2. Cycles modulo algebraic equivalence

Let \( X \) be a smooth projective variety over \( \mathbb{C} \).

We first recall the story for divisors. As in the proof of Thm. I.B.1, the exponential exact sequence on \( X \) gives

\[
0 \rightarrow \frac{\mathcal{H}^1(\mathcal{O})}{\mathcal{H}^1(\mathcal{O}_X)} \rightarrow \mathcal{H}^1(\mathcal{O}_X) \xrightarrow{\partial} \ker \left\{ \mathcal{H}^1(\mathcal{O}_X) \rightarrow \mathcal{H}^2(\mathcal{O}_X) \right\} \rightarrow 0
\]

\[
\text{or} \quad \mathcal{L}c(X) [L_0] \quad \mathcal{L}y(X)
\]

\[
\text{or} \quad \mathcal{H}^1(\mathcal{O}_X) \cong [D]
\]

In particular, we have

1. \( \mathbf{AJ}_X : \mathcal{H}^1(\mathcal{O}_X) \xrightarrow{\cong} J^1(X) \)

in which (esp. for \( \dim X = 1 \)) injectivity is usually referred to as Abel's theorem and surjectivity as Jacobi inversion. Note that

\[
( J^1(X) = \mathcal{C}h^1_{\text{hom}}(X) = \mathcal{C}h^1_{\text{alg}}(X) ,
\]

since any two points in \( J^1(X) \) can be connected by a curve (with a tautological cycle over it). Also, as the Jacobian of a level-one PHS \( H^1(X) \), \( J^1(X) \) is an abelian variety;

by the H-R bilinear relations, the polarization induces a Kähler metric \( h(u,v) = -iQ(u,v) \) on \( J^1(X) \) with rational Kähler class, and so \( J^1(X) \) is projective algebraic by the Kodaira embedding theorem.
Moving on to higher codimension, we begin with the

restriction

\[ AJ_{\text{alg}, x} : C_{\text{H}^k}(x) \to J^k(x) \]

of the Abel–Jacobi map to cycles algebraically equivalent to zero.

The first main point is that Jacobians of Hodge strata of
level > 1 are "generically" non-algebraic complex tori.

Exercise: Show this for level/weight 3, of type \((1, 1, 1, 1) = \frac{1}{2}\).

(a) Given \((V, \phi, \Psi)\) plus of this type, we may view \(\phi\) as
factorizing through a compact 2-torus \(T \leq \text{Sp}_g(\mathbb{R}) = \text{Aut}(V_{\text{irr}, \mathbb{Q}}), \quad \phi|_T \)
acting as \(z^2, z^{-1} z^1, z^3\) on \(V^0, V^1, V^2, \) resp. \(V^3, 0\). We can
also consider \(\hat{\phi}(\mathbb{Q})\) defined to (factor through the same \(T\) and ) have
eigenvectors \(z^2, z^{-1} z^1, z^3\) on the same spaces. This defines the HS \(\hat{\phi}\) on
\(H^1(J^2(V))\), where \(J^2(V) = V_G / (F^2 V_G + V_2)\). Show that if it is
polynomial, then \(M_{\hat{\phi}} \neq \text{Sp}_g\) (i.e. smaller).

(b) Prove that for \(\Phi\) off a proper analytic subset of \(D_\Phi\), \(M_{\Phi} \neq \text{Sp}_g\),
and conclude that (by (a)) "most" \(\Phi \in D\) have nonalgebraic Jacobian.
[You'll need to state/use Kodaira embedding theorem or the equivalent.]

(c) Show in particular that \(\text{Sym}^3 H^1(E)\) (\(E\) an elliptic curve) has nonalgebraic
Jacobians if \(H^1(E)\) is not CM.

\[ \text{Theorem 1 (Lieberman): } \quad \mathfrak{m} (AJ_{\text{alg}, x}) = : J^k_{\text{alg}}(x) \text{ is an abelian variety}. \]

Proof: \(C_{\text{H}^k}(x) = \sum_{C \text{ curve}} W_x (\mathbb{Z}^k_{\text{hom}}(C)) \]

\[ \text{defn. of algebraic equivalence in } 0 \]

\[ W \in \mathbb{Z}^k(\mathbb{C} \times X) \text{ cycles}. \]
\[ \text{Im} \left( \mathcal{C}^k \right) = \sum \text{Im}(W, J^k(\mathbb{C})) \bigg| (\mathbb{C}, W) \text{ so } A = \text{Im} \left( \mathcal{C}^k \right). \]

Let \( A \subseteq J^k(\mathbb{C}) \) be a maximal abelian subvariety in this image. If \( z \in \text{Im}(W, J^k(\mathbb{C})) \) is not in \( A \), then the image of \( \mu \times W : A \times J^k(\mathbb{C}) \to J^k(\mathbb{C}) \) is abelian and contains \( A \) and \( z \). 

