Appendix A: Results from Complex Algebraic Geometry

M compact Kähler, $L \to M$ holo. line bundle

proofs will be sketchy. Griffiths-Harris & Voisin are obvious ref's.

Preliminaries on line bundles

(1) Given a Hermitian metric h on L, and the sections $\sigma_\alpha \in \mathcal{O}(U_\alpha, L)$ arising from the local trivializations of L, we had

$$\rho_\alpha := h(\sigma_\alpha, \sigma_\alpha) \quad \text{and} \quad \omega_{(L,h)} := \frac{-\partial\bar\partial \log \rho_\alpha}{2\pi i}$$

$\Big\} \leftarrow$ these pieced together on the $\{U_\alpha\}$ to give a global closed real $(1,1)$ form

We also defined $c_1(L)$ as the image of $L \leftrightarrow \{g_{\alpha\beta}\} \in H^1(M, \mathcal{O}^*)$ under $H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z})$, and had the result

$$\boxed{c_1(L) = [\omega_{(L,h)}] \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})}.$$

(2) Given $N \subset M$ codim. 1 submanifold, we defined a line bundle $L_N \to M$ with sheaf of sections $\mathcal{O}_M(N)$.$^{(\dagger)}$ Recall that $\frac{Gy}{2\pi i} : H^k_{dR}(N) \to H^{k+2}_{dR}(M)$ was defined by taking $\exists$ a log-form on M with $2\pi i \times$ residue equal to the given form on N, and applying ∂. If $\{f_\alpha = 0\}$ are the local holomorphic equations for N in each U_α, then the $\{|f_\alpha|^2 \rho_\alpha\}$ agree on overlaps, and $\partial\bar\partial \log |f_\alpha|^2 = 0$. Hence, $\omega_{(L,h)} = -\frac{\partial\bar\partial \log \rho_\alpha}{2\pi i}$ $= \frac{+\bar\partial \partial \log |f_\alpha|^2 \rho_\alpha}{2\pi i} = \bar\partial \left\{ \frac{1}{2\pi i} \partial \log |f_\alpha|^2 \rho_\alpha \right\}$. But the bracketed form has $2\pi i \, \mathrm{Res}_N \{ \cdots \} = 1_N$. We conclude that

$$\boxed{c_1(L) = \frac{Gy}{2\pi i}([1_N])}$$

$(\dagger)$ with the natural generalization to divisors $\underset{\substack{\cap \\ \mathrm{Div}(M)}}{D} \leftrightarrow L_D \leftrightarrow \mathcal{O}_M(D)$.

(3) $L > 0 \underset{\text{defn.}}{\Longleftrightarrow} \exists\, h$ s.t. $\omega_{(L,h)} > 0$

④ $\boxed{L = L_D \text{ for some } D \in \mathrm{Div}(M) \iff L \text{ has a meromorphic section}}$

Proof: recall $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$

(from Exercise in PS #4(?))

$$\Longrightarrow \quad \to \mathrm{Div}(M) \to \mathrm{Pic}(M) \xrightarrow{\delta} H^1(M, \mathcal{M}^*) \to$$

holo. line bdls. $\cong$ $\mathrm{Pic}(M)$

$$L \longmapsto 0$$

$$\Updownarrow$$

$\exists \{f_\alpha\}$ mero. fcns. s.t. $g_{\alpha\beta} = f_\alpha / f_\beta \quad \forall \alpha, \beta$

but this is the definition of a mero. section!

$\square$

⑤ $\boxed{\exists L > 0 \iff \exists \text{ integral Kähler class}}$

i.e. $\exists$ a positive d-closed real $(1,1)$-form ω with $[\omega] \in H^2(M, \mathbb{Z})$.

$\uparrow$ this is the Kähler class

Proof: $(\Longrightarrow)$ is trivial: take $\omega = \omega_{(L,h)}$, & use ③ & ①.

