Algebraic cycles and representation theory

PART I: Algebraic cycles and their Hodge-theoretic invariants

A. Cycle groups

1. Operations on cycles

Let \( k = \overline{k} \) be a field (algebraically closed), \( X/k \) a smooth quasi-projective variety of dimension \( d \). We define the groups of algebraic cycles:

**Definition 1:** \( Z_q(X) = \mathbb{Z}^{d-q}(X) := \) the free abelian group generated by subvarieties \( W \subseteq X \) (irred over \( k \)) of dim \( q \) (cd. \( d-q \))

\[
\mathbb{Z} = \sum_{m \in \mathbb{N}} V_i \quad (m; \in \mathbb{Z})
\]

Given a morphism \( f: X \to Y \) of varieties, we define the pushforward of cycles by

\[
f_*: Z_q(X) \to Z_q(Y)
\]

\[
W \mapsto \begin{cases} 0 & \text{if } \dim f(W) < \dim W \\ (\text{irred}) \left\lceil \frac{\text{deg}(W) \cdot \text{dim}(W)}{\text{dim}(f(W)) \cdot \text{dim}(f(W))} \right\rceil & \text{if } \dim f(W) = \dim W \end{cases}
\]

This preserves dimension.

Next, given two cycles \( Z_1 \) and \( Z_2 \), we want to define an intersection product. We define on the level of irreducible subvarieties \( V, W \subseteq X \) of cd. \( i \) resp. \( j \), then extend by linearity.
First write $V \cdot W = UV^j$, $V$ mod. of cd. $< i + j$.

This intersection is proper $\implies$ cd $V = i + j (V)$, in which case

$$i_1 (V \cdot j; V \cdot W ; x) := \sum_{r=0}^{\ell} \{ \text{Tor}_r \left( \mathcal{O}/I_V \mathcal{O}/I_W \right) \}$$

and

$$V \cdot W := \{ i_1 (V \cdot j; V \cdot W ; x) \} V_x \in \mathcal{Z}^{i+j} (X).$$

There is a more geometric way to understand the intersection multiplication (2) : given a fiber space

$$3 \subset f^{-1} (W) \to \gamma \to \left( \begin{array}c \text{mapping into} \ V \\ \text{et of same dimension} \\ \text{e.g. resolution of singularities} \end{array} \right)$$

$$W \to X \quad \text{regular ends of codim. j}$$

with $V$, $W$ as above, write $I_X \subset \mathcal{O}_{\gamma} \mathcal{O}_{\gamma}$ for the ideal of $\gamma$.

The quotient has finite length and

$$1 \leq i_1 (f_{\gamma}, V \cdot W ; x) \leq \ell \left( \mathcal{O}_{\gamma} \mathcal{O}_{\gamma}/I_X \right)$$

(this is one of the $V_x$'s)

when (x) is an equality if $3 \to V$ is regular.

Ex/\n
$X = \mathbb{A}^4$

$V = \text{image of } f : \mathbb{A}^2 \to \mathbb{A}_4$

$W = \mathbb{A}_4$

$\sigma = \text{point } \sigma \in \mathbb{A}^2$, so (x) is equality

$$\Rightarrow i_1 (\sigma; V \cdot W ; x) = \ell \left( \mathcal{O}_{\sigma} \mathcal{O}_{\sigma}/I_{\sigma^{-1} (W)} \right) = \ell (k[s, t]/(s^2, t^2)) = 16.$$
Now we turn to cycle-theoretic pullback along a morphism \( f : X \to Y \). Write \( \pi_X, \pi_Y \) for the projections \( X \times Y \to X \) and \( \Gamma_f \subset X \times Y \) for the graph of \( f \). Given any \( z \in \mathcal{Z}^p(Y) \) for which
\[
(5) \quad \Gamma_f \cap (X \times z) \text{ is proper},
\]
we may define
\[
(6) \quad f^*(z) := (\pi_X)_* \{ \Gamma_f \cap (X \times z) \} \in \mathcal{Z}^p(X).
\]
(Notice that this preserves codimension.) The issue with (5) is that we need \( z \) to intersect properly the loci in \( Y \) along which \( f \)
the fiber-dimension of \( f \) jumps (up), as well as its image \( f(X) \).

Proposition 1: If \( f \) is dominant, then \( f^*: \mathcal{Z}^p(Y) \to \mathcal{Z}^p(X) \) is
defined on all cycles.

Proof: For \( f \) dominant, flatness is equivalent to equidimensionality. Since (5) \( \Leftrightarrow \) (4), the Prop. follows.

This is often called "flat pullback", but the criterion (4) is much
more general (and useful).

Remark 1: • \( f \) dominant means \( \bar{f}(X) = Y \)

• \( f \) flat means that any exact sequence of \( \mathcal{O}_Y \)-modules
remains exact upon tensoring \( \mathcal{O}_{Y \times X} \) with \( \mathcal{O}_{Y \times X} \) \((y \in Y, \sigma \in f^{-1}(y) \subset X)\).
Remark 2: Chap. 8 of Fulton's *Intersection Theory* describes a refined intersection product which does not require proper intersection. If $V$ and $W$ are subvarieties of $X$ (smooth), then one may define a cycle class $\sim_{V \cap W} \in CH_{i+j-d}(V \cap W)$ whose push-forward into $CH_{i+j-d}(X)$ is the correct class. (See §32-3 below.) Likewise there is a "refined pullback": given $f: X \to Y$ and $Z \subset Y$, this is a class $f^*\sim Z \in CH^*(f^{-1}Z)$ which maps to $f^*\sim Z \in CH^*(X)$. 