B. Cycle class map and the Hodge Conjecture

1. Cycle class ad lekschutz (1,1)

Recall the decomposition of $\mathbb{C}$-valued $C^0$ $k$-forms on $X_{\mathbb{C}}$:

\[ A_p^k(X) = \bigoplus_{p+q=k} A_p^q(X), \quad A_p^q = A^{p,q} \]

\[ d + \delta : A^k \rightarrow A^{k+1}, \quad H^k(X; \mathbb{C}) = H^k\{ A^*(X), d \} \]

\[ H^{p,q};(X; \mathbb{C}) \supset H^{p,q};(X; \mathbb{C}) = \frac{\ker(d) \cap A^{p,q}(X)}{\text{im} \delta \cap A^{p,q}(X)} \]

Assume $X$ proper (or at least compact without!)

Each class in $H^{p,q}$ has a unique $\Delta$-hermitian representative $\alpha = \sum x^{p,q}_i \delta^i$, and $\Delta = 2 \Delta = 2 \cdot \delta \Rightarrow \Delta x^{p,q} = 0 \Rightarrow H^k = \bigoplus H^{p,q}$ (hermitian forms) \Rightarrow (1) descends to cohomology

\[ H^k(X; \mathbb{C}) = \bigoplus H^{p,q}(X) \quad (\Rightarrow h^k(X) = h^{p,q}(X)) \]

This already says some interesting thing about the topology of alg.

varieties: $H^{p,q} = H^{q,p} \Rightarrow h^{p,q} = h^{q,p} \Rightarrow$

(3) for $X$ proper & $k$ odd, $h^k(X)$ is even.

Recall the Hodge filtration

\[ F^p A^k(X) = \bigoplus_{p+q=k} A^q,^{p-q}(X), \quad F^p H^k(X; \mathbb{C}) = \bigoplus_{p+q=k} H^q,^{p-q}(X) \]
We also get a refinement of Poincaré duality; recall this says that for $X$ proper/smooth

$$
\begin{align*}
H_k(X) \times H^{2n-k}(X) &\rightarrow \mathbb{C} \\
(x, \alpha) &\mapsto \#(x \cap \alpha) \quad (\text{it must moreover})
\end{align*}
$$

$$\begin{align*}
&H_k(X) \times H^k(X) \rightarrow \mathbb{C} \\
&\quad (x, \omega) \mapsto \int_X x \cdot \omega \\
&H^k(X) \times H^{2n-k}(X) \rightarrow \mathbb{C} \\
&\quad (\eta, \omega) \mapsto \int_X \eta \cdot \omega
\end{align*}
$$

are perfect pairings (i.e. nondegenerate). The refinement is that

$$
\text{perfect as well (up to)} \quad (\text{up to})
$$

Now take $\gamma \in \tilde{Z}^r_\text{top}(X, \mathbb{Z})$ a topological cycle (real dim. $r$), $[\gamma] \in H_r(X, \mathbb{Z})$ its class. Integrating over $\gamma$ gives a function on $r$-forms here in $H^r(X, \mathbb{C})$. In particular, if $\gamma \in \mathfrak{X}$ is a subvariety of (complex) codim $k$, then

$$
\begin{align*}
[\gamma] \in H^{2n-k}_r(X, \mathbb{Z}) &\quad \text{P.D.} \quad \rightarrow H^{2n-k}_r(X, \mathbb{C}) \\
&\quad \downarrow \\
H^{2n-k}_r(X, \mathbb{C}) \cong H^{2n-k}_r(X, \mathbb{C}) &\quad \text{not nec. injective: torsion may be killed} \\
&\quad \downarrow \\
&\quad \sum
\end{align*}
$$

* I take $\mathbb{C}$ here — they could also be $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{R}_{\text{an}}$.  

5. **5.**
Now consider the decomposition of $f_V$ and

$$H^{2k}(X, \mathbb{C}_x) = \bigoplus H^{p,2k-p}(x)$$

$$\cong \bigoplus H^{n-k}(x) \cong \bigoplus \left( H^{n-p, n-2k+p}(x) \right)$$

The point is that forms of type $(n-p, n-2k+p) \neq (n-k, n-k)$ have too many derivatives and pull back to zero on $V$. So

$$f_V \in \text{image} \left( H^{n-k}(x) \to H^{2k}(X, \mathbb{C}) \right),$$

and (7) extends by linearity to give a map

$$\hat{\xi}_x : H^{n-k}(x) \to H^{2k}(X, \mathbb{C})$$

Actually, the definition of $H^{n-k}(x)$ is really to shear w/torsion

$$H^{n-k}(x) = \ker \left( H^{2k}(x, \mathbb{Z}) \otimes H^{n-k}(x, \mathbb{C}) \to H^{2k}(x, \mathbb{C}) \right)$$

so $\hat{\xi}_x : H^{n-k}(x) \to H^{2k}(X, \mathbb{C})$

and $\hat{\xi}_x (V) = ( [V]_{\text{top}}, f_V )$.

Now if $x \in X$, then $\exists T \in t^k(x) x \in T(x)$

$\Rightarrow \delta T(\infty) = T(\partial(\infty)) = T(0 - 0) = T(0 - T(0)) = T \in \mathbb{Z}$

$\Rightarrow x \in X \Rightarrow x \in X$.

So $\hat{\xi}_x$ assigns to each $x$ and we have

**Definition 1 (cycle class map):** Define

$$cl^k_x : CH^k(x) \to H^{n-k}(x)$$

**to be the map induced by $\hat{\xi}_x$.**
The group \( \text{Hg}^1(X) \) indicates "symmetry of periods", which is an analytic/topological phenomenon. Evidence suggests that this propertysomething about the algebraic structure of \( X \):

Hodge Conjecture: \( \text{Cl}^1_X \) is normally surjective.

*Theorem 1 (Faltings, (1,1)): \( \text{Cl}^1_X : \text{Div}(X) \to \text{Hg}^1(X) \) is surjective.

**Proof (Kodaira/Serre):** Use hard Lefschetz theorem: every holomorphic bundle is of the form \( L_D \), \( D \in \text{Div}(X) \) — equiv., \( L \) has a mer. section.

Write down the exact sequence

\[
0 \to \mathcal{O}_X \to \mathcal{O}_X^* \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathcal{O}_X^*) \to 0
\]

of sheaves on \( X \), and look at the long exact sequence part

\[
\to H^1(X, \mathcal{O}_X^*) \overset{\text{Lefschetz}}{\to} \ker \{ H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^*) \} \to 0
\]

\describeto\[H^0(X, \mathcal{O}_X^*) \overset{\delta}{\to} \ker \{ H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^*) \} \to 0\]

\describeto\[L_D \{ \text{holo. line bundles} \} \overset{\delta}{\to} \text{Hg}^1(X) \]

\describeto\[\text{Cl}^1_X \in [L_D] = \text{Cl}(D) \]

\describeto\[
D \in \text{Cl}^1(X)
\]

*Faltings' integrality in general — rest of Atiyah & Hirzebruch only the special sequence for topological K-theory. Deligne suggests a refinement of HC taking their method of proof into account.