Problem Set # 3 (Solutions)

(1) (i) On the trivializations $U_\alpha \times \mathbb{C}^n \cong \pi^{-1}(\mathcal{U}_\alpha) \subset E$, there is a Hermitian metric h_α, just by putting one on the $\mathbb{C}^n$. Now consider $h = \sum \eta_\alpha h_\alpha$, where $\begin{cases} \sum \eta_\alpha \equiv 1 \\ \eta_\alpha \in C^\infty(\mathcal{U}_\alpha) \\ \eta_\alpha \geq 0 \end{cases}$. It is clearly still sesquilinear ($\mathbb{C}$-linear on left; $\mathbb{C}$-antilinear on right), since the transition maps preserve the $\mathbb{C}$-vector-space structure of fibers of E. The issue is whether it is still positive definite on the overlaps, which is fine: $h(\sigma, \sigma) = \sum \eta_\alpha h_\alpha(\sigma, \sigma) > 0$ (for $\sigma \neq 0$).

(ii) If σ_α are generating (nowhere-zero) sections of L over the $\mathcal{U}_\alpha$, then $\sigma_\alpha^\vee$ (defined by $\sigma_\alpha^\vee(\sigma_\alpha) \equiv 1$) are sections of $L^\vee$. If h is a metric on L, we can define a metric h^* on $L^\vee$ by

$h^*(f,f) \cdot h(\sigma,\sigma) = |f(\sigma)|^2$; this gives

$$\rho_\alpha^* := h^*(\sigma_\alpha^\vee, \sigma_\alpha^\vee) = \frac{1}{h(\sigma_\alpha, \sigma_\alpha)} = \frac{1}{\rho_\alpha}, \text{ and so}$$

$$\partial\bar\partial \log \rho_\alpha^* = -\partial\bar\partial \log \rho_\alpha \Longrightarrow c_1(L^\vee) = -c_1(L).$$

(2) (i) TWO PROOFS

1st proof: $\sigma \in \Gamma(\mathbb{P}^n, \mathcal{O}(-1))$ is a holo map

$$\sigma : \mathbb{P}^n \longrightarrow \mathbb{C}^{n+1} \quad \text{s.t.} \quad [\sigma([\underline{z}])] = [\underline{z}].$$

Let $\langle , \rangle$ = Euclidean inner product on $\mathbb{C}^{n+1}$ and $\underline{e} := (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$. Then $F([\underline{z}]) := \langle \underline{e}, \sigma(\underline{z}) \rangle$ defines some $F \in \mathcal{O}(\mathbb{P}^n)$, which by MMP must be constant. But for (say) $[\underline{z}_0] = [0 : \cdots : 0 : 1]$, $\sigma([\underline{z}_0]) = (0, \dots, 0, \mu)$ hence $F([z_0]) = 0$. So $F \equiv 0$

$$\Rightarrow 0 = F([1 : w_1 : \cdots : w_n]) \Rightarrow \sigma = 0 \text{ on } U_0 \Rightarrow \sigma \equiv 0.$$

2nd proof: $\sigma \in \Gamma(\mathbb{P}^n, \mathcal{O}(-1))$ consists of $\{\sigma_i \in \mathcal{O}(U_i)\}$ s.t. $\sigma_j = \frac{z_j}{z_i} \sigma_i$ on U_{ij}. But then $\frac{z_0}{z_1} \sigma_1$ is holo on U_1 and agrees with σ_0 on U_{01} $\Longrightarrow$ they glue to give $F \in \mathcal{O}(U_0 \cup U_1) = \mathcal{O}\left(\mathbb{P}^n \setminus \left\{\begin{smallmatrix}\text{codimension} \\ \text{2 subset}\end{smallmatrix}\right\}\right)$. By Hartogs, F extends to $\tilde{F} \in \mathcal{O}(\mathbb{P}^n)$; then by MMP $\tilde{F}$ is constant. But $C = \frac{z_0}{z_1} \sigma_1 \Longrightarrow C \frac{z_1}{z_0} = \sigma_1$ on U_1 $\Longrightarrow$ σ_1 has pole at $\{z_0 = 0\} \cap U_1$ (contradiction) unless $C = 0$. So $C = 0$, which implies $\sigma \equiv 0$.

