3. The Chow group

Given a smooth variety $V/k$ and function $g \in k(V)^*$, define

$$\text{div}(g) := \sum_{W \in V_{\text{red}}} \text{ord}_W(g) \cdot W \in \text{Div}(V)$$

(1)

(whence writing $g = g_1/g_2$, $g_j \in O_{W,V}$, $\text{ord}_W(g) = \text{ord}_W(g_1) - \text{ord}_W(g_2)$ and $\text{ord}_W(g_j) = \log(0_{W,V}/(g_j))$). (1) is also valid for $V$ singular.

Setting $D(V) \subset \text{Div}(V)$ the subgroup generated by divisors of functions, we write

$$\text{CH}^1(V) := \text{Div}(V)/D(V).$$

(2)

Understanding the structure of this group is equivalent to solving the problem "when is a divisor the divisor of a rational function?". (There is also the analogy to ideal class groups in number theory.)

This definition was generalized to higher codimension through the work of Severi, followed by Chow and Samuel.

To define $\equiv_{\text{rat}}$ (rational equivalence), for any subvariety $Y/k$ of $X/k$ we shall write

$$Y \xrightarrow{i}$$

(3)

$\xrightarrow{\sim}$

$$Y \hookrightarrow X$$
for a resolution of singularities. Then

$$ (\mathbb{P}(X)^{<})_{\text{rat}}(X) := \left\{ \sum_{i=1}^{n} z_i D_i \mid \exists Y_i \in X \text{ irred of } d, y_i \in k(Y_i)^* \right\} $$

$$(3) \quad \left\{ \exists T \in \mathbb{P}(\mathbb{P}^1 \times X), a, b \in \mathbb{P}^1(\mathbb{R}) \right\}$$

(Note that we can always take $\{a, b\} = \{\infty, 0\}. \text{ The } \text{ } \mathbb{P}^1 \text{ is where the name } \text{ "rational equivalence" comes from.} \text{ To see (3), note that if } T = \text{dim}^* (\phi), \text{ then } T = (\text{id}_X \times i)^* \Gamma^* \text{ will suffice; whereas, given } T \text{ and, with } \{a, b\} = \{\infty, 0\}, \text{ } \pi_{\mathcal{P}_1} \text{ yields a further } \text{ } \Phi \in k(T)^*, \text{ and setting } \phi := \text{Norm}_{k(T)/k(\mathcal{P})} \text{ we have } \text{dim}^* (\phi) = \text{dim}^* (\phi)(\text{dim} (\Phi)) = T(0) - T(\infty). \text{ }

(RA)} \text{ is clear for } \mathbb{R}, \text{ and I leave } (RA_2) \text{ as an exercise. (Theorem 1:)} \text{ That leaves checking } (RA_2), \text{ which is Chow's moving lemma:}

Let $X \subseteq \mathbb{P}^N$, $Z \subseteq X$ irred dim $p$, $W \subseteq X$ irred dim $q$. We must show that $\exists \tilde{Z} \equiv Z \text{ s.t. } \tilde{Z} \cap W \text{ is proper.}$

If $X = \mathbb{P}^N$ we can do this by a $\text{PGl}^{n+1}$-induced automorphism $\tau: \mathbb{P}^N \to \mathbb{P}^N$; indeed,

- there is a path — in fact, a sequence of lines — connecting any $\tau$ to $\text{id}_{\mathbb{P}^N}$, so that $\tau(Z) \equiv Z$; and

- we can choose $\tau$ so that irred components of $\tau_0(Z) \cap W$ contain a smooth point of both, then $\tau_i$ so that the tangent spaces there meet properly.
For $X \subseteq P^n$ we apply the following inductively:

\[ (4) \left\{ \begin{array}{l}
\text{If } Z \setminus W \text{ not proper (i.e. } \dim > p+q-n), \\
\text{then } \exists Z' \subset P^n \text{ s.t. } Z' \cap X \text{ is proper and}
\end{array} \right. \\
\begin{array}{c}
Z' = Z', X = Z + \sum_{j} Z_j \text{ with } \dim Z_j = d \text{ and } \dim Z_j \cap W < \dim Z_j \cap W.
\end{array} \]

(Applying $t$ as above to $Z'$ so that $t(Z') \cap W$ and $t(Z') \cap X$ are proper, $Z = Z' - \sum_{j} (t_j \cap W) X - \sum_{j} Z_j$ is "more proper" against $W$ than was $Z$, and we now repeat on the $Z_j$.)

