5. Cycles on projective hypersurfaces

We now apply the infinitesimal and topological invariants of normal functors to study homologically trivial cycles on very general projective hypersurfaces, beginning with a fundamental lemma on Normal Functors associated to primitive Hodge classes, which states (roughly) that the topological invariant recovers the Hodge class.

To state this lemma, let $X$ be smooth, projective of dimension $2k$, with a pencil of hyperplane sections

$$
\begin{align*}
\mathcal{X} \cdot \mathcal{H}_{c} &= X_{c} \subset X^{*} \\
&\xrightarrow{\beta} X \\
&\xrightarrow{\pi} \bar{X}
\end{align*}
$$

where $\beta$ is the blow-up along the base locus (assumed smooth).

Consider the VHS $V$ corresponding to $W := \frac{R^{2k-1}}{H^{2k-1}(\mathcal{X}, \mathbb{Z}(k)) \leftarrow \text{constant}}$ over $U$. Given any $\mathfrak{z} \in H_{prim}^{k}(\mathcal{X}) := H^{k}(\mathcal{X}) \cap H_{prim}^{k}(\mathcal{X}, \mathbb{Z}(k))$, we may use the diagrams

$$
\begin{align*}
0 \to J^{k}(X) &\xrightarrow{I_{X}} H_{2}^{k}(X, \mathbb{Z}(k)) \\
&\xrightarrow{\sim} H_{2}^{k}(X, \mathbb{Z}(k)) \\
&\xrightarrow{\psi} H_{2}^{k}(X_{c}, \mathbb{Z}(k)) \to H_{2}^{k}(X_{c}) \to 0 \\
0 \to J^{k}(X_{c}) &\xrightarrow{I_{X}} H_{2}^{k}(X_{c}, \mathbb{Z}(k)) \to H_{2}^{k}(X_{c}) \to 0
\end{align*}
$$

(2)

\begin{align*}
\phi_{\mathfrak{z}} &:= \frac{\psi^{*}(\mathfrak{z})}{\psi^{*}(\mathfrak{z})}
\end{align*}

\begin{align*}
\text{to obtain a section of } \frac{J^{k}(X_{c})}{J^{k}(X)} &= J(V), \text{ which we denote } \psi_{\mathfrak{z}} \in NF(V).
\end{align*}
Henceforth all coefficients are rational unless otherwise noted. Writing \([\cdot]_4\) for the composition

\[
\begin{align*}
H_{g_{\mathfrak{p}}^k}(X) & \xrightarrow{\sim} H_{g_{\mathfrak{p}}^k}(X, \mathbb{Q}(k)) \xrightarrow{(\text{Bor})^*} H^1(X, \mathbb{Q}(k)) = H^1(U, R^{2k-1} \pi_* \mathbb{Q}(k)) \xrightarrow{\psi} \{\mathbb{Z}\}_4
\end{align*}
\]

we have the

**Lemma on Normal Functions:** The diagram

\[
\begin{array}{ccc}
\mathfrak{g}_k \in H_{g_{\mathfrak{p}}^k}(X) & \xrightarrow{[\cdot]_4} & H^1(U, R^{2k-1} \pi_* \mathbb{Q}(k)) \\
\downarrow & & \downarrow \\
\mathfrak{g}_k \in NF_U(V) & \xrightarrow{[\cdot]_J} & H^1(U, V)
\end{array}
\]

commutes and if the pencil is Lefschetz then the top arrow is injective. In particular, if \(\{X_t\}\) is Lefschetz and \(H^{2k-1}(X) = \{0\}\), then

\[
H_{g_{\mathfrak{p}}^k}(X) \rightarrow NF_U(V).
\]

**Proof:** Of course the last line is clear since in that case \(\psi \circ [\cdot]_4\) is injective (since \(\psi\) is an \(\mathbb{Z}\)). For Lefschetz pencils, the singular fibers have a single node and \(H_{2k}\) identical to that of a smooth fiber. So if \(U = R^{1k} \times (\text{and } X_t = \text{union of singular fibers})\) we have

\[
\begin{align*}
H^{2k}(X) \rightarrow H^{2k}(X_t) \rightarrow H^{2k}(x^*) \rightarrow H^{2k}(X) \rightarrow H^{2k}(x^*)
\end{align*}
\]