these are superscripts not powers

(2)(ii) It's instructive to do this by brute force in local coordinates $(z^i_1, \dots, z^i_n) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$ on $U_i = \{z_i \neq 0\} \subset \mathbb{CP}^n$. The transition functions are (yuck!)

$$z^j_k = \begin{cases} z^i_k / z^i_{j+1}, & 1 \leq k \leq j \\ z^i_{k+1} / z^i_{j+1}, & j < k < i \\ 1/z^i_{j+1}, & k = i \\ z^i_k / z^i_{j+1}, & i < k \leq n \end{cases}$$

A local nonvanishing section of $K_{\mathbb{P}^n}$ on U_j (i.e., a local trivialization of this line bundle) is

$$(-1)^j dz^j_1 \wedge \dots \wedge dz^j_n \underset{\substack{\uparrow \\ \text{on } U_{ij}}}{=} (-1)^j d\left(\frac{z^i_1}{z^i_{j+1}}\right) \wedge \dots \wedge d\left(\frac{z^i_j}{z^i_{j+1}}\right) \wedge d\left(\frac{z^i_{j+2}}{z^i_{j+1}}\right) \wedge \dots$$

$$\wedge\, d\left(\frac{z^i_i}{z^i_{j+1}}\right) \wedge d\left(\frac{1}{z^i_{j+1}}\right) \wedge d\left(\frac{z^i_{i+1}}{z^i_{j+1}}\right) \wedge \dots \wedge d\left(\frac{z^i_n}{z^i_{j+1}}\right)$$

$$\uparrow = -\frac{dz^i_{j+1}}{(z^i_{j+1})^2}$$

Can't apply d to z^i_{j+1} in the other terms b/c of "$\wedge$"

$$= (-1)^j \cdot \frac{-1}{(z^i_{j+1})^{n+1}} dz^i_1 \wedge \dots \wedge dz^i_j \wedge dz^i_{j+2} \wedge \dots \wedge dz^i_i \wedge dz^i_{j+1} \wedge dz^i_{i+1} \wedge \dots \wedge dz^i_n$$

need to move to here

$$= (-1)^{i-j-1} (-1)^j \frac{(-1)}{(z^i_{j+1})^{n+1}} dz^i_1 \wedge \dots \wedge dz^i_n$$

$$= \left((-1)^i dz^i_1 \wedge \dots \wedge dz^i_n\right) \times \left(\frac{z_i}{z_j}\right)^{n+1}$$

$\leftarrow$ identifies with $\frac{1}{z^i_{j+1}}$

in effect, all I've done here is to compute the determinant of $(\phi_{ij})_*$

When one has nowhere-zero sections σ_i over $\mathcal{U}_i$, an arbitrary section is a collection $\{f_i \sigma_i\}$ agreeing on overlaps. So if $\frac{\sigma_i}{\sigma_j} = \left(\frac{z_j}{z_i}\right)^{n+1}$ (as here), $\Phi_{ij} := \frac{f_i}{f_j} = \left(\frac{z_i}{z_j}\right)^{n+1}$, which are →

flip

precisely the transition functions for $\mathcal{O}(-(n+1))$. Bundles with the same transition functions identify, and so $K_{\mathbb{P}^n} \cong \mathcal{O}(-(n+1))$.

(3) (i) I'll be excessively detailed here. Consider the composite map of sheaves (on N)

$$\begin{tikzcd}
\underbrace{\Lambda^n T_M^{\vee(1,0)}|_N}_{\text{i.e. } K_M|_N} \otimes T_M^{1,0}|_N \arrow[r, "\lrcorner"] \arrow[rr, bend right, squiggly, "\Theta"'] & \Lambda^{n-1} T_M^{\vee(1,0)}|_N \arrow[r, "\text{pullback of forms}"] & \underbrace{\Lambda^{n-1} T_N^{\vee(1,0)}}_{\text{i.e. } K_N}
\end{tikzcd}$$

That

$$0 \to K_M|_N \otimes T_N^{1,0} \hookrightarrow K_M|_N \otimes T_M^{1,0}|_N \xrightarrow{\Theta} K_N \to 0$$

is (everywhere) short-exact can be checked in local coordinates, e.g. $(z_1, \ldots, z_n)$ s.t. $z_n = 0$ defines $N \subset M$. (You have to "contract away" the dz_n to get something pulling back to $\neq 0$ on N.) (here, contract against $\partial/\partial z_n$)

