E. The Green-Voisin theorem

In our final application of Hodge theory to algebraic cycles, we shall use a lemma in commutative algebra to prove a vanishing result for Abel-Jacobi images of cycles on "sufficiently general" projective hypersurfaces. To do this, we shall have to go beyond the topological invariant $[\nu]$ of a normal function ν and introduce something called its infinitesimal invariant $\delta\nu$.

Let $\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \mathcal{S} \end{array}$ be a smooth projective morphism of relative dimension $2m-1$ (over a smooth quasi-projective base).

Write sheaves on $\mathcal{S}$:

$$H_{\mathbb{Z}}^{2m-1} = R^{2m-1}\pi_* \mathbb{Z}$$

$$\mathcal{H}^{2m-1} = H_{\mathbb{Z}}^{2m-1} \otimes \mathcal{O}_{\mathcal{S}}$$

$$\cup$$

$$\mathcal{F}^m \quad \text{(as usual)}$$

$$\mathcal{Q}^m = \frac{\mathcal{H}^{2m-1}}{\mathcal{F}^m + H_{\mathbb{Z}}^{2m-1}}$$

A normal function is a holo. section $\nu \in H^0(\mathcal{S}, \mathcal{Q}^m)$.

$0 \to H_{\mathbb{Z}} \to \mathcal{H}/\mathcal{F}^m \to \mathcal{Q} \to 0$ induces

$\partial : H^0(\mathcal{S}, \mathcal{Q}) \to H^1(\mathcal{S}, H_{\mathbb{Z}})$ or $[\cdot]$

(topological invariant)

$\nabla : \mathcal{H} \to \Omega^1_{\mathcal{S}} \otimes \mathcal{H}$ restricts to

$\nabla : \mathcal{F}^m \to \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{m-1}$, hence induces

$\overline{\nabla} : Gr_{\mathcal{F}}^m \to \Omega^1_{\mathcal{S}} \otimes Gr_{\mathcal{F}}^{m-1}$ ($\mathcal{O}_{\mathcal{S}}$-linear!)

We also get
$$\overline{\nabla}_{\mathcal{S}} : \mathcal{J}^m \to \Omega^1_{\mathcal{S}} \otimes \frac{\mathcal{H}^{2m-1}}{\mathcal{F}^{m-1}}$$
and set $\mathcal{J}^m_{hor} := \ker(\overline{\nabla}_{\mathcal{S}})$, $NF_{hor} := H^0(\mathcal{S}, \mathcal{J}^m_{hor})$.

Introduce complexes of sheaves on $\mathcal{S}$

- $\Omega^\bullet(\mathcal{H}) := \{\mathcal{H} \xrightarrow{\nabla} \Omega^1_{\mathcal{S}} \otimes \mathcal{H} \xrightarrow{\nabla} \Omega^2_{\mathcal{S}} \otimes \mathcal{H} \to \cdots\}$

  $\cup$
- $\Omega^\bullet(\mathcal{F}^m) := \{\mathcal{F}^m \xrightarrow{\nabla} \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{m-1} \xrightarrow{\nabla} \Omega^2_{\mathcal{S}} \otimes \mathcal{F}^{m-2} \to \cdots\}$
- $\tilde{\Omega}^\bullet(\mathcal{F}^m) := \{\mathcal{F}^m \oplus \mathbb{H}_{\mathbb{Z}} \xrightarrow{\nabla} \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{m-1} \xrightarrow{\nabla} \Omega^2_{\mathcal{S}} \otimes \mathcal{F}^{m-2} \to \cdots\}$
- $\Omega^\bullet(\mathcal{H}/\mathcal{F}^m) := \{\mathcal{J}^m \xrightarrow{\overline{\nabla}_{\mathcal{S}}} \Omega^1_{\mathcal{S}} \otimes \mathcal{H}/\mathcal{F}^{m-1} \xrightarrow{\overline{\nabla}} \Omega^2_{\mathcal{S}} \otimes \mathcal{H}/\mathcal{F}^{m-2} \to \cdots\}$

$$\underbrace{= \Omega^\bullet(\mathcal{H})/\tilde{\Omega}^\bullet(\mathcal{F}^m)}$$ [i.e. we have s.e.s. of cxs. $0 \to \tilde{\Omega}^\bullet(\mathcal{F}^m) \to \Omega^\bullet(\mathcal{H}) \to \Omega^\bullet(\mathcal{H}/\mathcal{F}^m) \to 0$]