Lefschetz decomposes \(H^{2k}(X, \mathbb{Q})\):

\[
\begin{align*}
Gr^0_t & = H^0(U, R^{2k-1} \pi_* \mathbb{Q}) \text{ (by \(H_{g_{\mathfrak{p}}^k}(X)\) vanishes here)} \\
Gr^1_t & = H^1(U, R^{2k-1} \pi_* \mathbb{Q}) \text{ (by \(H_{g_{\mathfrak{p}}^k}(X)\) vanishes in } H^{2k}(X)\) \\
Gr^2_t & = H^2(U, R^{2k-1} \pi_* \mathbb{Q}) = \{0\} \text{ for } U \text{ affine}
\end{align*}
\]
and hence $\beta^*(\ker(\beta_0)^*) \subset \text{im}(\beta_0^*)$ by \( \text{im}(H^2_c(X_k)) \to H^2_k(X_k) \).

Let \( \ker(\beta_0^*) \subset \text{im}(H^2_c(X_k)) \cap H^2_k(X_k) = \text{im}(U_1 H^1_c) \cap H^2_k(X_k) = U_{3,1} \).

So we turn to the commutativity of (4). Start with (closed) representatives $\gamma^Q \in C^2 \sigma(\Sigma_0)$, $\gamma^F \in F^{1,2}(\Sigma_0)$ of $\gamma$ and write $(\gamma^Q, \gamma^F, R)$ (where $R = \gamma^F - \gamma^Q$) for the lift to $H^2_c(\Sigma_0)$ in (2). By perturbations of the pullbacks of $\Sigma_0$ and $\Sigma^F$ over neighborhoods $\bar{U}_i$ covering $U$ we can "translating" $\Sigma^Q_{\bar{U}_i} = \Sigma^Q_{U_i}$, $\Sigma^F_{\bar{U}_i} = \Sigma^F_{U_i}$, $(\Sigma^{Q}_{U_i} \circ F^{1,2}(\Sigma^{1,2}_0)(X_{U_i}))$, $\Sigma^{F}_{U_i} \in C^2_{\Sigma^0}(X_{U_i})$. Here $\Sigma^Q$ is the relative differential 2-form, and we can restrict them to $X_k$ to get $\Sigma^Q_{X_k} = \Sigma^Q_{U_i}$, $\Sigma^F_{X_k} = \Sigma^F_{U_i}$.

Now, $\beta^0 \cdot \{\Sigma^f\}$ in (4) is computed by taking Čech coboundary of the collection $\{\Sigma^f\}$, to obtain $\{\Sigma^f_{i,j} \cdot \Sigma^f_{j,i}\}$ (where $\Sigma^f_{i,j}$ restrict to cycles on the $X_k \subset X_{U_i \cap U_j}$, and we consider their classes as sections of $V$).

The normal function $\gamma^N$ is obtained by restricting the Deligne triple to $X_k$, $(\gamma^Q, \gamma^F, R) |_{X_k} = (\Sigma^Q, \Sigma^F, R) \in (0, 0, R_{\Sigma^Q} - \Sigma^Q + \Sigma^F)$, and taking the projection of $R_{\Sigma^Q} - \Sigma^Q + \Sigma^F$ to $H^2(X_k, C)$ to get $\gamma^N$. Consider the short-exact sequence $0 \to V \to C/\text{poC} \to C/(\text{poC} \cap W) \to 0$ used to define $\gamma^N$; the connecting homomorphism is evidently obtained by lifting to the middle term over $U_i$'s and taking Čech coboundary $\gamma^N(\{\Sigma^f_{i,j} \cdot \Sigma^f_{j,i}\}) = \left\{ \left. \left( \gamma^N - \Sigma^f_{i,j} + \Sigma^f_{j,i} \right) \right|_{X_k} \right\}$, which restricts to $\{\Sigma^f_{i,j} \cdot \Sigma^f_{j,i}\} \in H^1(U_i, W)$ as desired.

Finally, quasi-homogeneity of $\gamma^N$ is just because $\delta[k]{\Sigma^f_{i,j} \cdot \Sigma^f_{j,i}} = (\delta[k]{\Sigma^f_{i,j}} - \delta[k]{\Sigma^f_{j,i}}) + (\Sigma^f_{i,j} - \Sigma^f_{j,i}) = (\Sigma^f_{i,j} - \Sigma^f_{j,i}) \in F^k D^{2k}$ for any hole.

A vector field $\Theta$ on $U_i$, $\nabla_\Theta \gamma^N = \left[ \Theta (\delta[k]{\Sigma^f_{i,j}} - \Sigma^f_{i,j}) \right] \in F^k H^{2,2}(X_k)$.
Remark: In the addendum we will discuss a more general and
panoramic version of this lemma, which will lead to Zucker's
Theorem on Minimal Fibrations.