Now we have some constraints on the image of (3) arising from the Exercise (or rather the statement preceding it). But the next result actually gives a stronger constraint still.

**Theorem 2:** Define \( J^k_{\text{alg}}(\mathbb{C}) := J^k \left( \mathcal{H}^{2k-1}_{\text{alg}}(\mathbb{C}) \right) \), where \( \mathcal{H}^{2k-1}_{\text{alg}}(\mathbb{C}) \) is the largest sub-alg. of \( H^{2k-1}(\mathbb{C}) \) with \( \mathcal{H}^{2k-1}_{\text{alg}}(\mathbb{C}) \subseteq H^{2k-1}(\mathbb{C}) \). Then we have that

\[ J^k_{\text{alg}}(\mathbb{C}) = J^k_{\text{alg}}(\mathbb{C}). \]

**Remark:** \( J^k_{\text{alg}}(\mathbb{C}) \) is of course an abelian variety. Equality in (4) is a conjecture, though it is easy to establish in examples.

**Proof (of Thm. 2):** Let \( W \subseteq X \times \mathbb{C} \) be an irreducible subvariety of codimension \( k \), with \( \pi_X, \pi_\mathbb{C} : \tilde{W} \to X, \mathbb{C} \) the projections from a desingularization of \( W \) to \( X \) resp. \( \mathbb{C} \). If we put \( z_i := \pi_X, \pi_\mathbb{C}(p_i) \),

*Exercise:* Use Poincaré Complex Reducibility to show that the image of an abelian variety \( A \), under a homomorphism \( A \to J = \text{cx. forms} \), is an abelian variety.
Then \( Z_1^x \subset Z_1 \to Z_1 \) is an abelian variety, and there is a 1-1 correspondence between the subvarieties of \( H^{2k-1}(X) \) and subvarieties of \( J^k(X) \). So the image of (3) must be contained in the Jacobian of a HS contained in \( H^{2k-1}_{	ext{log}}(X) \).

If \( X \) is projective of odd dimension \( 2k-1 \), we have the hard Lefschetz decomposition (writing \( L_\star \) for cup-product with hyperplane class):

\[
H^{2k-1}(X) = H^{2k-1}_{\text{perm}}(X) \oplus L_\star(H^{2k-3}(X))
\]
and one may have that $H^{2k-1}_{\text{Hdg}}(X) \subset L_{\mathfrak{g}}(H^{2k-3}(X))$ (exp. if $H^{2k-1}_{\text{prin}}$ is irreducible). For $X$ in a Lefschetz pencil on $\mathbb{P}^1$, we often have image $(H^{2k-1}(\mathbb{P}^1)) \subset L_{\mathfrak{g}}(H^{2k-3}(X))$, and so the following often gives a way to arrange these.

**Proposition 1:** Let $X$ be very general in a Lefschetz pencil on $\mathbb{P}^1$ (sm. proj. at dim $2k$), and assume $\text{level}(H^{2k-1}(X)) > \text{level}(H^{2k-1}(\mathbb{P}^1))$. Then $H^{2k-1}_{\text{Hdg}}(X) \subset \mathfrak{g}^{\times} H^{2k-1}(\mathbb{P}^1)$ and $\mathfrak{g}^{\times} J^k(X) \subset \mathfrak{g}^{\times} J^k(\mathbb{P}^1)$.