this produces a connecting homomorphism of cohomology sheaves
$$\mathcal{J}^m_{hor} = \mathcal{H}^0(\Omega^\bullet(\mathcal{H}/\mathcal{F}^m)) \xrightarrow{\delta} \mathcal{H}^1(\tilde{\Omega}^\bullet(\mathcal{F}^m)) = \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m))$$
or, taking sections over $\mathcal{S}$,
$$\delta : \underbrace{H^0(\mathcal{S}, \mathcal{J}^m_{hor})}_{NF_{hor}(\mathcal{S})} \to H^0(\mathcal{S}, \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m))),$$
which defines the <u>infinitesimal invariant</u> of a horizontal normal function.

<u>Lemma 1</u>: For $\nu \in NF_{hor}$, $\delta\nu = 0 \iff \nu$ has <u>flat</u> local liftings.

<u>Proof</u>: If $\tilde{\nu} \in H^0(U, \mathcal{H})$ is a local lifting, and $\nabla\tilde{\nu} \in H^0(U, \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{m-1})$ is ∇ of $f \in H^0(U, \mathcal{F}^m)$, then $\tilde{\nu} - f$ is a local lifting of ν and $\nabla(\tilde{\nu} - f) = 0$. The converse is trivial. $\square$

Writing $\Omega^\bullet(Gr_F^k) := \left\{ Gr_F^k \xrightarrow{\bar\nabla} \Omega^1_S \otimes Gr_F^{k-1} \xrightarrow{\bar\nabla} \cdots \right\}$,

the general formulation of spectral sequences of filtered complexes gives

(in this case, of sheaves on S)

$$\mathcal{E}_0^{p,q} := \Omega^{p+q}(Gr_F^{m+q}), \qquad \xrightarrow{d_0 = \bar\nabla}$$

$$\mathcal{E}_1^{p,q} = \mathcal{H}^{p+q}_{\bar\nabla}\{\Omega^\bullet(Gr_F^{m+q})\} \qquad \uparrow d_1$$

$$\vdots$$

$$\mathcal{E}_\infty^{p,q} = \text{“}Gr_F^q\text{”}\, \mathcal{H}^{p+q}_{(\nabla)}\{\Omega^\bullet(F^m)\}.$$

<u>Lemma 2</u>: If all $Gr_F^a \xrightarrow{\bar\nabla} \Omega^1_S \otimes Gr_F^{a-1} \xrightarrow{\bar\nabla} \Omega^2_S \otimes Gr_F^{a-2}$ $(a \geq m)$

are exact at the middle term, then $\mathcal{H}^1(\Omega^\bullet(F^m)) = 0$ $(\Rightarrow \delta\nu = 0)$.

<u>Proof</u>: Set $m+q := a$ $(q \geq 0)$, then this simply says all $\mathcal{E}_1^{1-q,q} = \{0\}$

$(\Rightarrow \mathcal{E}_\infty^{1-q,q} = \{0\} \Rightarrow$ done$)$.

$\square$

Let $S = \mathbb{P}\left(H^0(\mathbb{P}^{2m}, \mathcal{O}(d))\right) \setminus \underbrace{\Sigma}_{\text{discriminant locus (singular hypersurfaces)}}$ and $\mathcal{X} \xrightarrow{\pi} S$ be

the tautological (or "universal") family. (A point of S is an element $f \in S^d := S^d_{X_0,\ldots,X_{2m}}$ up to scale, and we set $X_f := \pi^{-1}(f) = \{f(\underline{X}) = 0\} \subset \mathbb{P}^{2m}$.)

Recall that for $p+q = 2m-1$,

$$H_f^{p,q} := H^{p,q}(X_f) \cong R_f^{(q+1)d-2m-1} := \frac{S^{(q+1)d-2m-1}}{\left(\frac{\partial f}{\partial X_0}, \ldots, \frac{\partial f}{\partial X_{2m}}\right)^{(q+1)d-2m-1}}$$

$\left(J_f^{(q+1)d\cdots} :=\right)$

by residue theory; and

the sequences of Lemma 2

$$H_f^{a,m-a-1} \xrightarrow{\bar\nabla} \Omega^1_{\mathcal{S},[f]} \otimes H_f^{a-1,m-a} \xrightarrow{\bar\nabla} \Omega^2_{\mathcal{S},[f]} \otimes H_f^{a-2,m-a+1} \to \cdots$$

dualize to

$$\cdots \to \Lambda^2 \Theta^1_{\mathcal{S},[f]} \otimes H_f^{m-a+1,a-2} \to \Theta^1_{\mathcal{S},[f]} \otimes H_f^{m-a,a-1} \to H_f^{m-a-1,a}$$

which is computed by

$$\cdots \to \Lambda^2 S^d \otimes R_f^{(a-1)d-2m-1} \to S^d \otimes R_f^{ad-2m-1} \to R_f^{(a+1)d-2m-1}$$

where the maps are given by polynomial multiplication (cf. Thm. III.B.6).