$(\Longleftarrow)$: from exp exact sequence,

$$\left(\mathrm{Pic}(M) = H^1(M, \mathcal{O}^*)\right) \xrightarrow{c_1} \underbrace{H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O})}_{\ker \,=\, H^{1,1}(M) \cap H^2(M, \mathbb{Z})}$$

(Hodge/Dolbeault) $H^2(M,\mathcal{O}) \cong H^2(M,\mathbb{C}) / F^1 H^2(M,\mathbb{C})$

So given $\omega \in A^{1,1}_{\mathbb{R}}(M)_{d\text{-closed}}$ with $\omega > 0$ & $[\omega] \in \ker$,

$\exists \ L \in \mathrm{Pic}(M)$ s.t. $c_1(L) = [\omega]$. But then for any h on L,

$$[\omega_{(L,h)} - \omega] = 0 \underset{\partial\bar\partial\text{-Lemma}}{\Longrightarrow} \omega_{(L,h)} - \omega = \frac{1}{2\pi i} \partial \bar\partial \phi$$

$$\Longrightarrow \omega_{(L, e^{\phi} h)} = \omega$$

$$\Longrightarrow L > 0.$$

$\square$

Vanishing theorem + Consequences

Continue to assume:
M compact Kähler, L holo. line bdl.
$\dim_{\mathbb{C}}(M) = n$

6) Kodaira-Nakano Vanishing theorem:

$$\boxed{L > 0 \implies H^q(M, \Omega^p(L)) = \{0\} \quad \forall\, p+q>n}$$

Proof: Take $\omega_{(L,h)}$ as Kähler metric, $\nabla :=$ Chern connection, $\langle\ ,\ \rangle_L$ as in the notes.

In notation of (1),

$$\nabla \sigma_\alpha = \nabla^{1,0}\sigma_\alpha = \left(\frac{h(\nabla^{1,0}\sigma_\alpha, \sigma_\alpha)}{h(\sigma_\alpha,\sigma_\alpha)}\right)\sigma_\alpha = \Big(\partial \log \underbrace{h(\sigma_\alpha,\sigma_\alpha)}_{\rho_\alpha}\Big)\sigma_\alpha \implies$$

$$(*) \qquad \nabla^2\sigma_\alpha = (\overset{\text{"}\bar\partial\text{"}}{d}\,\partial \log\rho_\alpha)\,\sigma_\alpha + \underbrace{\cancel{(\partial\log\rho_\alpha \wedge \partial\log\rho_\alpha)}}_{0}\,\sigma_\alpha = 2\pi i\,\omega_{(L,h)}\,\sigma_\alpha .$$

Now given $\eta \in \mathcal{H}^q(M, \Omega^p(L))$ $(\Rightarrow \bar\partial_L \eta = \bar\partial_L^* \eta = 0)$,

$$\nabla^2 \eta = (\nabla^{1,0} + \bar\partial_L)^2 \eta = \bar\partial_L \nabla^{1,0}\eta \implies$$

$$2i\langle \Lambda \nabla^2 \eta, \eta\rangle_L = 2i\langle \Lambda\bar\partial_L\nabla^{1,0}\eta,\eta\rangle_L = 2i\Big\langle \big(\bar\partial_L\Lambda - \tfrac{i}{2}(\nabla^{1,0})^*\big)\nabla^{1,0}\eta, \eta\Big\rangle_L$$

adjoint of $L = \iota[\omega_{(L,h)}]$

$$= \|\nabla^{1,0}\eta\|_L \geq 0$$

Kähler identity: $[\Lambda, \bar\partial_L] = -\frac{i}{2}(\nabla^{1,0})^*$

as $\bar\partial_L^*$ (gives 0)

$$2i\langle \nabla^2\Lambda\eta,\eta\rangle_L = 2i\langle\nabla^{1,0}\bar\partial_L\Lambda\eta,\eta\rangle_L = 2i\Big\langle\nabla^{1,0}\big(\Lambda\bar\partial_L + \tfrac{i}{2}(\nabla^{1,0})^*\big)\eta,\eta\Big\rangle_L$$

