: We need \( \mathcal{E} \in \mathcal{E}^{\mathcal{E}_+}^{2}(\mathcal{X}) \), with \( [\mathcal{E}] \neq 0 \). Let \( s = e^{2\pi i/5} \)

and

\[ P_1 = \{ \mathcal{E}_3 = -s \mathcal{E}_0, \mathcal{E}_4 = -s \mathcal{E}_1, \mathcal{E}_5 = -s \mathcal{E}_2 \} \subset \mathbb{P}^5 \]

\[ P_2 = \{ \mathcal{E}_3 = -s^2 \mathcal{E}_0, \mathcal{E}_4 = -s^2 \mathcal{E}_1, \mathcal{E}_5 = -s^2 \mathcal{E}_2 \} \subset \mathbb{P}^5. \]

Both lie in \( \mathcal{X} \), and \( P_1 \cap P_2 = \emptyset \). Let \( L_i : = \mathcal{H}_ + \cap P_i \) (2 lines).

We have \( h^4(\mathcal{X}_+^0) = 1 \) and \( h^4(\mathcal{X}_+, \mathcal{E}) \overset{\cong}{\longrightarrow} \mathbb{Z} \) for \( H \) a second hyperplane.
Since $H'$ meets $L_1$ & $L_2$ in one point each, we have

$$[L_i] \neq 0 \quad (i=1,2) \quad \text{and} \quad [L_1-L_2] = 0 \quad \Rightarrow \quad [P_i] \neq 0 \quad (i=1,2)$$

and $[P_1-P_2] \in H^4(X)_{prim}$. If $[P_i] = [P_j] \quad (i.e. \quad (P_i, P_j) = 0)$

then $P_i \cap P_j$ would have to have the same intersection pairing with any other cohomology class represented by a cycle. But setting

$$P_0 := \{ z_3 = -5z_1, z_4 = -5z_2, z_5 = z_3z_2 \},$$

we find $[P_0], [P_1], [P_2] = 1$. So $[P_1-P_2] = 0 \in H^4(X)_{prim}$.

Finally, we just need to verify (D.1.2). We have

$H^3(X) = 0$ simply because $X$ is a smooth projective hypersurface

of dimension 4 (by the Lefschetz hypersurface theorem). The

$X_t$ are quintic 3-folds in $H_t \subset P^4$, hence have

$$h^{3,0} = 1 \quad \Rightarrow \quad H^3(X_t) \cong H^{3,0} \oplus H^{1,2}.$$ 

So we are now done by Corollary 1. \( \square \)

Tracing through the proofs of Theorems 1 & 2, we find that

for sufficiently general $X_t$, $A_{X_t}(L_1-L_2)$ is non-torsion in

$J^2(X_t) \quad (= J^2(X_t) / \text{Gr}^2(X_t)^0 )$, proving $L_1-L_2$ is non-torsion

in $\text{Gr}^{\text{h}}^2(X_t)$. 

\( \square \)
Here are some results & conjectures which complete the panorama of cycles & AJ maps. We have (X smooth proj.):

\[ Cl^m_x : CH^m(x) \to H^m_g(x) \]

\[ AJ^m_x : ker(Cl^m_x) = CH^m(x)_{hom} \to J^m(x) \]

Hodge Conj: \( Cl^m_x \otimes Q \) surjective.

Do we have an analogue of Abel's theorem? **No**

Manin (1963): If \( \Omega^i(x) \neq \emptyset \) for some \( i \in \{2,m\} \cap \mathbb{Z} \), then \( ker(AJ) \) is huge (the technical term is non-prorepresentable).

For arithmetic cycles, one still hopes that AJ is enough to detect anything.

Bloch-Kato Conj. If \( X \) is defined over \( \overline{\mathbb{Q}} \), then on \( Cl^m(x)_{hom} \otimes \overline{\mathbb{Q}} \) (the cycles with defining equs. having coeffs. in \( \overline{\mathbb{Q}} \), AJ is still injective.

How about an analogue of Jacobian inversion? **Nope**

Oda's - Griffiths: The image of \( Cl^m(x) \) in \( J^m(x) / H^m_g(x) \) is in general countable and not finitely generated.

This is actually good, because whenever you have a countable image in transcendental AG, the image points have arithmetic meaning (because, being rigid, they don't change as we spread the cycle to one defined over \( \overline{\mathbb{Q}} \).