To prove (4), we choose proper $Z$ by choosing $P_k \neq X$ at each step.

\[ \begin{array}{c}
\tau_n : P^n \rightarrow P^{n-1} \\
\tau_n = \tau_{n-1} \circ \cdots \circ \tau_2 = P^n \setminus L \rightarrow P^{n-1}
\end{array} \]

Pick $x_0 \in Z \subseteq X$, and $x_j$ in each irred. component of $Z \cap Y$. We can choose them so that

(a) $\tau$ is finite \[ \tau_k \] has fibers $= \{ \text{line } (x \cap X) \cap X \}$

(b) $\tau$ étale in $\tau_k(X)$ open about each $x_j$ \[ \text{ choose } \tilde{P}_k \text{ so that } \text{span } \{ L_k, x_j \} \text{ meets } X \text{ transversely at } x_j(\pi(x)_j) \]

(c) $\tau_k : \tilde{Z} \rightarrow \pi_k(\tilde{Z})$ over open neighborhood of each $\pi_k(\tilde{Z})$ \[ \text{ chosen } \tilde{P}_k \text{ so that } L_k \text{ avoids cone } x_j(\tilde{Z}) \]

(d) $W \cap \pi_k(\tilde{Z}) = (W \cap \tilde{Z}) \cup E$, $E \subseteq W \text{ closed}$ and $E \cap X \supseteq \text{ cone } x_j \cap X$, $\dim E \leq p+q-n,

\[ \text{ let } T = \{ (y, \tilde{P}_k, \cdots, \tilde{P}_m) \mid \omega \omega \omega, \omega \subseteq E, \tilde{P}_i \in \tilde{P}_i \}, \text{ we span } \langle L, \tilde{Z} \rangle \]

\[ \text{ we span } \langle L, \tilde{Z} \rangle \]
Compute \( \dim T = p + q + (N-n) - n \), so for general \( \phi \)

\[
\dim \pi^{-1}(\phi) = \dim T - \dim B = p + q - n
\]

Since \( L \cap W = L \cap X = \emptyset \), \( \dim \langle L, x \rangle \cap W = 0 \),

so \( \pi^{-1}(\phi) \) is finite over \( W \) and image of \( \dim p + q - n \)

Now let \( \phi' = \prod_{i} \pi_{*} X \subset P^{N} \). Then

\[
\begin{align*}
& (b) \Rightarrow \exists \, \phi' \mid X = \emptyset + \sum_{i} V_{i} \text{ with multiplicity 1} \\
& \Rightarrow \exists \, \phi' \mid X = \emptyset + \sum_{i} \mathbb{V}_{i} \\
& \Rightarrow \exists \, \phi' \mid W = \prod_{i} \pi_{*} \mathbb{V}_{i} \text{ since } \dim \mathbb{V}_{i} = 1
\end{align*}
\]

Let \( \phi' \mid W \) be a sum of irreduces. \( \exists \, V_{i} \).

2 possibilities:

\[
\begin{align*}
& (c) \Rightarrow V_{i} = E \Rightarrow \dim E = 1 + q - n \\
& (d) \Rightarrow V_{i} \not\subset E \Rightarrow \dim V_{i} < \dim E = 1 + q - n
\end{align*}
\]

Definition: \( CH^{p}(X) := Z^{p}(X) / Z_{rat}^{p}(X) \). [Also: \( CH^{p}(X) := Z^{p}(X) / Z_{rat}^{p}(X) \)]

We now turn to an example of the computation of a Chow group.

Note that if \( X \subset \mathbb{P}^{d+1} \) has degree \( D \), then

\[
K_{X} \cong K_{\mathbb{P}} \otimes \mathcal{O}(X) \big|_{x} \cong \mathcal{O}(-d+2) \otimes \mathcal{O}(D) \big|_{x} = \mathcal{O}_{x}(D-d+2)
\]

\[
\Rightarrow h^{d,0}(X) = \dim \Gamma(X, K_{X}) = \binom{D-d+2}{d+1}
\]

[homogeneous polynomials of degree \( D-d+2 \) in \( d+2 \) variables]

Theorem (Rojtman): For a hypersurface \( X \subset \mathbb{P}^{d+1} \) of degree \( D \leq d+1 \),

\[
CH^{d}(X) \xrightarrow{\text{ch}} \mathbb{Z}
\]

\[
\xrightarrow{\text{hom}} \mathbb{E}^{m+1}
\]
Proof: Assume $1 < D \leq d$.