Let $X = \{ \sum_{i=0}^{5} z_i^5 = 0 \} \subset \mathbb{P}^5$ be the Fermat quintic 4-fold $X$
and contain the 2-planes $x = e^{2\pi i/5}$

$P_0 = \{ z_3 = -x_5 z_1, z_4 = -x_5 z_0, z_5 = -x_5^3 z_1 \}$

$P_1 = \{ z_3 = -x_5 z_0, z_4 = -x_5 z_1, z_5 = -x_5 z_2 \}$

$P_2 = \{ z_3 = -x_5 z_2, z_4 = -x_5 z_1, z_5 = -x_5 z_0 \}$

For any 2 hyperplanes $H, H'$, we have $H \cdot H' (P_1 - P_2) = 0 \Rightarrow H_1(X) = \mathbb{Z}$

$H_1(P_1 - P_2) \cong 0 \Rightarrow (P_1 - P_2) \in H^{\text{prim}}_0(X, \mathbb{Z})$.

Exercise: $P_0 \cdot (P_1 - P_2) = 1$, hence $S := [P_1 - P_2] \neq 0$.

Now take a Let's take a pencil $X_t = \overline{\mathbb{P}^1 \cdot H_t}$ of (quartic 3-fold) hyperplane
sections (which are CY by adjunction); then $\text{lem} (H^3(X_t)) = 3 > 1$
and (by Lemma 4.3.2), we therefore have (for very general $t$) that $\text{lem} (H^3(X_t)) = 3$.

Moreover, $\text{H}^3(X_t)$ also gives (5) by the
lemma. Consider the "difference of lines" $\mathcal{Z}_t := H_t \cdot (P_1 - P_2) \in \mathbb{T}_{\text{mon}}(X_t)$,
and note that (by definition of $\mathcal{Z}_t$) $\mathcal{Z}_t = A T_{X_t}^1(\mathcal{Z}_t) \in T^2(X_t) = T^2(\mathcal{Z}_t)$

By (5) (and $t \neq 0$), this is nonzero for very general $t$,
and so $\mathcal{Z}_t \neq 0$. We conclude:

Theorem 1 (Griffiths): $\text{Griff}^2(X_t, \mathbb{Q}) \neq 0$ for very general $t$. 


Having seen an application of the topological invariant, we briefly turn to an application of the infinitesimal invariant $d$.

**Theorem 2 (Green–Voisin):** Let $X \subset \mathbb{P}^m$ $(m \geq 2)$ be a very general smooth hypersurface of degree $d \geq 2 + \frac{4}{m-1}$. Then $\text{image } (AJ^m_X) \subset J^m(X)_{\text{c}}$ is trivial.

**Remark:** This doesn't conflict with Griffiths's result since it only rules out "interesting" 1-cycles on 3-folds of degree $\geq 6$.

It also leaves open the door for some other "low degree" cases like cubic 7-folds (on which the Italian school has some nice work).

**Sketch:** Given $\mathcal{Z} \in \mathcal{Z}^m(X)_{\text{c}}$, spread out to $\mathcal{Z} \in \mathcal{Z}^m(X)$

\[
\mathcal{Z} = X \times \mathcal{F} \xrightarrow{\pi} \mathcal{F} \xrightarrow{\psi} \mathcal{G},
\]

where $\pi$ is a universal family on $\mathcal{F}$, and $\psi$ is an open immersion. The $\mathcal{Z}_x$ produce a morphism $\nu_2 \in \text{NE}_\eta^{-j}(\mathcal{G})_{\text{c}}$, $\nu$ con. to $R^{2m-1}_x \in \mathbb{Q}(m)$. It is sufficient to show $\text{c}-j \geq 0, j \geq 0$.

Since $\nu_2 = 0$ implies $\nu_3 = 0$, we have $\delta_{ij} = 0 = \nu_{i+1}(j) (\forall j \geq 0)$, since $\nu_2 = 0$.