(gives 0)

$$= -\|(\nabla^{1,0})^*\eta\|_L \leq 0 .$$

Hence,

$$0 \leq 2i\langle[\Lambda,\nabla^2]\eta,\eta\rangle_L \overset{(*)}{=} (2\pi i)\,2i\langle[\Lambda, L]\eta,\eta\rangle_L$$

$$= -4\pi^2\langle Y\eta,\eta\rangle_L = 4\pi^2(n-(p+q))\,\|\eta\|_L^2$$

multiplies by degree of form η minus n = $p+q-n$

$n < p+q$
$\implies \|\eta\|_L^2 = 0$
$\implies \eta \equiv 0$. $\square$

⑦ **Weak Lefschetz Theorem:** $\boxed{\begin{array}{l} N \overset{\imath}{\hookrightarrow} M \text{ smooth hypersurface } (\dim_{\mathbb{C}} N = n-1), \\ \text{with } L_N > 0. \text{ Then } \imath^* \text{ is} \\ \left\{\begin{array}{l}\cong \\ \hookrightarrow\end{array}\right. \text{ on } H^k \text{ for } \left\{\begin{array}{l} k \le n-2 \\ k = n-1 \end{array}\right. \end{array}}$

[Key example: $M = X \subset \mathbb{P}^N$, $N = \underbrace{X \cap \left(\begin{smallmatrix}\text{smooth} \\ \text{hypersurface in } \mathbb{P}^N\end{smallmatrix}\right)}_{\text{assumed smooth: see ⑨ below.}}$]

Proof:

$$\begin{array}{ccccc} H^k(M,\mathbb{C}) & \overset{\text{by Hodge theory,}}{=} & \bigoplus_{p+q=k} H^{p,q}(M) & = & \bigoplus H^q(M,\Omega^p) \\ \downarrow \imath^* & & \downarrow & & \downarrow \imath^* \\ H^k(N,\mathbb{C}) & = & \bigoplus H^{p,q}(N) & = & \bigoplus H^q(N,\Omega^p) \end{array}$$

[It will suffice to prove $\cong$ resp. $\hookrightarrow$ on these gps.

We have the s.e.s.'s:

$(*)$ $\quad 0 \to \Omega^p_M(-N) \to \Omega^p_M \to \imath_*(\Omega^p_M|_N) \to 0$ $\quad$ (sheaves on M)

$(**)$ $\quad 0 \to \underbrace{\mathcal{N}^\vee_{N/M} \otimes \Omega^{p-1}_N}_{\mathcal{O}(-N)|_N \otimes \Omega^{p-1}_N = \Omega^{p-1}_N(L_N^{-1})} \overset{\wedge}{\to} \Omega^p_M|_N \xrightarrow{\imath^*} \Omega^p_N \to 0$ $\quad$ (sheaves on N)

Kodaira vanishing $(p+q<n)$ $\Rightarrow$ $H^q(M,\Omega^p_M(L_N^{-1})) \underset{\text{duality}}{\overset{\text{Serre}}{=}} H^{n-q}(M,\Omega^{n-p}_M(L_N)) \overset{⑥}{=} \{0\}$ $\quad$ (†)

$H^q(N,\Omega^{p-1}_N(L_N^{-1}|_N)) = H^{n-q-1}(N,\Omega^{n-p}_N(L_N|_N)) \overset{⑥}{=} \{0\}$ $\quad$ (††)

For $p+q<n-1$,

$$\begin{array}{ccc} & H^q(N,\Omega^p_M|_N) & \underset{\imath^*}{\overset{(**)+(††)}{\xrightarrow{\cong}}} H^q(N,\Omega^p_N) \\ & \| & \\ H^q(M,\Omega^p_M) \underset{|_N}{\overset{(*)+(†)}{\xrightarrow{\cong}}} & H^q(M,\imath_*(\Omega^p_M|_N)) & \end{array}$$

and "$\imath^* \circ |_N$" is $\imath^*$ on Dolbeault coh. groups.

So done in this case ($k \le n-2$). The other case ($k=n-1$) is similar. $\square$

[N.B.: The next result makes it clear that weak Lefschetz is a result for *projective* M.]

⑧ Kodaira embedding theorem: $\boxed{\exists\, L>0 \implies M \text{ projective}.}$

[Important Remark: This proves Remark I.6.4 (the polarizable cx. tori are precisely the abelian varieties) and Example II.6.5.]