**Proof:** If $\{X_s\}_{s \in \mathbb{P}^1}$ is the pencil (of hyperplane sections, $X_s = H_{X_s} \mathbb{P}^1$, in which singular fibers have one ODP each), let $B = X_0 \times X_\infty$ be the base locus and $\tilde{X}$ the blow-up of $\mathbb{P}^1$ along $B$. We have a diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\beta} & \mathbb{P}^1 \\
\downarrow{\pi} & & \uparrow{\mathfrak{g}} \\
X & \rightarrow & X_s \\
\end{array}
$$

and $X = X_{s \in \mathbb{P}^1}$ is a very general fiber. It is known that $\mathfrak{g}^{\times}$ is image of $\text{Hdg}$

$$
\rho : \pi_1(U) \rightarrow \text{Aut}(H^{2k-1}(X_s, \mathbb{Q}), \mathfrak{g})
$$

acts irreducibly on $(H^{2k-1}_{\text{Hdg}}(X_s, \mathbb{Q}), \mathfrak{g})$, and $H^{2k-1}_{\text{Hdg}} = \mathfrak{g}^{\times} H^{2k-1}(\mathbb{P}^1)$.

Since the MTC of $H^{2k-1}(X_s, \mathbb{Q})$ is generic, $H^{2k-1}_{\text{Hdg}}(X)$ extends to a level 1 sub VHS of $R^i\pi_*\mathbb{Q} \cong (\mathfrak{g}^{\times}) H^{2k-1}(\mathbb{P}^1) \oplus H_{\text{var}}$. Assumptions imply that $H_{\text{var}}$ has level $> 1$. Since $H_{\text{var}}$ has no sub VHS, $H^{2k-1}_{\text{Hdg}}(X) \subset \mathfrak{g}^{\times} H^{2k-1}(\mathbb{P}^1)$, and we are done by Theorem 2. □
Now we turn to the "quasitri" or "reduced" AJ map on the Cartier map
\[ \text{Griff}^k(X) : \text{Griff}^k(X) / \text{Z}_0(X) \to \text{CH}^k(X) / \text{CH}^k_{\text{crys}}(X), \quad \text{This is,} \]
\[ (5) \quad \overline{AJ}_X : \text{Griff}^k(X) \to J^k(X) / J^k_{\text{hdg}}(X) \to J^k(X) / J^k_{\text{hdg}}(X) =: \overline{J}^k(X). \]

**Proposition 2:** \( \text{Griff}^k(X) \) (and thus \( \text{im} (\overline{AJ}_X) \)) is at most countable.

**Proof:** Given a cycle \( \overline{z} / L \) on \( X / K \) (\( L \supset K \) both are fields)

consider the \( \overline{Q} \)-spread

\[ (6) \]

\[ X \supset X \times \mathbb{A}^1 \]
\[ \downarrow \quad \downarrow \]
\[ B \supset B_{\text{to}} \]
\[ \downarrow \quad \downarrow \]
\[ \overline{y} \in \{ \overline{a} \} \] (all defined in \( \overline{a} \)),

where \( \overline{Q}(5) = K \) and \( \overline{Q}(8) = L \). Now there are only countably manifolds like (6), and all cycles \( \overline{z} \in \text{Griff}^k(X / \overline{a}) \) occur as some fiber \( \overline{z}_s^i \), \( s \in B_{\text{to}}(C) \), in some situation "(6)". But all the fibers \( \overline{z}_s^i \), \( s \in B_{\text{to}}(C) \), in any given situation "(6)" are algebraically equivalent as there is always a (chain of) curves connecting any \( s, s' \in B_{\text{to}}(C) \)!

\[ \square \]

**Remark** (on "cardinalities"): Note that \( \underset{\overline{a}}{\text{Griff}}^k(X) \) would still be a countably \( \Omega \)-dim \( \overline{Q} \)-vector space. Contrast this...
to the image of $A^*_{\text{alg}}$: any complete torus (e.g. elliptic curve $C$) over $\mathbb{Q}$ is of countably infinite dimension as a $\mathbb{Q}$-vector space. But $\text{im}(A^*_{\text{alg}})$ is still parameterized by a finite dimesional algebraic variety. Later we will speak of cycle groups not being pro-representable", which means this is impossible. So there are 3 levels of “boyness” here. One should think of $\text{Ch}_{k^*}(X)$ as the discrete/totally disconnected part, and $\text{Ch}_{k^*}^\text{cont}(X)$ as the “continuous” part, of $\text{Ch}_{k^*}(X)$.

**Corollary:** Jacobian inversion fails for $\text{Ch}_{k^*}^\text{cont}(X)$. If $H_{2u-1}(X)^{k^*} \neq H_{u-1}(X)^{k^*} \oplus H_{u-1}(X)^{k^*}$

\[ \Gamma \]

We now turn to an example due to B. Harris, which will be our first computation of an AJ map.

