Problem Set 1 (sol'n sketches)

(1) Since M is compact and f (being holo.) continuous, it attains a (absolute value) maximum in some $\mathcal{U}_\alpha \cong \mathcal{D} \subset \mathbb{C}^n$. By (iii) on p.11 (maximum principle), $f|_{\mathcal{U}_\alpha}$ is constant. Using (ii) on p.11 + connectedness of M, we see that it has to be constant everywhere.

(2) The map $\bigoplus_{p+q=k} \Lambda^p V \otimes \Lambda^q W \xrightarrow{\theta} \Lambda^k (V \oplus W)$ is surjective because any element of the RHS can be written $\sum_i (v_1^i + w_1^i) \wedge \cdots \wedge (v_k^i + w_k^i)$, expanded by bilinearity, then the v's moved (by alternating) so they all come first — hence

$$\Lambda^k V \oplus W = \sum_{p+q=k} \Lambda^p V \wedge \Lambda^q W \quad (= \text{image}(\theta)).$$

To see injectivity, let $\begin{cases} e_1, \ldots, e_m & \text{be a basis for } V \\ e_{m+1}, \ldots, e_{m+n} & \text{be a basis for } W. \end{cases}$

A basis element for $\Lambda^p V \otimes \Lambda^q W$ maps to an element of the form $e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}$, where $\underbrace{i_1 < \cdots < i_p}_{\in [1,m]} < \underbrace{j_1 < \cdots < j_q}_{\subseteq [m+1, n]}$, i.e. a basis element of $\Lambda^k (V \oplus W)$.

Clearly the forms of these basis elements imply linear independence of the $\theta(\Lambda^p V \otimes \Lambda^q W)$ (for diff. p & q) in $\Lambda^k(V \oplus W)$, so we are done.

NOTE: A neat corollary of (2) is the identity

$$\boxed{\sum_{p+q=k} \binom{n}{p}\binom{m}{q} = \binom{n+m}{k}.}$$

(3) (a) Writing $\bar{I}$ for I (= multi-index) backwards,

$$\left(\frac{\partial}{\partial x_j} \lrcorner\, dx_I\right)\left(\frac{\partial}{\partial x_{\widetilde{I \setminus \{j\}}}}\right) \underset{\substack{\text{defn. of } \lrcorner \\ \text{on p. 2 of} \\ (X.4)}}{=} dx_I\left(\frac{\partial}{\partial x_{\widetilde{I\setminus\{j\}}}} \wedge \frac{\partial}{\partial x_j}\right) = (-1)^{\ell-1} dx_I\left(\frac{\partial}{\partial x_{\tilde{I}}}\right) = (-1)^{\ell-1}$$

if j appears as i_ℓ,

while

$$\left(\frac{\partial}{\partial x_j} \lrcorner\, dx_I\right)\left(\frac{\partial}{\partial x_{I'}}\right) = dx_I\left(\frac{\partial}{\partial x_{I'}} \wedge \frac{\partial}{\partial x_j}\right) = 0 \quad \text{unless } \begin{cases} j \notin I' \\ \{j\} \cup I' = I. \end{cases}$$

We conclude $\dfrac{\partial}{\partial x_j} \lrcorner\, dx_I = (-1)^{\ell-1} dx_{I \setminus \{j\}}$.

(b) We can do this on the level of monomial terms.

$$\xi = \frac{\partial}{\partial x_k}, \quad \mu = dx_I, \quad \omega = dx_J$$
$$|I| = \ell \qquad |J| = p$$

We may $\begin{cases} \text{assume } I \cap J = \emptyset \text{ so } \mu \wedge \omega \neq 0 \\ \text{Choose an ordering so that } I < J. \end{cases}$

If k appears as ℓ^{th} term in I,

$$\frac{\partial}{\partial x_k} \lrcorner\, (\underbrace{dx_I \wedge dx_J}_{dx_{I \cup J}}) \underset{(a)}{=} (-1)^{\ell-1} dx_{I \setminus \{k\} \cup J} = \underbrace{(-1)^{\ell-1} dx_{I\setminus\{k\}}}_{\left(\frac{\partial}{\partial x_k} \lrcorner\, dx_I\right)} \wedge dx_J$$

while if k appears as ℓ^{th} entry in J,

$$\frac{\partial}{\partial x_k} \lrcorner\, (\underbrace{dx_I \wedge dx_J}_{dx_{I \cup J}}) \underset{(a)}{=} (-1)^{\ell+\ell-1} dx_{I \cup J \setminus\{k\}} = (-1)^{\ell} dx_I \wedge \underbrace{(-1)^{\ell-1} dx_{J \setminus\{k\}}}_{\left(\frac{\partial}{\partial x_k} \lrcorner\, dx_J\right)}.$$