Proof: Show that the map $M \xrightarrow{\varphi} \mathbb{P}\left(H^0(M, \mathcal{O}(L^{\otimes N}))^\vee\right)$

$$p \longmapsto \left\{\text{hyperplane of sections vanishing at } p\right\} \; *$$

is well-defined, separates points, and kills no tangent vector ($\widehat{d\varphi}$), for $N \gg 0$.
(Then Chow's theorem (GAGA) $\Rightarrow$ the resulting smooth compact cx. submanifold of $\mathbb{P}$ is actually algebraic.)

I'll just sketch the well-definedness: by compactness, all you really have to show is $\exists$ of $\{N\gg 0$ & $\sigma \in H^0(M, \mathcal{O}(L^{\otimes N}))$ with $\sigma(p) \neq 0$, for a given p.

Let $\tilde{M} = B_p(M) \overset{i}{\supset} E$ be blow-up of M at p (note $E \cong \mathbb{P}^{n-1}$),

$$\begin{array}{ccc} \tilde{M} & \supset & E \\ \downarrow\beta & & \downarrow \\ M & \ni & p \end{array}$$

and $\tilde{L} = \beta^{-1}L$ be the pullback bundle. (note $\tilde{L}|_E = \mathcal{O}_E$).

also trivially restricts to E.

One computes that $\beta^* K_M = K_{\tilde{M}}((-n+1)E)$ & $\mathcal{O}_{\tilde{M}}(-E)|_E = \mathcal{O}_E(1)$ (cf. Voisin),

so $$\mathcal{O}_{\tilde{M}}(\tilde{L}^{\otimes N})(-E) = K_{\tilde{M}} \otimes \Big\{\underbrace{\beta^* K_M^{-1} \otimes \mathcal{O}_{\tilde{M}}(-nE) \otimes \mathcal{O}(\tilde{L}^{\otimes N})}_{}\Big\};$$

$$= \Omega^n_{\tilde{M}}(\text{positive})$$

restriction to E is $\mathcal{O}_E(n)$, and it follows that for $N \gg 0$, this is positive on $\tilde{M}$

and using

$$0 \to \mathcal{O}_{\tilde{M}}(\tilde{L}^{\otimes N})(-E) \to \mathcal{O}_{\tilde{M}}(\tilde{L}^{\otimes N}) \to \underbrace{\mathcal{O}_{\tilde{M}}(\tilde{L}^{\otimes N})|_E}_{\cong\, \mathcal{O}_E} \to 0,$$

⑥ $\Rightarrow$ ($H^1(\Omega^n(\text{pos.})) = \{0\}$)

$$\begin{array}{ccc} H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^{\otimes N})) & \twoheadrightarrow & H^0(E, \mathcal{O}_E(\tilde{L}^{\otimes N}|_E)) \\ \| \text{ Hartogs} & & \| \\ & & H^0(E, \mathcal{O}_E) \\ & & \| \\ H^0(M, \mathcal{O}_M(L^{\otimes N})) & \twoheadrightarrow & L^{\otimes N}|_p \end{array}$$

i.e. not every section evaluates to 0 at p. $\square$

$*$ note: if you take basis $\{\sigma_0, \ldots, \sigma_r\} \subset H^0(M, \mathcal{O}(L^{\otimes N}))$, then $\varphi(p) = \{0 = \Sigma\, \sigma_i(p)\, \underbrace{\sigma_i^*}_{\text{dual basis}}\}$ $\Rightarrow$ can think of φ as

$$p \longmapsto [\sigma_0(p) : \sigma_1(p) : \cdots : \sigma_r(p)] \in \mathbb{P}^r.$$

9) <u>Bertini's Theorem (baby case):</u> Given $X \subset \mathbb{P}^n$ smooth projective.

Then for $H \subset \mathbb{P}^n$ a general hypersurface (of some degree d), $X \cap H$ is smooth.

<u>Proof</u>: I'll just do the case of H a hyperplane ($d=1$).

