We now give a second application of infinitesimal invariants.

To begin, we construct the homogeneous variation of Hodge structure (over locally symmetric varieties). Let \( g \) be a semisimple Lie algebra with complexification \( g_\mathbb{C} \), and compact Cartan subgroup \( t_\mathbb{C} \);

write \( \Delta = \Delta_c \sqcup \Delta_n \) for the decomposition into compact and noncompact roots, and pick \( E \in \bigoplus_{\Delta_c} \mathbb{Z} \) with \( \frac{1}{2} E(\Delta_c) \subset \mathbb{Z} \), \( \frac{1}{2} E(\Delta_n) \subset \mathbb{Z} + 1 \).

The Theorem of the highest weight gives a 1-to-1 correspondence

\[
\begin{array}{c|c}
\text{finite-dimensional} & \text{dominant integral} \\
\text{irreps of } g & \text{weights } \lambda \\
\text{" } \nabla^{\lambda} \text{"} & \text{in } \mathbb{C} \cap \Lambda \\
\text{Borel-Weil thm.} & \text{weight lattice}
\end{array}
\]

which

\[
\lambda = \sum m_i \alpha_i \quad (m_i \geq 0)
\]

by fundamental weights \( \alpha_i \)

(\( \Lambda \) as above for \( g \)).

So the (finite-dim.) reps. of \( g \) take the form \( (\text{Lie } g, \nabla^{\lambda}) \).

\[
\nabla^{\lambda} = \begin{cases}
\nabla^{\lambda} & \text{"real" case} \\
\nabla^{\lambda} \oplus \nabla^{t(\lambda)} & \text{"quaternionic" case} \\
\left( \frac{1}{\sqrt{2}} \right) \begin{pmatrix} \nabla^{\lambda} \\ \nabla^{t(\lambda)} \end{pmatrix} & \text{"complex" case}
\end{cases}
\]

where \( t = -w_0 \) (\( w_0 \in W = W(\text{GC}_k) \) the longest element). Assuming \( E(\lambda) \in 2\mathbb{Z} + 1 \), the decomposition

\[
\nabla^{\lambda} = \bigoplus_{\rho \in \mathbb{Z}} E_{(2\rho + 1)} \text{Lie } \rho \text{ (}\rho \text{th \textit{w}eigenvalue \textit{2} in \textit{for } \text{Lie } \rho \text{)}}
\]
defines an $R$-HS of weight $(-1)$ and level $-E(\lambda)$ on $\mathcal{V}$. (One can show this is polarized by a unique $g_0$-invariant alternating form $\Omega$.)

Now take $G = \text{semisimple } R$-algebraic group of Hermitian type, s.t. $G_{\mathbb{R}}$ contains a compact Cartan subgroup $T_{\mathbb{R}}$. Choose a cocharacter $\chi_0 : G_{\mathbb{C}}^\times \to T_{\mathbb{C}}$ s.t. $E := \chi_0'(1)$ satisfies

\begin{equation}
E(\Delta_0) = 0 \quad E(\Delta_n) = \{ \pm 2 \},
\end{equation}

That is, the $(\text{ad } E)$-Hodge decomposition of $g_{\mathbb{C}}$ takes the form

\begin{equation}
g_{\mathbb{C}} = g_{\mathbb{C}}^{-1} \oplus g_{\mathbb{C}}^0 \oplus g_{\mathbb{C}}^{+1}.
\end{equation}

Then $\Delta_n \cap \Sigma = \{ \delta_i \}$ is a special simple root, i.e. $\lambda_{ad} = \delta_i + \sum_{j \neq i} m_{ij} \delta_j$ and $E(\delta_i) = -2 \delta_i$. [In this way, the choice of $\Sigma$ (for amongst the special nodes of the Dynkin diagram of $g_{\mathbb{C}}$) determines the real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$.] The $\rho_\lambda \circ \chi_0$, resp. Ad$\circ \chi_0$, eigenspaces recover the (composible) Hodge decompositions (3) resp. (5). To verify them, compose

\begin{equation}
G_{\mathbb{C}} \xrightarrow{\phi_0} T_{\mathbb{C}} \xrightarrow{} G_{\mathbb{C}}
\end{equation}

and take the orbit of (6) under conjugation

\begin{equation}
D := G(\mathbb{R}) \cdot \phi_0 \cong G(\mathbb{R}) / (G_0(\mathbb{R})^{0}) \langle \text{center of } g_{\mathbb{C}} \rangle \cong \text{Lie } (G_0) = g_{\mathbb{C}}^{0,0} \cong g_{\mathbb{C}}^{0,0}.
\end{equation}

This is a Hermitian symmetric domain of dimension $D = \text{dim } g_{\mathbb{C}}^{-1}$. Taking

$\Gamma \subseteq G(\mathbb{Z})$ torsion-free of finite index, the quotient

\begin{equation}
\bar{X} := \Gamma \backslash D
\end{equation}

is a quasi-projective (weakly symmetric) variety by the Baily-Borel theorem.
Varying the choice of root system and special node, we get
the classification of irreducible Hermitian symmetric domains into types
(9) \( I_{p,q} \) (elliptic), \( II_{n} \) (quaternionic), \( III_{n} = \text{Siegel } h_{n} \), \( IV_{n} \) (orthogonal),
EIII \& EVII (exceptional).

