Problem Set 4 (Solutions)

(1) \[ 0 \to \beta \to \gamma \to \delta \to \epsilon \to 0 \]

Given \( c \in C \), \( j_4(g^{-1}(i_3(c))) = i_4(k(c)) = 0 \Rightarrow \[
\exists \bar{c} \in Y \ s.t. \ j_3(\bar{c}) = g^{-1}(i_3(c)) = \]
\[ l_3(f(\bar{c})) = g(j_3(\bar{c})) = i_3(c) \Rightarrow l_3(f(\bar{c})-c) = 0 \Rightarrow \]
\[ \exists b \in B \ s.t. \ i_2(b) = f(\bar{c})-c \text{ and } b \text{ s.t. } e(b) = b \Rightarrow \]
\[ f(\bar{c}-j_2(b^\ast)) = f(\bar{c}) - f(j_2(\bar{c}^\ast)) = f(\bar{c}) - (i_2(e(b))) = f(\bar{c}) - (f(c) - c) = c. \]

(2) (a) \[ 0 \to A \to B \to C \to 0 \text{ exact} \]

\[ 0 \to \operatorname{Hom}(X, A) \to \operatorname{Hom}(X, B) \to \operatorname{Hom}(X, C) \]

To prove exactness here, main point is: given \( f \in \operatorname{Hom}(X, B) \) with \( \beta \circ f = 0 \), define \( g \in \operatorname{Hom}(X, A) \) by

\[ g(x) = \text{unique element of } A \text{ mapping by } \alpha \text{ to } f(x) \left( \exists \alpha \text{ s.t. } \beta(f(c)) = 0 \right) \]

by injectivity of \( \alpha \).
(b) $0 \to \mathbb{Z}_m \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$

(c) \[ \text{Ext}^1(\mathbb{Z}_m, \mathbb{Z}_n) = \left( R^1 \text{Hom}(\mathbb{Z}_m, -) \right)(\mathbb{Z}_n) \]

\[ = \frac{H^1 \{ \text{Hom}(\mathbb{Z}_m, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}_m, \mathbb{Q}/\mathbb{Z}) \}}{n \cdot \text{Hom}(\mathbb{Z}_m, \mathbb{Q}/\mathbb{Z})} \]

identified with \( \left\{ \frac{\mathbb{Z}}{m} \left| \alpha \in \mathbb{Q}/\mathbb{Z} \right. \right\} \)

With (b) or when \( \mathbb{Z}_m \) is exact

\[ = \frac{\mathbb{Z}_m}{n \cdot \mathbb{Z}_m} \cong \mathbb{Z}(m, n). \]

(3)

- For surfaces \( X \) and divisors with support in \( X \).

These are also divisors with support in any open subset \( U \), so \( D(W)\rightarrow D(U) \). Note that this is only true for \( \dim M = 1 \)

- We have \( 0 \to \Omega^1 \to M^* \to D \to 0 \) (show \( M \))

So \( 0 \to H^0(\Omega^1) \to H^0(M^*) \to \Gamma^0(D) \to H^1(\Omega^1) \to H^1(M^*) \to H^2(D) \)

\[ = 0 \to 0 \to \mathfrak{M}(M)^* \to \text{Div}(M) \\xrightarrow{\Delta} \text{Div}(M) \]

\[ \xrightarrow{\text{splitly}} \beta = 0 \iff H^1(M^*) = 0 \]

and \( \Delta \) surjective \( \Rightarrow \beta = 0 \iff H^1(M^*) = 0 \)

...what we want

- need a little argument to show that \( \Delta \) sends a divisor \( D \to \Theta(D) \). Done in Čech cohomology, this is almost a tautology (left to you).
(4) Identity \( \omega(D) = H^0(X(-D)) \).

(a) \( \omega \in \mathcal{K}^1_m \), \( \omega + \mathcal{O}(D) \) then

\( \omega \geq 0 \), which yields

\[ 2g - 2 = \deg(\omega) \geq \deg D \, \text{ at } \mathcal{O} \]

If \( \deg D > 2g - 2 \), this is impossible since there are no (nonzero) meromorphic forms in \( \omega \).

(b) Likewise, if \( f \in \mathcal{O}(D) \) then \( \deg f + D \geq 0 \)

\[ \Rightarrow 0 = \deg(f) \geq -\deg D \]

so if \( \deg D < 0 \) (i.e., \( -\deg D > 0 \)), we again get a contradiction.

(5) (a) \( p \in M \)

\[ \lambda((g_1)(p)) = \overline{(g_1)(p))} + (g_1) - g_1 \geq 2 \]

\[ \Rightarrow \lambda((g_1)(p)) \text{ contains a nonconstant term } f \]

The mapping degree is the "Cardinality with multiplicity" of \( f^{-1}(x) \) for any \( x \in \mathcal{P} \). For \( x = 0 \) as to that number counting order of 0's or poles.

Now \( f \) has only pole at \( p \), so

\[ \deg(f) = \deg_0(f) \leq g + 1 \]

or \( \deg(f_{\{x_0\}}) \).
(b) By Riemann-Roch, for any $p \in M$,

$$\ell((g-2)[p]) = g - (g-2) - 1 + \chi((g-2)[p])$$

$$\geq 1 \quad \text{(constans constant term)}$$

$$\geq 2.$$ 

Hence, there are 2 linearly independent holomorphic forms $\omega_1, \omega_2$ with $\ell((2j - (g-2))[p]) = D_j > 0 \quad (j = 1, 2)$, and $\deg D_j = (2g - 2) - (g-2) = g$. Set $f := \frac{\omega_1}{\omega_2} \in M(M)^*$, so that $(f) = D_1' - D_2' = D_1 - D_2$ where $D_1', D_2'$ are effective divisors of degree $\leq g$ with no points in common.

We have that the mapping degree

$$\deg(f) = \deg(f^{-1}(0)) = \deg D_1' \leq g.$$