To prove $\otimes$, i.e., the vanishing of the 0th and 1st cohomology of the complexes $\mathcal{F}_x \otimes \mathcal{E}^j \otimes \mathcal{O}(i-1) \mathcal{V} \xrightarrow{\mathcal{F}_x \otimes \mathcal{E}^j \otimes \mathcal{O}(i-2) \mathcal{V} \rightarrow \ldots}$, one uses the complexes and uses residue theory to write them in terms of polynomial algebra, at which point we use Hodge–Lefschetz symmetries hence deduce the truth. (See the Hodge Theory notes.)
The upshot of this result is that if you want $AJ$ to detect a cycle in (say) $\text{Griff}_3$ at a CY 5-fold, it can't be a very general degree 7 hypersurface in $\mathbb{P}^6$ — it has to be more special than that. Just as we have loci where the space of Hodge classes jumps up (and MTC jumps down), we have loci where the "image of $AJ$" jumps up, but these are less well-understood.

Finally, we look at Clemens's infinite generation result for $\text{Griff}_2$ of a very general quintic 3-fold (the situation of Griffith's theorem above), which was the first such result (predating Mori's). The story, which I will only sketch, due to the many technical details, begins with

$Q = \text{quartic K3 surface in } \mathbb{P}^3 \text{ having an elliptic fibration }$

$\{Q(x) = 0\}$

with section $L_0$, and additional section $L_1$ (deg 4) such that $L_1 - L_0$ is non-torsion:

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Using the group structure on the fibers (with $L_0 = "0"$) we set $L_n := "nL_1" \subset Q$ to get an infinite sequence of smooth rational curves.

Exercise: Show that the $(L_n)$ are necessarily rigid, and that their degrees $d_n \to \infty$. 

Now form the 3-fold \( X_{s, \lambda} \subset \mathbb{P}^4 \) given by \((f, F)\) generically.

\[
(6) \quad \lambda S(x_0, \ldots, x_5) \mathcal{L}(x_0, \ldots, x_5) + x_4 f(x_0, \ldots, x_5) + \lambda F(x_0, \ldots, x_5) = 0.
\]

For general \( s \), \( X_{s, 0} \) has 16 ODPs (nodes), along \( s_5 = x_4 = f = 0 \).

Since \( \{q = x_4 = f = 0\} \subset C \subset \mathbb{Q} \) intersects the \( \{L_n\} \) transversally, we may choose the pencil \( \lambda S \) so that for any \( s \) at most one \( L_n \) hits a node. As \( s \) moves, the nodes sweep out \( C \), so they eventually hit every \( L_n \) (more than once). For each \( n \), pick an \( s_n \) such that \( L_n \subset X_{s_n, 0} \) hits a node. Clemens shows that, in the 1-disc about \( (s_n, 0) \), the \( \{L_{m+n}\} \) deform, but \( L_n \) deforms to a multi-valued family (in \( \sqrt{\lambda} \)). So we pull back under \( \mu \rightarrow \mu^2 = \lambda \), which gives 16 ODPs in the total space over the \( \mu \)-disc through \( (s_n, 0) \). Clemens blow up these up so as to have a semi-stable degeneration (the nodes get replaced by additional components of the singular fiber — 16 generic 3-folds).

Here is the key point: we consider, over the complement \( U \subset \mathbb{P}^1 \times \mathbb{C}^n \) of the discriminant locus of the family of varieties, the normal function

\[
(7) \quad v_n(s, \lambda) := \operatorname{AJ}_{X_{s, \lambda}}^2 (5L_n - \lambda_n H).
\]

We will show that \( \Gamma = \mathbb{Z} \langle v_n \rangle \subset \text{ANF}(V) \) is infinitely generated \( \otimes \mathbb{Q} \), by computing monodromies — more precisely,
The integer singularity classes described at the end of §4.

Write $T_j$ for the monodromies in $H^g(X)$ attached to the $p$-counter-clockwise loops about $(s_j, \theta)$. (These loops are not homotopic, as there are many divisors other than $d = 0$ in play.) Writing

$$5\Gamma_n = 5 \Gamma_n - d_n H \simeq 0$$

(\text{of course}), we compute

$$\sigma_j := \frac{(T_j - I) H^3(X, \mathbb{Q}) \cap H_3(X, \mathbb{Q})}{(T_j - I) H^3(X, \mathbb{Q})}$$

Then as mentioned in §4, just the restriction of the topological invariant to a $\Delta^k$ about $(s_j, \theta)$.