So

$$K_M|_N \otimes \underbrace{\frac{T_M^{1,0}|_N}{T_N^{1,0}}}_{\mathcal{N}_{N/M} \text{ (holo normal bundle!)}} \cong K_N .$$

(ii) Same game as in 2(ii) but easier. Local generators of $\mathcal{N}^\vee_{X/\mathbb{P}^n}$ (= the conormal bundle) are given by

$$\sigma_j := df_j \in \Gamma(U_j \cap X, \mathcal{N}^\vee_{X/\mathbb{P}^n})$$

(note: $\mathcal{N}^\vee_{X/\mathbb{P}^n} \subset \Omega^1_{\mathbb{P}^n}|_X$)

where $F \in S^d_{n+1}$ is the homogeneous polynomial defining* X and $f_j := \frac{F}{Z_j^d} \in \mathcal{O}(U_j)$ are the local equations defining** $X \cap U_j$. We clearly have $f_j = \left(\frac{Z_i}{Z_j}\right)^d f_i$ on U_{ij}, and so on $U_{ij} \cap X$

$$\sigma_j = df_j = d\left(\left(\frac{Z_i}{Z_j}\right)^d f_i\right) = \underset{0\ (f_i = 0 \text{ on } X \cap U_i)}{\cancel{f_i \cdot d\left(\left(\frac{Z_i}{Z_j}\right)^d\right)}} + \left(\frac{Z_i}{Z_j}\right)^d \underbrace{df_i}_{\sigma_i}$$

$$= \left(\frac{Z_i}{Z_j}\right)^d \sigma_i .$$

By the same logic as at the end of 2(ii), we get

$$\mathcal{N}^\vee_{X/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-d)\big|_X$$

$$\Longrightarrow \mathcal{N}_{X/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(d)\big|_X$$

$$\underset{(i)}{\Longrightarrow} K_X \cong \underbrace{K_{\mathbb{P}^n}|_X}_{\mathcal{O}(-(n+1))} \otimes \underbrace{\mathcal{N}_{X/\mathbb{P}^n}}_{\mathcal{O}(d)} \cong \mathcal{O}_{\mathbb{P}^n}(d-n+1)\big|_X \quad \left(\text{or just } ``\mathcal{O}_X(d-n+1)"\right)$$

(iii) $d = n+1 \Rightarrow K_X \cong \mathcal{O}_X$ (is trivial), so $\exists$ a nowhere vanishing section (take the image of $1 \in \mathcal{O}(X)$ under the $\cong$). Moreover, $\Omega^{n-1}(X) = H^0(K_X) = H^0(\mathcal{O}_X) = \mathbb{C}$, as desired.

---

* i.e. $X := \{F = 0\}$

** i.e. $X \cap U_j = \{f_j = 0\}$

(4) $$\nabla_{\partial/\partial x_i} \partial/\partial x_j = \sum \Gamma^{\ell}_{ji} \partial/\partial x_\ell$$

$$\nabla_{\partial/\partial x_j} \partial/\partial x_i = \sum \Gamma^{\ell}_{ij} \partial/\partial x_\ell .$$

so $$\nabla_{\partial/\partial x_i} \partial/\partial x_j - \nabla_{\partial/\partial x_j} \partial/\partial x_i = [\partial/\partial x_i, \partial/\partial x_j] \ (=0)$$

$\Longrightarrow$ all $T^{\ell}_{ji} = \Gamma^{\ell}_{ji} - \Gamma^{\ell}_{ij} = 0$.

Conversely, if T^{ℓ}_{ji} vanish, then for arbitrary vector fields

$X = \sum_i f_i \partial/\partial x_i$, $\xi = \sum g_j \partial/\partial x_j$, we have

$$\nabla_X \xi - \nabla_\xi X = \sum \underset{0}{\cancel{T^{\ell}_{ji}}} f_i g_j \partial/\partial x_\ell$$
$$+ \sum X(g_j) \partial/\partial x_j - \sum \xi(f_i) \partial/\partial x_i$$
$$= [X, \xi] .$$

(5) Recall $B_0 = \{Z_0 z_1 = Z_1 z_0\} \subset \mathbb{C}^2 \times \mathbb{P}^1$

This is covered by charts:

(B₀ = union)

