We begin with the basic

Example 1: \( A = \text{abelian 4-fold} \quad (V = H^1(A), \eta, \Omega) \),

\[ E(4) \cong F \text{ imaginary quadratic} \quad (so, \text{ type IV}) \]

\[ \Rightarrow g \text{-} \text{split} \quad V_F = V_+ \oplus V_- \quad \text{compatible with the} \]

\[ H^1(y) \not\subset H \quad \text{Ih} \quad \text{II} \quad \text{III} \quad \text{IV} \]

Assume \( \dim V_+ = \dim V_- = \ldots = 2 \), so that the resulting Hermitian form \( h \) on \( V_+ \) has signature \((2, 2)\).

Then \( \varphi \) acts through \( L(A) = Sp(V, \Omega) \cap \text{Res}_{F/Q}(GL(V_+)) \cong U(V_+, h) \). Considering a general HS in the 4-dim. *Hermitian symmetric domain* \( D_{2,2} = L(A)(\mathbb{R}), \varphi \cong U(2,2)/U(2) \times U(2) \),

we are in the situation of Example 4.3. Applying Proposition 4.6, we have

\[ SU(2,2) \leq M(A)(\mathbb{R}) \leq U(2,2) \ldots \]

* In general, since \( \dim_{\mathbb{R}} V(p,q) = \min \text{ GL}(p+q) = (p+q)^2 \),

\[ \dim_{\mathbb{C}} (V(p,q)/\langle \xi(p,q) \rangle) = \frac{1}{2} \dim_{\mathbb{R}} (V(p,q)) = \frac{1}{2} \min \langle \xi(p,q) \rangle = \frac{1}{2} \log (p+q)^2 - 1 \cdot 1 - 2 \leq 1 \]
So which is \( \mu \neq 0 \) if \( \mu(\mathfrak{g}) = 0 \)? If we let \( \xi^\vee = \mathbf{3} \in \mathfrak{a}^\vee(\mathfrak{g}) \) be a basis such that \( \omega_1, \omega_2, \in V_1^1, 0 \), then \( B_+ = \{ \omega_1, \omega_2, \omega_3, \omega_4 \} \subset V_+ \) is a basis. Since

\[
\begin{bmatrix} \gamma \end{bmatrix}_{B_+} = \begin{bmatrix} \xi \end{bmatrix}_{B_+} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ 1 & 1 & 1 & 1 \end{bmatrix},
\]

\( \gamma \) actually factors through \( SU(V_+, 1) \cong SU(2, 2) \). (Notice that for signature \((1, 3)\) this would NOT happen.) So a general \( A \) "in" \( D_{2,2} \) is \underline{degenerate}, and has an exceptional Hodge class.

To "see" the Hodge class, notice that \( \mu(\mathfrak{g}) \) acts trivially on

\[
(2) \quad \omega_1 \omega_2 \cdot \omega_3 \omega_4, \quad \omega_3 \omega_4 \cdot \omega_1 \omega_2 \in H^{2,2}(\mathfrak{g}^\vee(\mathfrak{g})).
\]

Moreover, these 2 classes span the space \( \text{Reg}_{\mathbf{F}/\mathbf{Q}} \Lambda^4 V_+ / \text{det} V_+ \), which is defined \( \mathbf{Q} \). Appropriate linear combinations of (2) therefore give classes in \( H^2(\mathfrak{g}) \); as they are not fixed by \( \text{Lor} \subset U(2, 2) \), these are the desired exceptional classes.

More generally, let \((V_1, \varphi, A)\) be a PHS of any weight, with \( F \)-multiplication \( (F \mu \otimes \mathcal{E}_\mathfrak{g}, \varphi) \).
not necessarily imaginary quadratic). Let $h := \dim_F V$,
and consider the $1$-dimensional $F$-vector space $\Lambda^h_F V$,
which is in fact a $\mathbb{Q}$-HS of rank $[F: \mathbb{Q}]$.
(This is by the compatibility of $\mu$ with $\rho$, and because
we really mean $\text{Res}_{F/\mathbb{Q}} \Lambda^h_F V$.) The canonical projection
\[ p : \Lambda^h_{(\mathbb{Q})} V \to \Lambda^h_F V \]
is a morphism of $\mathbb{Q}$-Hodge structures, and so $\ker(p) \subset \Lambda^h_F V$
is a sub-HS.