Consider the Fermat quartic curve $C = \{x^4 + y^4 = 1\} \subset \mathbb{P}^2$.

By the degree-genus formula, it has genus 3, with holomorphic forms given by Griffiths's residue formula:

\[ \frac{\partial(x, y) \ dx \wedge dy}{x^4 + y^4 - 1} = P \text{ F}_y \frac{dx}{F} = \frac{2\omega}{dy} \frac{dx}{F} = P \frac{dx}{4y^3} \quad \text{deg} \leq \text{deg}(x) - 2 - 1 \]

\[ P = 1, x, y. \]

After normalizing, we get the basis of $\mathbb{P}(F)$

\[ \omega_1 = \frac{1 - i}{4b} \frac{dx}{y^2}, \quad \omega_2 = \frac{1}{25} \frac{dx}{y^3}, \quad \omega_3 = \frac{1 - i}{4b} \frac{x dx}{y^3} \]
where \( b = \int_0^1 \frac{dt}{(1 + t^2)^{3/2}} \), \( b' = \sqrt{2} b = \int_0^1 \frac{dt}{(1 + t^2)^{3/2}} \).

The reason for this normalization is that we have three morphisms from \( E \) to the elliptic curve given by \( E : \{ v^2 = 1 - w^3 \} \), given by

\[
\pi_1 (x, y) = (x, y^2), \quad \pi_2 (x_1, y_1) = \left( \frac{-1 + i}{\sqrt{2}}, \frac{x_1}{x^2} \right), \quad \pi_3 (x_1, y_1) = (-y, x^2),
\]

and the form \( dz = \frac{1 - i}{\sqrt{2}} \frac{du}{v} \) on \( E \) has \( \pi_j^* (dz) = \omega_j \) (\( j = 1, 2, 3 \)).

Exercise: (i) Check this.

Moreover, \( dz \) has periods \( 1 \) \& \( 2 \) on a cycle generating \( H_1 (E, \mathbb{Z}) \).

(ii) Check this too! \[ \text{So } E \cong \mathbb{C} / \mathbb{Z} \langle 1, i \rangle. \]

Since \( \pi = (\pi_1, \pi_2, \pi_3) : E \to \mathbb{C}^3 \) induces an isomorphism (surjection)
\( \pi^* : H^1 (\mathbb{C}^3, \mathbb{C}) \to H^1 (E, \mathbb{C}) \), we have that \( \pi_k : H_1 (\pi_k, \mathbb{Z}) \to H_1 (E^3, \mathbb{Z}) \) is an injection. (If \( \pi_j (y) \equiv 0 \) (\( y_j \)), then \( 0 = \pi_j (y) \cdot dz = \int_{y_j} \pi_j^* dz = \int_{y_j} \omega_j (y_j) \), so \( y_j \equiv 0 \).)

Recall the Abel map
\[
\psi : E \to J^1 (E) = \frac{\mathfrak{M}^1 (\mathcal{F})}{\mathcal{M}_1 (\mathcal{F}, \mathbb{Z})} \cong \mathbb{C}^3 / \Lambda
\]
given by \( \psi : \mathcal{A} J (\mathcal{F} - \mathcal{O}_E) \) \( \psi (p - \mathcal{O}_E) = \int_0^p (\cdot) = \left( \int_0^p \omega_1, \int_0^p \omega_2, \int_0^p \omega_3 \right) \)
where \( \mathcal{O}_E = (1, 0) \). But the RHS clearly \((\pi_1 (p), \pi_2 (p), \pi_3 (p)) \), and so

\[
\begin{align*}
\psi & \quad \pi \quad \mathcal{E}^3 \\
\psi & \quad \pi \\
\psi & = \text{isogeny (induced by } (\pi^*)^* )
\end{align*}
\]

(9)

\[
\begin{align*}
\phi & \quad \mathcal{E}^3 \\
\phi & \quad \pi \\
\phi & \quad \psi = \text{isogeny (induced by } (\pi^*)^* )
\end{align*}
\]

commutes.
Now we consider the Ceresa cycle

\[(10) \quad \frac{\tau_m}{m} : = \phi'(\bar{\tau}) - \phi(\bar{\tau})^- \in \mathbb{Z}_{\text{hom}}^2(J'(\bar{\tau}))\]

where \((-)^-\) means to apply the involution \(y \mapsto -y\) on \(J'(\bar{\tau})\).