Now for $n \neq j$, one can choose $\Gamma_n$ so it doesn't pass through the (resolved) node. So $T_j \Gamma_n = \Gamma_n$. For $n = j$, a well

\text{known (essentially)} Reidemeister lemma tells you that since $\Gamma_n$ has an $m$th (a node, $(T_n - I) \Gamma_n = 5 \sigma_j(n)$ where $\sigma_j(n) \in \{1, \ldots, 16\}$ is the vanishing cycle attached to the node (prior to blowing up). So, all told,

$$\{\text{Sing}_{(s_j, 0)}(\mathbb{Q}) \}_{j \in \mathbb{N}} = \{5 \sigma_j \}_{j \in \mathbb{N}} \in \bigoplus_{j \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z} < \sigma_j - \sigma_j >$$

$\Rightarrow$ image of $\Gamma$ in the $\mathbb{Z}$-group $G$ is so generated. Now

$$\Gamma_{\text{tors}} \subset J(X_{s_j})_{\text{tors}} \subset (\mathbb{Q}/\mathbb{Z})^r$$

for some $r$.

Lemma: Let $\Gamma'$ be an abelian group with $\Gamma'_{\text{tors}} \subset (\mathbb{Q}/\mathbb{Z})^r$.

$$\text{dim}_{\mathbb{Q}}(\mathbb{F}/2 \Gamma') \leq \text{dim}_{\mathbb{Q}}(\mathbb{Q} \otimes \mathbb{Q}) + r$$

Proof: Exercise.
If \( \mathcal{O} \) is finitely generated, then so would \( \mathcal{O}/\mathcal{O}' \) be, of which the image of \( \mathcal{O} \) in \( \mathcal{O}' \) is a quotient. (Contradiction!)

We conclude:

**Theorem 3 (Clemens):** \( \text{Griff}^2(X) \otimes \mathbb{Q} \) and image \( \overline{\text{Pic}^2} \) are (countably) infinite-dimensional for \( X \) a very general quintic 3-fold in \( \mathbb{P}^4 \).

---

**Addendum (on a refinement of the Lemma on Normal Functions):**

We mentioned previously that normal functions arising from families of algebraic cycles are admissible. In fact this now holds for a NF arising from a primitive Hodge class. In (4), we can replace NF by \( \text{ANF}^2 \) and both of the right-hand terms by the rational \((\Theta, 0)\) classes in \((\text{the canonical pure weight } 0 \text{ HS on}) \ H^1(\mathbb{P}^1, j^*V)\). More precisely, we have

\[
(4') \quad \frac{H_g^{\text{prim}}(X) \otimes H_g^{k-1}(\Theta)(-1) \otimes H_g^{k-1}(\mathcal{O}_x)(-1)}{\mathbb{Z}} \xrightarrow{\cdot j} H_g(H^1(\mathbb{P}^1, j^*V))
\]

(\( j \) here means \( \mathcal{O}_x \) maps into this)
the base locus, \( B \) of the pencil is assumed smooth.

- \( W = \mathbb{P}^{2k-2} \times \pi^* \mathcal{Q}(1) / H^{2k-1}(X, \mathcal{Q}(1)) \) is assumed to have no constant sub-local system.

- The numerator of the "big" term is \( H^k_g(X)_{\text{prim}} = \ker \{ H^k_g(X) \to H^k_g(X_\epsilon) \} \)
  and \( H^k_g(B)^0 = \ker \{ H^k_g(B) \to H^k_g(X_\epsilon) \} \) is mapped into it by pulling back to \( B \times \mathbb{P}^1 \) (the exceptional divisor) and pushing forward.

- \( H^k_{_{X_\epsilon}}(X) \cong H^k_{_{X_\epsilon}}(X_\epsilon) \), \( X_\epsilon \) is union of singular fibers.

- The right-hand term includes into those of (4) via the short exact seq.

\[
0 \to H^1(P^1, j^* W) \to H^1(P^1, R^j_\mathcal{E} W) \to H^0(P^1, R^j_\mathcal{E} W) \to 0
\]

(basically Leray's spectral sequence).

- If the pencil is Lefschetz, then \( H^k_g(B)^0(1) \) and \( (\mathcal{E}) \ast H^k_{_{X_\epsilon}}(X) \) cancel, and the diagram becomes

\[
(4'') \quad H^k_{g_{\text{prim}}} (X) \oplus H^{k-1}(B)^0(-1) \xrightarrow{\cdot \mathcal{E}_L} H^k_{g_0} (H^1(P^1, j^* W)),
\]

\[
\xrightarrow{\cdot \mathcal{E}_L} \quad \xrightarrow{\cdot \mathcal{E}_L} \quad \xrightarrow{\cdot \mathcal{E}_L}
\]

\[
\mathcal{N}^0_\mathcal{F} (W)
\]

which will be important in \( \S 7 \).

As we shall see, \([\cdot, \cdot]\) in (4') is the refinement of the topologies invariant referred to at the end of \( \S 4 \).