**Definition 1:** The Weil classes associated to $(V, \rho, \mathbb{Q}, \mu)$
are the nonzero elements of the $\mathbb{Q}$-I-complement $W_F$
to $\ker(p)$ inside $\Lambda^h_V$. We have $p|_{W_F} : W_F \cong \Lambda^h_F V$.

**WARNING:** These are not yet Hodge classes!!

**Example 1 (cont'd)**
Let $\{x_i\}_{i=1}^4 \subset V$ be an $F$-basis. Writing $F = \mathbb{Q}(f)$, we may extend $\mu$ to a $\mathbb{Q}$-basis
by taking $x_{i+4} := \mu(f) x_i$. $\Lambda^4_F$ means that we may more
$\mu(f)$'s (and $\mu(f)^{-1}$, etc.) across the $\Lambda$. There are $16 (= 2^4)$
independent classes $\{\{x_1\} \wedge \{x_2\} \wedge \{x_3\} \wedge \{x_4\}\} \in \Lambda^4_F V$ mapping
to some combination of $\ldots \wedge d_2 \wedge d_3 \wedge d_4 \ldots$ and $p(f) \wedge d_1 \wedge d_2 \wedge d_3 \wedge d_4 \ldots$

in $\Lambda^4 V$. The linear combination of these 16 will
generate $W_F \subset \Lambda^4 V$.

If $V = H^1(A)$ has $F$-multiplicativity
for an abelian $d$-fold $A$, then

$h = 2d/[F : \mathbb{Q}], \quad$ and $W_F \subset \Lambda^h V = H^h(A)$. Also note

$V_F = \bigoplus_{\text{permutation}(F, g)} V_p$

for the decomposition.

Proposition 1 (Mumford-Tate) : TFAE:

(i) $W_F \cap \ker_{H^2(A)} \neq \{0\}$

(ii) $W_F \subset \ker_{H^2(A)}$

(iii) $\dim V_p^{(0)} = \dim V_p^{(1)}$ (Xp)

(iv) $M(A) \subset \text{Res}_{F/Q}(SL(V_p))$.

(We won't prove this but it should seem plausible from the above examples.)

Definition 2 : In this case, $W_F$ is called a space of Weil-Hodge classes for $A$. 
Note that well-Hodge classes need not be exceptional. If $A$ is simple of type I and $E(A) = F$ is real quadratic, then $L(A)$ satisfies (iv).

The proposition is most effective for producing exceptional Hodge classes on abelian varieties of type III and IV. Unfortunately we really have no idea how to approach the resulting cases of the Hodge conjecture, even in Example 1. There just isn’t enough “accessible geometry” in such abelian 4-folds.

The only exception to this statement (so far) is for the few families of type $(2,2)$ 4-folds that come from a generalized Prym construction, which must necessarily have $F = \mathbb{Q}(S_3)$ or $\mathbb{Q}(i)$. We now describe Schoen’s proof of the Hodge conjecture in this case.

---

* The point is that because the Prym construction involves a $\mathbb{Z}/m\mathbb{Z}$ étale curve cover, you have to have “multiplication by” $\mathbb{Q}(S_m)$ ($S_m = e^{2\pi i/m}$) in whatever results. But for $(2,2)$ well-Hodge classes on an abelian 4-fold, $d = 4 = h \implies [F : \mathbb{Q}] = 2$. The only quadratic cyclotomic fields are $\mathbb{Q}(S_3)$ and $\mathbb{Q}(i)$, though we naturally wonder if anything can be made of the fact that every $\mathbb{Q}(S_m)$ coincides with $\mathbb{Q}(i)$. 
Let 
\[ C \xrightarrow{\pi \; m : 1} X \]
be a cyclic étale cover, with \( \text{Aut}(C/X) = \langle \alpha \rangle \cong \mathbb{Z}/m \), and 
\[ \sigma : \mathbb{Z}/m \to \text{Aut}(H^1(C)) \]
\[ \alpha \mapsto \sigma(\alpha) := (\alpha^a)^* \]
the induced action on cohomology. We have the characters 
\[ X_r : \mathbb{Z}/m \to (\mathbb{Q}(\zeta_m))^* \quad [\zeta_m = e^{2\pi i / m}] \]
\[ a \mapsto \zeta_m^{ar} \]
and the subspaces \( H^1(C)^\chi \subset H^1(C, \mathbb{Q}(\zeta_m)) \) on which \( \sigma \) acts through \( \chi \).

