2. Equivalence relations

The algebraic cycle groups are best studied modulo some equivalence relation, so that we can always take intersection products of cycle classes, take pullbacks, etc. Assume $X, Y$ are projective (or just proper) and smooth.

**Definition 1 (Samuel):** An equivalence relation $\equiv$ on cycles is adequate if the following hold:

1. $(RA_1)$ \[ \{ z \in \mathbb{Z}^i(X) \mid z \equiv 0 \} \subset \mathbb{Z}^i(X) \] is a subgroup $(\forall i, X)$

2. $(RA_2)$ \[ \exists z' \equiv z \text{ such that } z' \cap W \text{ is proper } (\forall z \in \mathbb{Z}^i(X), W \in X) \]

3. $(RA_3)$ for any $z \in \mathbb{Z}^*(Y)$ and $T \in \mathbb{Z}^*(X \times Y)$, such that $T a(X \times z)$ is proper, and $z \equiv 0$ \[ \text{ we have } T(z) := (\pi_X)_* \{ (X \times z) \cdot T \} \equiv 0 \]

**Proposition 1:**

1. $z \equiv 0 \Rightarrow z \times X'' \equiv 0 \quad \text{(a)}$

2. $z \equiv 0 \quad \text{and } z \cap W \text{ proper } \Rightarrow z \cdot W \equiv 0 \quad \text{(b)}$

3. $z \in \mathbb{Z}^*(X \times Y), \, z \equiv 0 \Rightarrow (\pi_X)_* \cdot z \equiv 0 \quad \text{(c)}$

4. $z \equiv 0, \quad W \in X \text{ nonsingular, } \, z \cap W \text{ proper } \Rightarrow z \cdot W \equiv 0 \quad \text{(d)}$

5. $z \equiv 0, \quad X \in Y \quad \Rightarrow (\pi_Y)_* (\pi_X)_* z \equiv 0 \quad \text{ (e)}$

[Note: converse to (e) is *false* — can have $z \in \mathbb{Z}^i(X), \, X \subset Y$, $z \equiv 0 \quad \text{on } Y, \, z \neq 0 \quad \text{on } X$.]


Proof of (a) & (b): (a) follows from \((RA_3)\) by taking \(T = \Delta_{x''} x x'', Y = x', X = X' x x''\). \([\Delta_{x''} \text{ is the diagonal in } x' x'']\)

(b) follows from \((RA_3)\) by taking \(T = (x x W) \cdot \Delta_{X^2}\)

on \(X x X\), so

\[
X^2 W \cdot \Delta_{X^2} = (X x W) \cdot (X x W) \cdot \Delta_{X^2} = (X x W) \cdot T \\
\]

Exercise: try another

Proposition 2: Write \(C^i_x(x) := \mathcal{Z}^i(x) / \mathcal{F}^i_x(x)\) for the quotient group of equivalence classes. Then

(a) \(C^i_x(x) := \bigoplus C^i_x(x)\) forms a commutative ring, graded by codimension, with identity \(\langle x \rangle\).

(b) Given \(T \in \mathcal{Z}^i(x x y)\), \(T(\cdot) : C^i_x(x) \rightarrow C^i_y(y)\) is a homomorphism of abelian groups, depending only on the class \(\langle T \rangle \in C^i_j(x x y)\). A special case is when \(T = t_f\), \(f : X \rightarrow Y\), \(T(\cdot) = f_x^*\).

(c) Given a morphism \(f : X \rightarrow Y\), \(f^* : C^i_x(y) \rightarrow C^i_x(x)\) is a homomorphism of commutative rings.

Proof:

(a) use Prop 1 (b), \(RA_1, RA_2\)

(b) use Prop 1 (c), \(RA_2, RA_3\)

(c) take \(T = \Gamma_f\), use (b); the ring assertion follows from associativity of interaction product. \(\square\)
Remark 1: In general it is false that $f^\#$ gives a ring homomorphism. Consider the case of a blow-up

\[ \begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow & & \downarrow \text{f} \\
V_0 & \xrightarrow{f^\#} & W_0
\end{array} \]

Then $f^\#_V : f^\#_W = V_0 : W_0 = p$, but $f^\#(V \cdot W) \neq f^\#(V) \cdot f^\#(W)$.

(To see that $f^\#(V \cdot W_0) = f^\#(V_0) \cdot f^\#(W_0)$, you have to either use Elm's refund product [a puzzle!] or move $V_0$, $W_0$.)

Remark 2: We can define cycle groups for quasi-projective varieties as follows: if $X \subset \overline{X}$ is a smooth compactification, then $Z_i(X) = Z^i(\overline{X})/Z^i_c(\overline{X} \setminus X)$ (where I assume $\overline{X} \setminus X$ is closed of codim k in $\overline{X}$). We then put $C^i_c(X) = \frac{Z^i(\overline{X})}{Z^i_c(\overline{X} \setminus X) + Z^i_c(\overline{X})}$, and remark that $C^i_c(\overline{X} \setminus X) \to C^i_c(\overline{X}) \to C^i_c(X) \to 0$ is exact.

Moreover, in the more general case where $X \cup Y$ are quasi-projective: Prop 2(c) still holds (for $f^\#$), but for $f^\#$ to be defined we must assume that $f$ is proper ("proper push-forward").