We now turn to the proof, which we shall use to introduce some concepts. First consider the logarithmic version of the complexes \( C \varepsilon \mathcal{E} F^\bullet C \varepsilon \) defined in \( \S 4 \), where for simplicity we
will stick to a VHS $V$ over a curve $\Delta \subset \overline{\Sigma} (\xi = \overline{\xi} \not \subset 8)$ with canonical extension $V_e$ (all monodromies assumed unipotent). Write
\[(\mathbf{10}) \begin{aligned}
C^*_e &:= V_e \xrightarrow{\nabla} \mathcal{L}^{1}_{\xi} ( \xi ) \otimes V_e \\
F^0 C^*_e &:= f^0_e \xrightarrow{\nabla} \mathcal{L}^{1}_{\xi} ( \xi ) \otimes f^{-1}_e
\end{aligned} \]
and, recalling that $j^* V^\otimes Z$ has stalks equal to stalks of $W_Z$ at $x \in \Sigma$, consider the exact sequence
\[(\mathbf{11}) \quad 0 \to j^* V^\otimes Z \oplus F^0 C^*_e \to C^*_e \to \frac{C^*_e}{F^0 C^*_e \oplus j^* V^\otimes Z} \to 0 \]
of sheaves on $\overline{\Sigma}$. Now looking at
\[(\mathbf{12}) \quad j^*_e \text{ker}(V) := H^0\left( \frac{C^*_e}{F^0 C^*_e \oplus j^* V^\otimes Z} \right) = \ker \left\{ \frac{V_e}{j^0_e + j^* V^\otimes Z} \xrightarrow{\nabla} \mathcal{L}^{1}_{\xi} ( \xi ) \otimes \frac{V_e}{f^{-1}_e} \right\},
\]
the natural question is "what is it the sheaf of (quasi-horizontal) sections of?". Composing its restriction to $x \in \Sigma$, we get
\[(\mathbf{13}) \quad \ker \left\{ \frac{V^\text{lim}_{x,\Sigma}}{F^0 V^\text{lim}_{x,\Sigma} + \{\ker(T_x - I) \subset V^\text{lim}_{x,\Sigma}\}} \xrightarrow{\text{Res}_x(V)} \frac{V^\text{lim}_{x,\Sigma}}{F^{-1} V^\text{lim}_{x,\Sigma}} \right\}
\]
which looks at first puzzling until one realizes that $-\frac{1}{2\pi i} \text{Res}_x(V) = N_x = \log (T_x)$.

A brief inspection of the picture should convince you that
\[
\ker \left\{ \frac{V^\text{lim}_{x,\Sigma}}{F^0 V^\text{lim}_{x,\Sigma} \cap \ker(N)} \rightarrow \frac{V^\text{lim}_{x,\Sigma}}{F^{-1} \cap \ker(N)} \right\}
\]
which is to say

\[\text{ignoring degree shifts, this is a "baby" example of a perverse sheaf on } \overline{\Sigma} : \text{not quite a local system, but a logarithmic extension of the notion.}\]
that (8) becomes

\[
\frac{\ker(N) \cap V^\text{lin}_{x, e}}{F^0 \ker(N) + \ker(T_x - I)_x} = \text{Ext}^1_{\mathcal{MHS}}(\mathcal{Z}(0), \ker(T_x - I)) = \text{J}(\ker N_x).
\]

So if we define \( J_e(V) \) to be the "slit complex analytic space" which is \( J(V) \) over \( \mathcal{S} \) and \( J(\ker N_x) \) (of lower dimension!!) over \( x \in \Sigma \), then \( \mathcal{O}_x, \ker(V) \) is the sheaf of its quasi-holomorphic sections.

Next I want to convince you that (working notation again)

\[
A_N^e(\mathcal{V}) = \Gamma(\overline{\mathcal{S}}, \mathcal{O}_x, \ker(V)) = H^0(\overline{\mathcal{S}}, \frac{C_e}{p^m C_e + j e W}),
\]

so that in this case we see very concretely what admissibility means. Refer to Definition I.C.4.3 for notation, there were 2 criteria:

(a) and (b) (for admissibility):

\begin{enumerate}
\item (a): \( \exists \) lift \( \nu_e \) satisfying \( N_x \nu_e \in W_{-2} V^\text{lin}_x \) (\( \subset N_x V^\text{lin}_x \))
   \[ \Rightarrow \text{can modify it by } (\overline{\nu})_x \text{ to get } N_x \nu_e = 0. \]
   [i.e., the local monodromy of the NF in \( H^1(\Delta^*_x, W_e) \) is zero:
   we say it has (virtually) a local lifting at \( x \)]
\item (b): \( \exists \) lift \( \nu_e \) to \( \tilde{\mathcal{V}}_e \) with \( \nu_e |_{\Delta^*_x} \) in \( \mathcal{O}_x \)
   \[ \Rightarrow N_x \nu_e \in \Gamma(\Delta, \nu_e) \text{ and } \mathcal{D}(\nu_e - \tilde{\nu}_e)_{\Delta^*_x} = -\nabla \nu_e \in \Gamma(\Delta^*_x, F^{-1} W) \]
   \[ \Rightarrow \nu_e - \tilde{\nu}_e \text{ (suitably lifted) sends a value in } J(\ker N_x), \text{ as above.} \]
\end{enumerate}

Exercise: Show the converse (we have just done \( \subset \) in (a); you do \( \supset \), which is now easy).!!

One can think of (b) as saying the normal function \( \nu \) has logarithmic growth in the left-hand pencil setting. Together with local liftability, that's why
Now we can use (15) to "compute" $\text{ANF}_g(V)$: the
long-exact sequence associated to (11) gives (weakly formally)

$$
\begin{align*}
0 \to & \frac{H^0(C_e \cdot)}{H^0(F^0 C_e) \oplus H^0(j_V)} \xrightarrow{\varphi} \text{ANF}_g(V) \xrightarrow{\beta} \ker\{H^1(j_{g*} V) \oplus H^1(F^0 C_e) \to H^1(C_e)\} \\
& \xrightarrow{=} J(V_{\text{fix}}) \xrightarrow{H g} H^1(Y, j_V)
\end{align*}
$$

where $V_{\text{fix}}$ is the largest constant sub-VHS of $V$. The left-hand side is easy (Exercise), while the way to understand the right-hand side is this: the injectivity of the topological invariant $\alpha$ of $\beta$ in an ANF, together with the inclusion $H^1(S, j_* V) \hookrightarrow H^1(S, W_0)$, shows that in the long-exact sequence associated to $0 \to j_* V \to C_{e/F^0 C_e} \to \frac{C_{e/F^0 C_e}}{F^0 C_e \cdot j_* V} \to 0$, $H^0(j_* V) \to H^0(C_{e/F^0 C_e})$ is surjective. If $V_{\text{fix}} = \{0\}$, then $H^0(j_* V) = 0 = H^0(C_{e/F^0 C_e}) = 0 \Rightarrow H^1(C_{e/F^0 C_e}) \to H^1(C_e)$ (true also in general case, Exercise). Moreover, $C_{e/F^0 C_e} \cong R j_* V_0 \Rightarrow H^1(C_{e/F^0 C_e}) \cong H^1(S, W_0) \Rightarrow H^1(S, j_* V) \to H^1(C_e)$. So the RHS of (16) is really the intersection of $H^1(j_* V)$ and $H^1(F^0 C_e) = F^0 H^1(S, W_0)$ in $H^1(S, W_0)$. Of course, since $\beta$ refines the mapping from ANF to $H^1(S, W_0)$, $\beta$ is our refinement of the topological invariant and is henceform denoted $\{\cdot\}$.

Specializing back to the setting $S \cong \mathbb{P}^1$, we now examine the cohomology of $X$. One way to do this efficiently is with the Decomposition Theorem (originally conceived
by Beilinson–Bernstein–Deligne–Gabber (for perverse sheaves, and then for MHM by M. Saito–de Cataldo–Migliorini). \[ \text{[Be forewarned that I will not use the "shifts" necessary to make this computation completely kosher.]} \]

This theorem states in our setting that there is a non-canonical quasi-isomorphism in \( D^b_{\text{MHM}}(\mathbb{P}^1) \)

\[
(\text{17}) \quad R^i\pi_* Q_X \cong \bigoplus_i H^i \pi_* Q_X [-i],
\]

where furthermore (for each \( i \))

\[
(\text{18}) \quad H^i \pi_* Q_X \cong \bigoplus \bigoplus_{\pi} R^2 \pi_* Q_{X_{\pi}} \oplus \bigoplus_{\pi} R^2 \pi_* Q_{X_{\pi}}.
\]