Let \( V \subset H^1(C) \) be the \( \mathbb{Q} \)-subspace with
\[ (4) \quad V = \bigoplus_{\chi \, \text{primitive}} H^1(C)^\chi \subset H^1(C, \mathbb{Q}(\zeta_m)) \]
more formally, we would write \( V = \text{Res}_{\mathbb{Q}(\zeta_m)/\mathbb{Q}} H^1(C)^\chi \) (for some primitive \( \chi \)).

Compute
\[ h := \dim_{\mathbb{Q}(\zeta_m)} V = \dim H^1(C)^\chi \]
\[ (\text{character theory}) \quad = \frac{1}{m} \sum_{a \in \mathbb{Z}/m} \chi(-a) \cdot \text{tr} \left( \sigma(a) \right) \]
\[ = -\frac{1}{m} \sum_{i=0}^2 (-1)^i \cdot \frac{1}{m} \sum_{a \in \mathbb{Z}/m} \chi(-a) \cdot \text{tr} \left( \sigma_i(a) \right) \]
\[ = -\frac{1}{m} \sum_{a \in \mathbb{Z}/m} \chi(-a) \sum_{i=0}^2 (-1)^i \cdot \text{tr} \left( \sigma_i(a) \right) \]
\[ = -\frac{1}{m} \chi(\delta) \sum_{i=0}^2 (-1)^i \cdot \text{tr} \left( \sigma_i(\delta^{-1}) \right) \]
\[ = -\frac{1}{m} \chi(\delta) \cdot 0 \quad \text{for } \delta \neq 0, \text{ since } \delta \text{ acts w/o fixed pts. (apply Lefschetz fixed pt. theorem)} \]

\[ \text{Euler characteristic of } C \quad \chi = -\frac{1}{m} \chi(\delta) \cdot \frac{1}{h^1(C)} \]
\[
= \frac{1}{m} (2g_c - 2)
\]

Replacing \( \{\delta_i: i \in \mathbb{Z}\} \) by \( \{\delta_i: Z_m \to \text{Aut}(H^*(C, \mathbb{Q}))\} \), essentially the same computation yields \( \text{dim} \ H^0(C) \times = g_x - 1 = \frac{d}{2} \text{dim} \ H^1(C) \).

\[
\text{Setting}
\]

\[
A := J(V) := \frac{V_c}{F'V_e + V_z} \quad (V_z = V \cap H^1(C, \mathbb{Z}))
\]

we have

\[
d := \text{dim}_e A = \frac{1}{2} \text{dim}_Q V = \frac{[\mathbb{Q}(\zeta_m) : \mathbb{Q}] \cdot h}{2} = \frac{\varphi(m) h}{2} = \varphi(m) (g_x - 1),
\]

where \( \varphi(m) = \text{Euler \phi-function} \). Here are the "lowest-genus" examples of interest, keeping in mind that we want \( d \geq 4 \) \( \text{even} \).

\[
\begin{array}{cccccc}
m & g_x & g_c & h & d & \\
3 & 3 & 7 & 4 & 4 & (\text{in cases, } h \neq d) \\
4 & 3 & 9 & 4 & 4 & \\
3 & 4 & 10 & 6 & 6 & \\
4 & 4 & (3) & 6 & 6 & \\
\end{array}
\]