Step 1. Consider the local affine equation of $X$ about $p \in X(k)$:

$$0 = f_1(x) + \ldots + f_d(x)$$

To say that $x_p : t \mapsto (a_0 t, \ldots, a_d t)$ belongs to $X$ means

$$0 = t f_1(x) + \ldots + t^D f_D(x) \quad (\forall t),$$

i.e. $[a] \in V$, belongs to the complete intersection $V(f_1, \ldots, f_d) \subset \mathbb{P}^d$.

Since $D \leq d$, and $k = \overline{k}$, $V \neq \emptyset$.

Step 2. Let $p, q \in X(k)$, $L_p, L_q \subset X$ then $p, q$ resp.

$H^2(P^{d+1}) = \mathbb{Z}$, $CH^\text{hom}_{\mathbb{Z}}(P^{d+1}) = \{0\}$ \implies $L_p \equiv_{\mathbb{Q}} L_q$ on $P^{d+1}$.

Now $L_p$ and $L_q$ of course don't properly intersect $X$, but we can use Fulton's refined intersection product to have (on $X$)

$L_p \cdot X \equiv_{\mathbb{Q}} L_q \cdot X$, where $L_p \cdot X$ is a $0$-cycle of degree $D$

supported on $L_p \subset L_q$. Since every point on $L_p$ (resp. $L_q$) is

$\exists \ p \ (\text{resp. } \exists \ q)$, we get $D \cdot p \equiv_{\mathbb{Q}} D \cdot q$ on $X$.

Step 3. Since $CH^\text{hom}_{\mathbb{Q}}(X)$ is generated by differences of points $p-q$, we conclude that $D \cdot CH^\text{hom}_{\mathbb{Q}}(X) = 0$. Since $CH^\text{hom}_{\mathbb{Q}}(X)$ is divisible (see below), it follows that $CH^\text{hom}_{\mathbb{Q}}(X) = 0$. Hence

for any $p, q \in X(k)$, $p \equiv_{\mathbb{Q}} q$.

$\square$

$D = d+1$ is an exercise, using

$$\{x_p \in X \text{ so } x_p \cap X = D \cdot p\} \Rightarrow [a] \in V(f_1, \ldots, f_{d-1})$$
Remark 1:
The divisibility assumption is checked as follows: given $p,q \in X(h)$, take a curve $C$ from $p,q$. We then have

$$\mathcal{J}(C) \rightarrow CH^0_0(X),$$

which shows that

$$\langle p-q \rangle \mapsto \langle p-q \rangle$$

all the generators of $CH^0_0$ lie in the image of a torsion (even lacks).

Since torsor are divisible, so is $CH^0_0$.

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Remark 2:
We note the exact sequence (for $Y \subseteq X$ closed)

$$0 \rightarrow CH^0_q(Y) \rightarrow CH^0_q(X) \rightarrow CH^0_q(X/Y) \rightarrow 0$$

which follows from the generating of 2.I.A.2.

Remark 3:
$2^i$ is the "finest" admissible equivalence relation. Here are the subgroups canon to a few others:

$$2^i_{\text{vol}} \subseteq 2^i_{\text{alg}} \subseteq 2^i_{\text{hom}} \subseteq 2^i_{\text{vol}} \subseteq 2^i$$

- $2^i_{\text{vol}}$ is the "same" definition as (3) but with the $1^i$ replaced by an arbitrary sequence of algebraic curves
- $2^i_{\text{hom}}$ is homotopy equivalence (regard only cycles as topological ones: $2^i_{\text{hom}} \equiv 2^i_{\text{vol}}$ if $E$ (a chain $\gamma$ with $\partial \gamma = 2^i_{\text{vol}} - 2^i_{\text{vol}}$).
- You can look up the next in Moore's article

$2^i_{\text{vol}}/2^i_{\text{alg}}$ is called the Gottlieb group $Gott^i(X)$.

Note: divisible groups cannot have torsion, but they can be torsion.

Exercise: Show (in detail) how to use $3 \mathbb{P}^2$’s to prove Theorem 2 without the refined intersection product.

(Take $D \geq 1$ and $e \leq d$, and aim at $(D-1)(-p-q) \geq 0$.)

Exercise: prove Theorem 2 in the case $D = d+1$, using the hint on page 5.

As we shall see, Theorem 2 is a first instance of the way in which the Hodge theory of $X$ (in this case, vanishing of $h^{d,0}(X)$) “controls” the cycle groups on $X$. Indeed, as we shall see, if $h^{d,0}(X) \neq 0$ then $\text{CH}_0(X)$ becomes “huge”, while any sort of converse (see $\S$ II.3) is still conjectural — but with good evidence like Theorem 2.