By the Leray spectral sequence we then have an \( \cong \) of HS

\[
H^k(\mathbb{P}^1, R^i\pi_* Q_X) \cong H^k(\mathbb{P}^1, R^i\pi_* Q_X) \cong H^k(\mathbb{P}^1, R^2 \pi_* Q_{X_{\pi}}) \oplus H^k(\mathbb{P}^1, R^2 \pi_* Q_{X_{\pi}})
\]

\[
(\text{19}) \quad R^2 \pi_* Q_{X_{\pi}} \cong H^0(\mathbb{P}^1, i^* R^2 \pi_* Q_{X_{\pi}}) \oplus H^2(\mathbb{P}^1, i^* R^2 \pi_* Q_{X_{\pi}})
\]

In particular, we have

\[
(\text{20}) \quad H^2_{\text{prim}}(\mathbb{P}^1) \cong H^2_{\text{prim}}(\mathbb{P}^1, R^i\pi_* Q_X) = \bigoplus_{\pi} H^2_{\text{prim}}(\mathbb{P}^1, R^i\pi_* Q_{X_{\pi}}) \oplus H^2_{\text{prim}}(\mathbb{P}^1, i^* R^2 \pi_* Q_{X_{\pi}})
\]

Exercise: RHNS (20) can be rewritten \( H^2_{\text{prim}}(\mathbb{P}^1) \oplus H^2_{\text{prim}}(\mathbb{P}^1, i^* R^2 \pi_* Q_{X_{\pi}}) \oplus H^2_{\text{prim}}(\mathbb{P}^1, R^2 \pi_* Q_{X_{\pi}}) \).

\[ \text{[Use elementary blow-up formula, not DT]} \]

\[ \text{[\# Meaning of } R^i_{\text{prim}} \pi_* : H^i_{\text{prim}}(X_{\pi}) \cong H^i_{\text{prim}}(X_{\pi}) \text{ is } \ker \left( H^i_{\text{prim}}(X_{\pi}) \to H^i_{\text{prim}}(X_{\pi}) \right) \text{ for } \pi \text{ sing, simple, } H^0 \text{ trivial, tubular, etc.]} \]
Now using the Clemens-Schmid sequence
\[ \cdots \to H^{2k}(X_{\delta}) \to H^{2k}(X_{\delta}) \to H^{2k}(X_{\varepsilon}) \to 0 \]
we identify \( \bigoplus_{\sigma \in \Sigma} H^{2k}(X_{\sigma}) \) with \( \bigoplus_{\sigma \in \Sigma} H^{2k}(X_{\sigma}) \), and so quotienting \( H^{2k}_{\text{prim}}(X) \)
by this gives \( H^{2k}_{\text{prim}}(X) \oplus \bigoplus_{\sigma \in \Sigma} H^{2k-2}(X_{\sigma})(-1) \) by the Lemma and \( H^1(\mathbb{R}^1, I_+ V) \) by \( (20) \). Taking Hodge classes we get the \( \mathbb{C} \) on the top of \( (4') \).

At least for intermediate periods, this is a subdiagram of \( (4') \) so we don't need to prove it commutes. The only thing left is to check that \( \sigma(\cdot) \) actually goes into ANF and not just NF. All we need to show is that applying the procedure in diagram \( (2) \) yields an element of \( \tilde{J}(\text{ANF}) \) and not just \( \tilde{J}(\text{NF}) \). The trick is thus:

\[ \cdots \to J^k(X) \to H^{2k}(X, \mathbb{Q}(k)) \to H^{2k}(X, \mathbb{Q}(k)) \to 0 \]

the image of \( (x) \) is not zero, but it is the same as that of \( H^{2k}_{\text{prim}}(X) \) (i.e. of \( H^{2k}_{\text{prim}}(X) \)), see \( (21) \). Moreover, changing \( \sigma \in H^{2k}_{\text{prim}}(X) \) by an element of \( \tilde{J}(\text{ANF}) \) doesn't change it on smooth.
This so doesn't change $\psi$. So we may kill the image of $\delta$ by $(\delta)$ for free, and then $(\delta \delta)$ gives a well-defined element of $J^k(x) / \delta J^k(x) \cong J^k(\frac{H^{2k+1}(x)}{ \delta H^{2k+1}(x)})$. But the relevant version of Cleary-Schmidt

$$
(21') \quad \Rightarrow H^{2k+1}(x) \Rightarrow \Rightarrow H^{2k+1}(x_0) \Rightarrow \Rightarrow H^{2k-1}(x_0) \Rightarrow N
$$

tells us that this is just $J^k(\ker N)$. That was one long proof, but now we'll be ready to discuss the Hodge Conjecture in 27.