Now consider the action of the symmetry group \( S_n \) on \( C^h \), by permuting factors. We have

\[
H_1^1(C)^{S_n} \subset H_1^1(C^h)
\]

and

\[
\Lambda H_1^1(C) \equiv (H_1^1(C)^{S_n})^{S_n} \subset H_1^1(C^h)^{S_n}.
\]

\[\text{This is confusing. Simple example: Consider } x = w \circ \zeta \text{, and the projections } \pi_1, \pi_2: C^2 \to C. \text{ We have } \pi_1^* \omega \circ \pi_2^* \omega \text{ canonically form } i: H_1^1(C) \otimes H_1^1(C) \to H_1^1(C^2)) \text{, which is clearly; this becomes } \text{dual of a copy under exchange of factors.}\]
Writing \(E : (\mathbb{Z}/m)^h \to \mathbb{Z}/m\) for the argument and \((a_1, \ldots, a_h) \to \sum a_i\), let
\[
N := \ker(E) \subset (\mathbb{Z}/m)^h =: \hat{\mathbb{N}}.
\]

We have
\[
\mathfrak{h}_h V \cong (\bigvee h) N \subset H^h(C^h)^N_{Q(S_m)}
\]

Since cutting by \((\sum a) (a_1, a_2, 0, \ldots, 0) \in N\), we get \(\sigma_1 \sigma_2 \xi_1 \xi_2 \cdots \) (i.e., allow \(\sigma_i\) to move \(Q(S_m)\) across the \(\xi_i\)); hence
\[
\left(\bigwedge h^h V \cong (\bigvee h) W^h_{Q(S_m)} \right) U \subset H^h(C^h)^N_{Q(S_m)}.
\]

Since \(h = \dim_{Q(S_m)} V\), \(U\) has dimension 1 over \(Q(S_m)\) (dimension \(Q\) over \(Q\)), it also clearly identifies with
\[
\left(\bigwedge^h V^\vee = \left(\bigwedge^h V^\vee \right)^{x} \subset \left(\bigwedge^h H^h(A)^{\vee}\right)^{x} =: U^\vee \subset H^h(A)^{\vee},
\right. \quad x \quad \left(\text{principal} \right) \quad x \quad \left(\text{principal} \right)
\]
a space of Weil classes. That these are \(\text{Weil-Hodge}\) classes follows from \(h^0(C)^{\vee} = \frac{1}{h} H^0(C)^{\vee}\) (i.e., \(\dim(V^\vee) = \frac{1}{h} \dim(V^{\vee})\)), as observed above, which also implies that \(W_2\) is an integer.

**Theorem 1 (Schanuel):** (a) The Hodge Conjecture is true for these Weil-Hodge classes; that is,
\[
U' \subset \text{cl}(CH^{h/2}(A)_\mathbb{Q}) \subset H^h_{\text{alg}}(A).
\]

(b) Ditto for the corresponding classes on \(C^h:\)
\[
U \subset \text{cl}(CH^{h/2}(C^h)) \subset H^h_{\text{alg}}(C^h).
\]
Proof:

We shall first prove (b), then show that (a) follows from it.

Look at the big diagram (in which $S^k := S^k \mathbb{C}$)

$$
\begin{align*}
C^k & \Rightarrow Y \backslash C^k \Rightarrow Y \backslash C^k = X^k \\
& \Rightarrow W \Rightarrow W \backslash C^k = S^k X \\
& \Rightarrow W \Rightarrow W \backslash C^k = S^k X \\
& \Rightarrow W \Rightarrow W \backslash C^k = S^k X
\end{align*}
$$

[AT maps require a choice of base point on $X$ resp. $W$]

and consider the preimage of $|K^g_x| - h_p$ under $\overline{AJ}_x$: this is the linear system $|K^g_x| \subseteq IP^g_{x-1}$. The fiber of $\overline{AJ}_x$ is of dimension greater than the relative dimension $h - g_x = g_x - 2$. Since $P^g_{x-1}$ has no nontrivial connected strata, $
\psi^{-1}(|K^g_x|) \subseteq \bigcup_{j} IP^g_{x-1}$.