(If \(\bar{\tau}\) were hyperelliptic, this would just be zero; but, as we shall see, it isn't.) Note that this involution acts as the identity on \(H^4(J'(\bar{\tau}))\), which is why \(\frac{\tau_m}{m} \in \mathbb{Z}_{\text{hom}}\). Our question is: is it algebraically equivalent to zero?

One way to show it is not \(\equiv 0\) is to show it has no nonzero image under

\[(11) \quad AJ^2_{J'(\bar{\tau})} : \mathbb{Z}^2(J'(\bar{\tau})) \to \overline{J^2}(J'(\bar{\tau}))\]

but it turns out to be easier to work with

\[(12) \quad \overline{AJ}^2_{E^x^3} : \mathbb{Z}^2(E^x^3) \to \overline{J}^2(E^x^3)\]

and \(\phi'(\bar{\tau}) = \pi'(\bar{\tau}) - \pi(\bar{\tau})^-.\) Write \(d_{ij}(j = 1, 2, 3)\) for the copy of \(\mathbb{Z}\) on each factor of \(E^x^3\).

**Exercise:** Since \(E\) is CM, \(H^2_{\text{alg}}(E^x^3)\) is the \(L\)-complement of \(\langle d_2, nd_2, nd_3, d_3, nd_1, nd_3, \rangle\).

Conclude that \(\overline{J}^2(E^x^3) = \mathbb{C} \langle d_2, nd_2, nd_3, d_3, nd_1, nd_3, \rangle / \mathbb{Z}(1, 1) \subseteq \mathbb{C} / \mathbb{Z} \otimes \mathbb{Z}.

So, writing \(\Gamma\) for a 3-chain on \(E^x^3\) with \(\Delta \Gamma = \pi'(\bar{\tau}) - \pi(\bar{\tau})^-\), if we can show that

\[(13) \quad \int d_2, nd_2, nd_3 \notin \mathbb{Z} \otimes \mathbb{Z}\]

then we win.
Suppose first that we want to draw a chain $\Gamma^+_{\alpha}$ with boundary

$\pi^{+}(\mathcal{F}) = \{ \text{stuff supported on } \sigma \in \mathcal{F} \}$. To do this,

draw cuts

$\alpha \neq \beta$ on $E$:

and write $\varepsilon_j := \int_{0}^{x} dz_j$ on the shaded region with jumps along $\alpha \neq \beta$,

and $0 \neq \beta$ ($\beta \in E$) to draw the shortest path from the origin to $p$ (which changes as we pass through the cuts!). Then we set

$$
\Gamma^+_{\alpha} = \left\{ (0, \pi^-(\beta), \pi^+(\alpha), \pi^+(\beta)) \mid \beta \in \mathcal{F} \right\}
$$

$$
+ \left\{ (\beta, 0, \pi^+(\alpha), \pi^+(\beta)) \mid \beta \in \pi^+(\mathcal{F}) \right\}
$$

$$
+ \left\{ (\alpha, 0, \pi^-(\beta), \pi^+(\beta)) \mid \alpha \in \pi^-(\mathcal{F}) \right\}
$$

and

$$
+ \left\{ (\beta, \beta, 0, \pi^+(\alpha)) \mid \beta \in \pi^+(\mathcal{F}) \cap \pi^-(\mathcal{F}) \right\} + 3 \text{ more terms}
$$

where the idea is that $\beta$ of the first term gives $\pi^+(\mathcal{F})$ if we stay away from $\pi^+(\mathcal{F})$, but at the cuts, the jump in $\pi^-(\mathcal{F})$ creates an extra boundary term, which the next two terms' boundary cancel, and so on.