By a diagram chase, this will be exact at the middle term if

(E.1) $\boxed{S^d \otimes J_f^{ad-2m-1} \to J_f^{(a+1)d-2m-1} \text{ is surjective}}$

AND

(E.2) $\boxed{\Lambda^2 S^d \otimes S^{(a-1)d-2m-1} \to S^d \otimes S^{ad-2m-1} \to S^{(a+1)d-2m-1} \text{ is exact at the middle term}}$.

For $ad-2m-1 \geq d-1$, $J_f^{ad-2m-1}$ contains all the generators $\partial f/\partial X_i$; so (E.1) holds for $a \geq m$ provided

(E.3) $$md - 2m \geq d.$$

By the <u>Symmetrizer Lemma</u> (cf appendix to this section), for $ad-2m-1 > d$, (E.2) holds; so to have it for $a \geq m$ we need

(E.4) $$md - 2m \geq d+2 . \quad (\Rightarrow (E.3))$$

$$\Updownarrow$$

$$(m-1)d \geq 2m+2 = 2(m-1)+4$$

$$\Updownarrow$$

$$d \geq 2 + \frac{4}{m-1}.$$

We conclude:

<u>Lemma 3</u>: Normal functions $\nu \in NF_{har}$ for the universal family as above, have flat local liftings if (E.4) holds.

Theorem 1 (Green, 1989; Voisin, [unpublished] (for 3-folds)): Let $X \subseteq \mathbb{P}^{2m}$ (≥ 4) be a very general smooth hypersurface of degree $d \geq 2 + \frac{4}{m-1}$.

Then $\mathrm{image}(AJ) \subseteq J^m(X)$ is torsion.

1-cycles on 3-folds	$d \geq 6$
2-cycles on 5-folds	$d \geq 4$
3-cycles on 7-folds	$d \geq 4$
- - - - -	$d \geq 3$

Proof: Given $Z \in Z^m(X_{t_0})$ ⊂ (very general) we can spread it out to $\mathfrak{Z} \in Z^m(\tilde{\mathcal{X}})$ where

$$\begin{array}{ccc} \tilde{\mathcal{X}} = \mathcal{X} \times_g T & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{g} & \mathcal{S} \end{array} \Bigg\} \text{universal family}$$

(g = finite étale morphism)

Define $\nu(t) = AJ(\mathfrak{Z} \cdot X_t)$; then by Lemma 3, $\bar{\delta}\nu = 0$ and ν has a flat local lifting $\tilde{\nu}$ in any ball $\subset T$.

Now $S^d \otimes S^{ad-2m-1} \twoheadrightarrow S^{(a+1)d-2m-1}$ $(\forall a \geq m)$ (need only $md - 2m \geq 1$) $\Longrightarrow$

(dually) $F^m \mathcal{H}^{2m-1}_{t_0} \overset{\nabla}{\hookrightarrow} \Omega^1 \otimes F^{m-1}\mathcal{H}^{2m-1}_{t_0}$ $\Longrightarrow$

the flat local liftings are unique up to $H^{2m-1}_{\mathbb{Z}}$. Hence for $\gamma \in \pi_1(T, t_0)$, $\underbrace{\rho(\gamma)\tilde{\nu}_{t_0} - \tilde{\nu}_{t_0}}_{= \langle \tilde{\nu}_{t_0}, \delta_\gamma\rangle \delta_\gamma} \in H^{2m-1}(X_{t_0}, \mathbb{Z})$.