The choice of the point $p \in X$ gives an embedding $S^k X \hookrightarrow S^{k+1} X$, and $S^{k+1} X = S^{2g_x-1} \overline{AJ}_x \backslash J(X)$ is a smooth projective bundle with $|K^g_{x+p}| \subseteq |K^g_x|$ as fiber. So the normal bundle $N|K^g_{x+p}| / S^{k+1} X$ is trivial, and

$$
0 \rightarrow N|K^g_{x+p}| / S^{k+1} X \rightarrow N|K^g_{x+p}| / S^{k+1} X \rightarrow N^{k+1} / S^{k+1} X|_{K^g_x} \rightarrow 0
$$

* Schenck actually proves more, considering non étale curve coverings.

But the étale case are the only ones that permit to apply general obstruction varieties of Weil type.
\[ \begin{align*}
&\text{given } c(N_{kx/l}^{1/k}) = c \left( N_{kx/l}^{1/k} \bigg|_{kx} \right)^{-1} = c \left( \Omega_{kx/l}^{1/k} \right)^{-1} = (1 + H)^{-1}
&= 1 - H + H^2 - \ldots \quad \text{(Prop. 1)},
&\Rightarrow \quad c_{kx/l} \left( N_{kx/l}^{1/k} \right) \neq 0 \Rightarrow \quad c_{kx/l} \left( N_{kx/l}^{1/k} \right) \neq 0
&\Rightarrow \quad (z, z) \neq 0. \quad \text{Define algebra cycles for each primitive } x
\end{align*} \]

\[ Z_x := \bigoplus_{x \in \mathbb{Z}_m} x \cdot z_x \in \text{CH}^{w_2} (W) \otimes (m) \]

and note that

\[ \begin{align*}
(z, z_x)_{w_2} &\neq 0 \Rightarrow \quad c_1 (z_x) \neq 0 \Rightarrow \quad c_1 (z_x) \neq 0
&\quad \text{by (2)}
\end{align*} \]

\[ Z_x = \chi (x) \cdot z_x, \quad \text{where } \chi (x) = \text{Aut}(w_2 / s \cdot x) \cong \mathbb{Z}_m. \]

Under the action of the $\mathbb{Z}_m$, we have the decomposition into $1$-dim eigenspace

\[ \begin{align*}
\mathcal{U} = \oplus_{x \in \mathbb{Z}_m} \mathcal{H}^n (W)^x \cong \mathcal{H}^n (W) = \mathcal{H}^n (C^x)^{w_2}
\end{align*} \]

which together with (10) and (11) now shows $\text{cl}(z_x) \in \mathcal{U}$. Since the various $Z_x$ have "all the eigenvalues $\chi"$ in (10) (by (2)), these classes span $\mathcal{U}$. Taking appropriate $\mathbb{Z}_m$-linear combinations of the $z_x$ now gives cycles with rational coefficients whose classes span $\mathcal{U}$.

For (a), we remark that $E^* P^*$ gives an isomorphism $\mathcal{U} \cong \mathcal{U}$. We now need an inverse operation that also works on the level of cycles.

Noting that, up to a mult. const., $\mathcal{E}_x \otimes E^* (\gamma) = [\mathbb{C}]^{g - h}$ for $\gamma \in \mathcal{H}^1 (C, \mathbb{C})$, and $[\mathbb{C}]^{g - h}$ has an algebraic inverse $\Lambda$, we have

\[ i^* \Lambda \otimes E^* (\mathcal{E}_x \otimes \gamma) = i^* P^* \gamma = \gamma \quad \text{(up to mult. const.)} \]

for $\gamma \in H^k (A)$. Hence $i^* \Lambda \otimes E_x^* : U \to U'$ allows us to transfer the cycles from $C^x$ to $A$.
But we aren't done, of course. Theorem 1 establishes the Hodge Conjecture for very general "generalized Prym varieties". What about a very general Weil abelian variety?