(4) In coords. (x,y), write $\begin{cases} \omega = f\,dx + g\,dy \\ \xi^i = F^i\,\partial/\partial x + G^i\,\partial/\partial y \quad (i=1,2) \end{cases}$

$$\overbrace{d\omega\,(\underbrace{\xi^0,\xi^1}_{\text{really } \xi^0\wedge\xi^1})}^{\text{LHS}} = \left((g_x - f_y)\,dx\wedge dy\right)\left((F^0G^1 - G^0F^1)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)$$

$$= (g_x - f_y)*(F^0G^1 - G^0F^1).$$

Checking the "RHS" = this, is a bit painful. First of all, the RHS is

$$\xi^0(\omega(\xi^1)) - \xi^1(\omega(\xi^0)) - \omega([\xi^0,\xi^1])$$

where
$$[\xi^0,\xi^1] = (F^0\partial/\partial x + G^0\partial/\partial y)(F^1\partial/\partial x + G^1\partial/\partial y) - (F^1\partial/\partial x + G^1\partial/\partial y)(F^0\partial/\partial x + G^0\partial/\partial y)$$
$$= (F^0F^1_x + G^0F^1_y - F^1F^0_x - G^1F^0_y)\,\partial/\partial x + (F^0G^1_x + G^0G^1_y - F^1G^0_x - G^1G^0_y)\,\partial/\partial y$$

NOTE: Casey B. had a better proof. You might ask to see it!

... rest is left to you (there are lots of cancellations).

~~Better idea: do a change of coordinates so that $\omega = f$~~

(5) We need only (as usual) check the formula on monomials.

Define $\Pi_*(f\,dt\wedge dx_I) := \left(\int_0^1 f(t,\underline{x})\,dt\right)dx_I$, $\Pi_*(f\,dx_I) := 0$.

Then
$$\Pi_*(d(f\,dx_I)) + d(\underbrace{\Pi_*(f\,dx_I)}_{0}) = \Pi_*\Big(\underbrace{\sum_{j\notin I}\frac{\partial f}{\partial x_j}dx_j\wedge dx_I}_{0}\Big)$$
$$+\ \Pi_*\left(\frac{\partial f}{\partial t}dt\wedge dx_I\right) = \left(\int_0^1\frac{\partial f}{\partial t}dt\right)dx_I = (f(1,\underline{x}) - f(0,\underline{x}))\,dx_I$$
$$= I_1^*(f\,dx_I) - I_0^*(f\,dx_I).$$
A similar computation on the $f\,dt\wedge dx_I$'s yields 0, which is indeed correct as $I_1^*(f\,dt\wedge dx_I) = 0$.
or 0

(from plus/minus of $\sum_{j\notin I}\left(\int_0^1\frac{\partial f}{\partial x_j}dt\right)dx_j\wedge dx_I$)

(6) I'll do this one in more generality. Let

$M = \mathbb{R}^n/\mathbb{Z}^n \ni p = (0)$

$G = \mathbb{R}^n/\mathbb{Z}^n$ acting by translation on M, hence (by pullback) on forms on M

$V = \Lambda^k T_p^\vee M = \mathbb{R}\langle\{dz_{i_1}|_p \wedge \cdots \wedge dz_{i_k}|_p \mid i_1 < \cdots < i_k\}\rangle.$

Sending $dz_I|_p \mapsto dz_I$ induces an isomorphism

$V \xrightarrow{\cong} (A^k(M))^G$.

Now consider the operation of "averaging" a form:

$\eta \mapsto \eta^G := \int_G g^*\eta \; d\mu(G)$.

Since the g^*'s are all homotopic to identity, if $d\eta = 0$ then $[\eta^G] = [\eta]$ in de Rham cohomology. Moreover, $d(dz_I) = 0$ $\Rightarrow$ all G-invariant forms are closed. Hence, we have a surjective map

$(A^k(M))^G \longrightarrow H^k_{dR}(M, \mathbb{R})$.

But it's also injective, for if $\omega \overset{\neq 0}{\in} (A^k(M))^G$ has $\omega = d\eta$, then "averaging both sides" gives $\omega = d(\eta^G) = 0$ ※.

Hence $V \cong H^k_{dR}(M, \mathbb{R})$,

and $\dim V = \dim \Lambda^k \mathbb{R}^n = \boxed{\binom{n}{k}}$

Many of you tried Mayer–Vietoris, which is OK. But note that this would be inconvenient at the above level of generality!