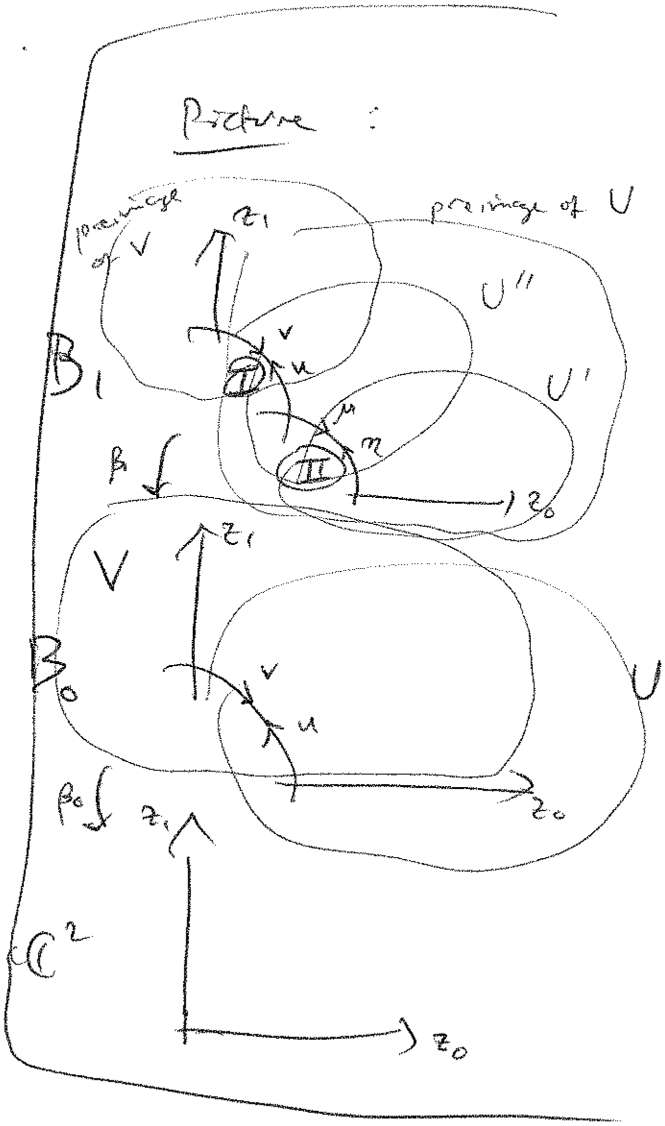
$U = \{Z_0 \neq 0\}$ w/ coords $(z_0, \overset{Z_1/Z_0}{u})$

$V = \{Z_1 \neq 0\}$ w/ coords $(\overset{Z_0/Z_1}{v}, z_1) \underset{\substack{\uparrow \\ \text{transition} \\ \text{on } U \cap V}}{=} (\frac{1}{u}, u z_0)$.

The map to $\mathbb{C}^2$ is given by

$$\left.\begin{array}{l} (z_0, u) \longmapsto (z_0, u z_0) \\ (v, z_1) \longmapsto (v z_1, z_1) \end{array}\right\} =: (z_0, z_1)$$

For the further blow-up, rather then writing everything out again, just repeat by noting $U = \mathbb{C}^2$ (the blow-up of "0" $\in U$ won't affect V).



Now compute pullbacks of $dz_0 \wedge dz_1$ to U: $dz_0 \wedge d(uz_0) = z_0 dz_0 \wedge du$ and V: $d(vz_1) \wedge dz_1 = z_1 dv \wedge dz_1$. which in each case vanishes to 1st order along the "exceptional divisor" $\beta_0^{-1}(\underline{0})$.

For the second blow-up, we send

$$\left.\begin{array}{l} U' \ni (z_0, \eta) \mapsto (z_0, z_0 \eta) \\ U'' \ni (\mu, u) \mapsto (\mu u, u) \end{array}\right\} = (z_0, u) \in U$$

so $z_0 dz_0 \wedge du$ becomes

$z_0 dz_0 \wedge d(z_0 \eta) = z_0^2 dz_0 \wedge d\eta$ (on U')

$\mu u\, d(\mu u) \wedge du = \mu u^2 d\mu \wedge du$ (on U''),

i.e. $\beta_1^* \beta_0^* (dz_0 \wedge dz_1)$ vanishes to 1st order on (I) and to 2nd order on (II).

see picture