Given a general <u>pencil</u> of hyperplanes $H_t = \{\underbrace{t_1 L_1 + t_2 L_2}_{L_t} = 0\}$ $\left(t = \frac{t_2}{t_1}\right)$,

$f + tg = L_t|_X \in \Gamma(X, \mathcal{O}(1))$. Set $H_t \cap X =: X_t = \{f + tg = 0\}$,

and suppose $\begin{cases} t \neq 0, \infty \\ X_t \text{ singular at } p \end{cases}$; then in <u>local coords.</u> $\left(\frac{\partial f}{\partial z_i} + t\frac{\partial g}{\partial z_i}\right)(p) = 0$ $(\forall i)$

Now suppose $p \notin B := H_0 \cap H_\infty \cap X$ $\left(= \bigcap_{t \in \mathbb{P}^1} X_t \right) =:$ "<u>base locus</u> of pencil".

Then f & g cannot <u>both</u> vanish at p $\Rightarrow$ <u>neither</u> can $\Rightarrow$ $t = -\frac{f(p)}{g(p)} =: t(p)$

$\Rightarrow g(p)\frac{\partial f}{\partial z_i}(p) = f(p)\frac{\partial g}{\partial z_i}(p)$ $(\forall i)$ $\Rightarrow \left(\frac{\partial}{\partial z_i}\left(\frac{f}{g}\right)\right)\Big|_p = 0$ $(\forall i)$. $(*)$

Let $V \subset X \setminus B$ be the locus of p where the unique $X_{t(p)}$ containing p is singular. Then $(*)$ is saying that each connected component of V has <u>constant</u> $t(p)$. Hence, $\exists$ only <u>finitely</u> many t with $X_t \cap V \neq \emptyset$, and so only <u>finitely many</u> $X_t \setminus B$ are singular.

Now, if a <u>general</u> hyperplane section was singular, then this would be true for any pencil <u>through</u> that hyperplane section. But the base loci of these pencils do not intersect. $\square$

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$$\boxed{X \text{ smooth projective} \implies \text{every line bundle is of the form } L_D,\ D \in \mathrm{Div}(X)}$$

Proof: By 4, we need only check that every line bundle admits a meromorphic section. We prove this by induction on $n = \dim X$. The inductive step uses Kodaira vanishing + Bertini.

Suppose that for every $\left\{ \begin{array}{l} Y \subset \mathbb{P}^N \text{ of } \dim < n \\ L' \to Y \text{ holo. line bundle} \end{array} \right\}$,

$H^0(Y, \mathcal{O}_Y(L')(\mu H)) \neq \{0\}$ for $\mu \gg 0$. By 9, there

general notation for hyperplane, hyperplane section, associated bundle, etc.

exists $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$ s.t. $Y := H \cap X$ is smooth. We have the s.e.s. (sheaves on X)

section of $\mathcal{O}(1)$ vanishing exactly on H

$$0 \to \mathcal{O}_X(L)((\mu-1)H) \xrightarrow{\otimes s} \mathcal{O}_X(L)(\mu H) \xrightarrow{1_Y} \underbrace{\iota_* \mathcal{O}_Y(L|_Y)(\mu H)}_{\substack{H^0 \text{ of this } \neq \{0\} \\ \text{by induction}}} \to 0$$

with associated l.e.s.

$$\underbrace{H^0(X, \mathcal{O}_X(L)(\mu H))}_{(*)} \xrightarrow{\alpha} \underset{\{0\}}{\underset{\neq}{H^0(Y, \mathcal{O}_Y(L|_Y)(\mu H))}} \to H^1(X, \mathcal{O}_X(L)((\mu-1)H))$$

$$H^1(X, \mathcal{O}_X(L)((\mu-1)H)) = H^1(X, \Omega^n_X(\underbrace{K_X^{-1} \otimes L \otimes H^{\otimes(\mu-1)}}_{>0 \text{ for } \mu \gg 0})) = \{0\} \text{ by } 6$$

$$(*) \neq \{0\} \impliedby \alpha \text{ surjective} \impliedby \{0\}$$

(Here $L \otimes H^{\otimes \mu}$ has a holo. section $(\neq 0)$. But $H^{\otimes \mu} = \mathcal{O}(\mu)$ obviously has one too; dividing them gives the desired mero. section of L.)

$\square$