← Picard-Lefschetz, if γ is a loop and δ_γ the corresponding vanishing cycle

We conclude that, for all vanishing cycles, $\langle \tilde{\nu}_{t_0}, \delta_\gamma \rangle \in \mathbb{Q}$.
But (another FACT from Picard-Lefschetz theory) these vanishing cycles generate the cohomology, so $\tilde{\nu}_{t_0} \in H^{2m-1}(X_{t_0}, \mathbb{Q})$. Hence $\nu_{t_0} = AJ(Z_{t_0}) \in$ image of $H^{2m-1}(X_{t_0}, \mathbb{Q})$ in $J^m(X_{t_0})$. □

Appendix: The Symmetrizer Lemma (Donagi–Green, 1984)

Theorem 2: $\Lambda^2 S^a \otimes S^{b-a} \xrightarrow{\mu^2} S^a \otimes S^b \xrightarrow{\mu} S^{b+a}$ is exact

at the middle term if $b > a$.

Proof: We check $\ker(\mu) \subseteq \operatorname{im}(\mu^2)$.

Let I, J be multi-indices st

$$\sum_{\substack{|I|=a \\ |J|=b}} c_{I,J}\, z_I \otimes z_J \in \ker(\mu).$$

Then $\forall$ fixed multi-indices K (with $|K| = a+b$), $\displaystyle\sum_{I+J=K} c_{I,J} = 0$.

Now by definition

$$\operatorname{im}(\mu^2) = \left\{ z_I \otimes z_{I'+L} - z_{I'} \otimes z_{I+L} \;\middle|\; |I| = |I'| = a,\ |L| = b-a > 0 \right\}.$$

If we can show the following

$$\boxed{\begin{array}{l} \text{CLAIM: } \operatorname{im}(\mu^2) \text{ contains all elements of the form} \\[4pt] \left\{ z_I \otimes z_{K \setminus I} - z_{I'} \otimes z_{K \setminus I'} \;\middle|\; |I| = |I'| = a,\ |K| = a+b,\ K \supseteq I, I' \right\} \end{array}}$$

then all $z_I \otimes z_J$ with $I+J=K$ are equivalent modulo $\operatorname{im}(\mu^2)$

and so from $\displaystyle\sum_{I+J=K} c_{I,J} = 0$ follows $\displaystyle\sum_{I+J=K} c_{I,J}\, z_I \otimes z_J \equiv 0 \mod \operatorname{im}(\mu^2)$, hence

$$\sum_{\substack{|I|=a \\ |J|=b}} c_{I,J}\, z_I \otimes z_J = \sum_{|K|=a+b} \left(\sum_{I+J=K} c_{I,J}\, z_I \otimes z_J \right) \equiv 0 \mod \operatorname{im}(\mu^2),$$

which is the same as saying our arbitrary element of $\ker(\mu)$ is in the image of μ^2.

We now turn to the proof of the CLAIM. It is enough to show that

$\mathrm{im}(\mu^2) \ni z_I \otimes z_{K\setminus I} - z_{I'} \otimes z_{K\setminus I'}$ for I & I' differing by one index.

(and the same conditions as in the CLAIM). So for

$$I - I' = (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0, \underset{j}{-1}, 0, \cdots, 0) =: \delta_i - \delta_j$$

we will show, working mod $\mathrm{im}(\mu^2)$, that $z_I \otimes z_{K\setminus I} \equiv z_{I'} \otimes z_{K\setminus I'}$.

More generally, for any I, I', K (satisfying the CLAIM's conditions) for which there are multi-indices J & I'' with

$$|J| = b, \quad J - I \geq 0, \quad J - I' \geq 0, \quad I'' + J = K,$$

we have

$$z_I \otimes z_{K\setminus I} \equiv z_{I''} \otimes z_J \equiv z_{I'} \otimes z_{K\setminus I'}$$

[using the definition of $\mathrm{im}(\mu^2)$, e.g. for the first "$\equiv$" replacing the L & I' of the definition by $J\setminus I$ and I'']. Since we only care about the end terms we can forget I'' and replace the condition $I'' + J = K$ by $J \leq K$.

What's left is to show we can choose such a J for $I\setminus I' = \delta_i - \delta_j$. Since $b > a$, there exists $J \geq \delta_j + I$ with $|J| = b$; we then have immediately $J\setminus I \geq \delta_j \geq 0$ and $(J\setminus I' =)\ J - I' = (J - I) + (I - I') \geq \delta_j + \delta_i - \delta_j = \delta_i \geq 0$. Now $K \geq I, I' \Rightarrow K \geq \delta_j + I = \delta_i + I'$, and so it is possible to choose this J both $\geq \delta_j + I$ and $\leq K$.

$\square$