Problem Set 2 (solutions)

1) \( \omega' = \omega + \delta \gamma \Rightarrow \int (\omega' \wedge \eta) = \int (\omega \wedge \eta + \omega \wedge \delta \gamma) = 0 \)

\( \omega' = \omega + \delta \gamma \Rightarrow \omega' \wedge \eta' = \omega \wedge \eta + \omega \wedge \delta \gamma + \delta \gamma \wedge \eta' = \omega \wedge \eta + \delta \gamma \wedge \eta' + \omega \wedge \beta \)

2) Since \( d\omega = 0 \), \( d -\text{Poincaré} \Rightarrow \int_{\Omega} \omega \wedge \eta \subset \Omega \) where \( \omega = d\alpha \) (for some \( 1\)-form \( \alpha \)).

So we can write \( \omega = d\alpha = d\beta + \delta \gamma + \delta \beta + \delta \bar{\beta} \),

but since \( \omega \) is of type \((1,1)\),

\( \omega = \delta \beta + \delta \bar{\beta} \) and \( \delta \beta = 0 = \delta \bar{\beta} \).

By \( d -\text{Poincaré} \), \( \int_{\Omega} \omega \wedge \eta \subset \Omega \) where \( \bar{\beta} = \bar{\delta}f \) for some \( C\)-valued \( C \) is of type \((1,2)\),

But then,

\( \omega = \delta \beta f + \delta \bar{\beta} f = -\delta \beta \bar{\beta} + \delta \bar{\beta} f = \delta \bar{\beta} (f \bar{f}) \)

\( = 2i \cdot \delta \bar{\beta} (\text{Im} f) \).

3)(a) Let \( \{\psi_i\} \) be a partition of unity and \( \{\xi_i^1, \ldots, \xi_i^m\} \) systems of coordinates at the \( (\phi_i^\beta) \)'s (the Jacobians) have determinants > 0.

Then set \( \omega := \xi_i^1 \wedge \cdots \wedge \xi_i^m \), and note that

\( \omega = \)
\[
\omega|_{\phi_\beta} = \sum_{\alpha \neq \beta} \eta_\alpha \phi_\alpha (dx_1 \wedge \ldots \wedge dx^\alpha) + \eta_\beta \, dx_1 \wedge \ldots \wedge dx^\beta
\]

where \(\eta_\alpha\) are defined

\[
= \left( \sum_{\alpha \neq \beta} \eta_\alpha \det (\partial \phi_\alpha / \partial \chi) \right) + \eta_\beta \, dx_1 \wedge \ldots \wedge dx^\beta
\]

and clearly \(\sum \eta_\alpha \) is positive. It follows that \(\int_M \omega = 0\).

Now \(d\omega = 0\) since \(\omega\) is of top degree, and we cannot have \(\omega = d\eta\) since then \(0 < \int_M \omega = \int_M d\eta = \int_M \eta = 0\).

(b) We can choose \(C^\infty\) coordinates \(\{x^i\}\) about any point \(p \in U_a\) such that

\[
J_a(\partial x_i / \partial x^m) = \partial x_i / \partial x^m \quad \text{at} \quad p,
\]

but not on the whole neighborhood. However, replacing the bases with respect to which we compute \(\det (\partial \phi_\alpha / \partial x^m)\) by another \(\{x_i\}\) (equal to \(\{x_i^m\}\) at \(p\), but not necessarily integrable) does not change the signs of the determinants. Then for every \(\alpha\) we have simply \([J_a] \phi_\alpha = (0, -I)\), and the compatibility conditions

\[
J_a \circ (\phi_\beta)_* = (\phi_\alpha)_\circ J_\beta
\]

read \(M_{\alpha \beta} = (\partial \phi_\beta / \partial \chi) \circ (\partial \phi_\alpha / \partial \chi)\) commute with \((0, -I_n)\),

\[M_{\alpha \beta} \text{ is of the form } \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \Rightarrow \det M_{\alpha \beta} = |\det (A+Bi)|^2 > 0.
\]

4) (a) antisymmetry is obvious and for \(f \in C^\infty(M)\),

\[
N(f, x, y) = f N(x, y) - (f(y) \chi(x) + y(f) \chi(x) + \chi(f) \chi(x))/x \quad \text{using } \chi.
\]

\[
(f(x, y) = x, y) = f N(x, y)
\]

which demonstrates (in one side, have the other)

\(C^\infty(M) - \text{linearity}.\)
(6) Recall that on $T_{i0} \text{ resp. } T_{0i}$, $J$ is multi- by i resp. $(\cdot)$. So if $x \in T_{i0}$, $y \in T_{0i}$, $N(x, y) = \frac{[ix, -iy] - [x, y]}{J([ix, y] - J[x, iy])} = 0$.

Now if $x, y \in T_{i0}$ then

$0 = N(x, y) = \frac{[ix, iy] - [x, y] - J([ix, y] - J[x, iy])}{-2[x, y] - 2iJ[x, iy]}$

$\iff i[x, y] = J[x, iy] \iff [x, y] \in T_{i0}$.

The same thing holds for $T_{0i}$. We conclude that

$N = 0 \iff (T_{i0}, T_{0i}) \subseteq T_{i0} \iff J$ is integrable.

(And its "conjugate".) (Neuhausen)

(Quasi-sketch)

$m = 8^2 = 6$

By the CMA-bilinearity of $N$ it turns out that we can compute it (i.e. $N$) for $J$ on the space of imaginary quaternion vectors and then evaluate on a pair of vectors in $TS^5$ resp. $TS^5$ at a point. This is much easier than trying to work on the sphere. Goal is simply to show that, for $S^6$, there is a point $p \in S^6$ of a pair of vectors $TS^6$ on which $N$ gives a nonzero answer. Also, I haven't had time to sit down with a table of the octonions...