Our family of abelian varieties $A$ in the above construction lives over a finite cover of the moduli space of genus $g_x$ curves:

$$
\begin{array}{c}
\Phi \\
\downarrow \Phi \\
\tilde{M}_{g_x} \\
\overset{\Phi}{\longrightarrow} \mathfrak{g} \ \text{Sh}_{g_x}
\end{array}
$$

The period mapping for $A(\tilde{M}_{g_x})$ forms the Shimura variety of PEL type:

$$\Xi \cong \Gamma_o \backslash \text{SU}(\frac{1}{2}, \frac{1}{2})^x / K \quad (l = \# \text{ of conjug. pairs of complex embeddings of } \mathbb{Q}(g_x) \to \mathbb{C})$$

since the MTG of $V = H^1(A)$ is clearly contained in $\text{SU}(V^\ast, h_x)$.

The issue is whether $\Phi$ dominates $\Xi$.

A quick dimension count shows that this can't happen in general:

$$\dim \Xi = \frac{\Phi(l)(1/2)}{2}(g_x-1)^2 = \frac{\Phi(l)}{2}(g_x-1)^2$$

while $\dim M_{g_x} = 3g_x - 3$. The only pairs $(g_x, m)$ for which the second dimension $\geq$ the first are the ones displayed in Table (6) [where we equate $m = 3, 8m = 6$]. We will prove the dominance in one case:

2. **Theorem (Schwartz):** There exist families of Weil abelian 4-folds with $A(\tilde{M})$-invariant cycles in which the Hodge Conjecture holds for a very general member.

*by "families" I mean the complete PEL family, and not a smaller dimensional one.
Proof: We need to show that the map on tangent spaces, $d\Phi$, is surjective.

Of course $T_0M_g \cong H^1(X, \Theta_X^1)$, while the tangent space to $\mathcal{X}$ is controlled by the movement of the Hodge flag in one piece $V^\bullet$ or $V = V^k \otimes V^{k-1}$:

$$T_0\mathcal{X} \cong \text{Hom}(H^0(C)^\times, H^0(C)^\times)$$

$$\cong H^0(C)^\times \otimes H^0(C)^{\times -1} \cong H^1(C, \Theta_C)^\times \otimes H^1(C, \Theta_C)^{\times -1}$$

$$\cong H^1(X, Z) \otimes H^1(X, Z^{-1})$$

where $Z$ is a 3-torsion sheaf $(Z^{3}=\mathcal{O}_X) \cong \pi_*\Theta_C \cong \mathcal{O}_X \otimes \Omega^1$. So

$$d\Phi^*: H^1(X, \Theta_C^{\times -1}) \to H^1(X, Z) \otimes H^1(X, Z^{-1})$$

has kernel

$$d\Phi^*: H^0(X, Z \otimes K_x) \otimes H^0(X, Z^{\times -1} \otimes K_x) \to H^0(X, K_x^{\Theta^2}),$$

which turns out to be given by multiplication. (But what else could it be?!) Schroeer proves that not every $X$ has a functor with divisor of form $3p-3q$, hence that there exist $X$ for which $|Z \otimes K_x|$ is born point free. (See p. 30 of his paper.) Writing $W = H^0(X, Z \otimes K_x)$, we therefore have that $W \otimes \mathcal{O}_X \to \mathcal{L}_X \otimes K_x$, with kernel $Z \otimes K_x^{-1}$, since $\mathcal{W} = L^1(C)^{\times -1}$ has dimension 2. Tensoring by $Z \otimes K_x$ we arrive at the exact sequence

$$0 \to Z^{-1} \to W \otimes Z \otimes K_x \to K_x^{\Theta^2} \to 0,$$

since $H^0(X, Z^{\times -1}) = 0$, we get that

$$H^0(W \otimes Z \otimes K_x) \cong H^0(X, Z \otimes K_x) \otimes H^0(X, \Theta_C^{\times -1})$$

injects into $H^0(X, K_x^{\Theta^2})$. Therefore $d\Phi^*$ is injective and $d\Phi$ is surjective, as desired.