Integrating $d\varepsilon_1 d\varepsilon_2 d\varepsilon_3$ over $\Gamma^+$ yields $\int_{0}^{x} d\varepsilon_1 d\varepsilon_2 d\varepsilon_3$ which is zero.
by type. The next two terms yield (since $\int_{\Delta} dx_1 = 0$, $\int_{\Delta} dx_2 = i$)

$$i \int_{\pi^{-1}(2)} z_2 \omega_3 + \int_{\pi^{-1}(\beta)} z_2 \omega_3$$

\text{(i.e. $z_2$ is pulled)}

\text{(back to $\overline{\sigma}$)}

where $z_j := z_j \circ \pi_j$, and the final terms yield

$$- \sum_{\rho \in \pi^{-1}(\alpha) \cap \pi^{-1}(2)} \frac{1}{\rho} \sum_{\rho \in \pi^{-1}(\beta)} \omega_2 \omega_3 \quad \text{for $V \in \pi^{-1}(\overline{\sigma})$, $\rho \in \overline{\sigma}$}$$

\text{Exercise: If $\int_{Y \beta} \omega_2 \omega_3 := \int_{\beta} (\ast \rho) \omega_2 \omega_3$ (for $V \in \pi^{-1}(\overline{\sigma})$, $\rho \in \overline{\sigma}$)}

\text{show that the final terms sum the function of correcting the top two terms to}

$$-i \int_{\pi^{-1}(\alpha)} \omega_2 \omega_3 + \int_{\pi^{-1}(\beta)} \omega_2 \omega_3$$

Next we do the same thing for $\pi^{-1}(\overline{\sigma})$, with $\Gamma_- = (\Gamma_+)^{-}$, and observe that the integrals $-\int_{\Delta} dx_1 \Delta_2 \Delta_3$ simply double the term above.

Finally, $\Gamma$ will be $\Gamma_+ - \Gamma_- + (\text{stuff supported on $\overline{\sigma} \times E \times E \times E \times E \times E \times E \times E \times E \times E$})$ on $\overline{\sigma}$, which determines $\Delta_3$.

$$\therefore \int_{\pi^{-1}(\alpha)} \omega_2 \omega_3 = -2i \int_{\pi^{-1}(\alpha)} \omega_2 \omega_3 + 2 \int_{\pi^{-1}(\beta)} \omega_2 \omega_3$$

$$= 8 (1+i) \int_{(1,0)} \omega_2 \omega_3 = 8 (1+i) \int_{(0,0)} \omega_2 \omega_3$$

\text{using automorphism $(\xi, \eta) \mapsto (\frac{\xi}{1+\eta x} \frac{1}{1+i})$?}

$$= 8 (1+i) \int_{0}^{1} \frac{(1-i)}{4b} \frac{dx}{\sqrt{1-x^2}} \frac{1}{2b} \frac{dx}{(1-x^2)^{3/2}}$$
\[ 4 \int_0^1 \frac{dt}{\sqrt{1-t^4}} \cdot \frac{d\tau}{(1-\tau^4)^{3/4}} = \kappa (\mathbb{C}/\mathbb{Z}(1,1)). \]

Of course, \( \kappa \in \mathbb{R} \) so we win if \( \kappa \not\in \mathbb{Z} \). Here's numerically compute that \( \kappa \approx 1.24108 \ldots \), and so we get that

\[ \mathbb{Z}_{\kappa} \]

is not algebraically equivalent to zero in \( \mathcal{J}^1(\mathbb{R}) \).

Unfortunately, one doesn't know whether \( \kappa \) is irrational, and so one does not get the

**Theorem 3 (Bloch):** \( \mathbb{Z}_{\kappa} \) is of infinite order in \( \text{Griff}^2(\mathcal{J}^1(\mathbb{R})) \).

Which is proved indeed using the 2-adic AL map \( (x/12) \)

\[ \text{AL}_2^2 : \text{CH}^2 \left( X \right) \to H^1_{\text{cont}}(\text{Gr}^1(\mathbb{R}/\mathbb{Q}), H^2_{\text{et}}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}(2))). \]

Formally speaking, this map is easy to construct: one has a cycle map

\[ \text{CH}^2(X) \to H^2_{\text{et}}(X, \mathbb{Z}(2)) \]

and a spectral sequence

\[ \text{H}^1_{\text{cont}}(\text{Gr}^1(\mathbb{R}/\mathbb{Q}), H^2_{\text{et}}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}(2))) \]

and then:

\[ \text{CH}^2_{\text{num}}(X) = \ker (\text{CH}^2(X) \to H^2_{\text{et}}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}(2))) \]

What is interesting about Bloch's proof is that \( \text{AL} \) of \( \mathbb{Z}_{\kappa} \), once shown to be nonzero, is almost "automatically" of infinite order (see p. 4 of his paper "Algebraic cycles and values of L-functions").

It though this does follow from Theorem 3 if the Bloch-Beilinson conjectures (which we'll discuss later) hold.