Now fix \( \Sigma \) in (8) and \( \varphi_{0} \in \text{D in (6)} \), as well as a
\( \mathbb{Q} \)-linear representation \( \rho \) : \( G \to \text{Aut}(V, \mathbb{Q}) \) s.t. \( \rho \varphi_{0} \) is a HS on \( V \)
polared by \( \mathbb{Q} \). Then the \( \{ \rho \circ \varphi \circ \rho \}_{\varphi \in \text{G}(\mathbb{Q})} \) give a variation of HS
over \( \Sigma \) with geometric monodromy (and derived MTC) \( G \).* Putting this
construction of Hermitian (homogeneous) VHS together with (1)-(3), we
get a bijection

\[
\text{(10) irreducible Hermitian } \mathcal{R} \text{-VHS } / \Sigma \text{ of weight } (-1) \quad \leftrightarrow \quad \{ \text{dominant integral } \lambda \} \text{ with } E(\lambda) \text{ odd} \}
\]

\[
\left\{ \lambda = \sum_{i} m_{i} \omega_{i} ; \quad m \geq 0 , \quad \sum_{i} m_{i} E(\omega_{i}) \in 2 \mathbb{Z} + 1 \right\}
\]

In all the situations we shall consider, \( \tilde{V}_{R} \) has an underlying
\( \mathbb{Q} \)-VHS \( \tilde{V}_{K} \).

Recalling the \( \{ \mathcal{H}^{k}_{j} \} \) from 8.4, where we shall denote
\( \mathcal{H}^{k}_{j}(j) \) for the irreducible \( \mathcal{C} \)-VHS \( \mathcal{V}_{ao}^{j} \) \( \neq \) (so that in the
"non-real" cases \( \mathcal{H}^{k}_{j}(j) = \mathcal{H}^{k}_{a0}(j) \oplus \mathcal{H}^{k}_{s}(j) \)), the key point
is that we can compute these via representation theory:

* This looks strange but the \( \mathcal{F}^{c} \subset \mathcal{V}_{ao} \) and \( \mathcal{C} \)-local system \( W_{c} \) are all

\( \mathcal{F}^{c} \subset \mathcal{V}_{ao} \) and \( \mathcal{C} \)-local system \( W_{c} \) are all
Theorem 1 (Kostant-K) : 
\[ H^k_j (\varphi) \cong \bigoplus_{w \in W(k,j)} V^w \cdot \lambda, \]

where \( \varphi \) means the fiber over \( \varphi_0 \in D \)

- \( W^0(k,j) = \{ w \in W \mid w(\Delta^+) \supset \Delta^+, \ w|\Delta^+ = k, \ \text{and} \ E(w \cdot \lambda) = 2j+1 \} \)

- \( w \cdot \lambda = w(\lambda + \rho) - \rho \), \( \rho = \frac{1}{2} \sum \alpha \in \Delta^+ \delta(\alpha) \)

- \( V^w \) is the irrept. of \( g_0 \) of highest weight \( \lambda \).

(The part I won't do, use Kostant's theorem on Lie algebra cohomology and the identification \( \bigoplus H^k_j (\varphi) \cong H^k_j (g_0, V) \).

For any \( \lambda \) as in (10) we have the

**Corollary:** \( H^0_j (\varphi) = 0 \) \( \forall j \geq 0 \)

- \( H^1_j (\varphi) \neq 0 \) \( \iff \) \( j = \mu(\lambda) := \frac{1}{2}(E(\lambda) - 1) \),

where \( w_\lambda \) is the reflection in the unique simple root \( \sigma_\lambda \) with \( E(\sigma_\lambda) = 1 \) \( \iff w_\lambda \).

Therefore, \( NF_{\tilde{J}} j^* V) = 0 \) for any étale neighborhood \( \tilde{J} \to \times \) if \( \mu(\lambda), \mu(\tau(\lambda)) < 0 \).

This turns out to rule out many functions outside of a short list of cases, including \( III_3, IV_3 \) for the Ceresa cycle (of course), part of which is given in Theorem 2 below.

There is one other important point: recall that for \( g=3 \), the Ceresa cycle was defined on a 2:1 cover of \( \mathbb{G}_{a_3}^3 = [\text{smooth space of 3-dim PPAVs}] \cong \text{Sp}_6(\mathbb{R}) / \mathbb{O}_3 \), but does it fail to
push it down to \( A_3 \). There is a reason for this: if \( G \) has \( \text{rank} > 1 \) (i.e. isn't \( SL_2 \) or \( U(n,1) \), for our purposes), then for any arithmetic \( \Gamma \leq G(\mathbb{Q}) \) we have

\[
G = H^1(\Gamma, \tilde{V}^\lambda) = H^1(\Sigma, \tilde{V}^\lambda)
\]

by a theorem of Raghunathan. Since admissible normal functions extend to the Zariski closure inside the domain where a VHS is defined (nonsingular), \( ANF_U(\tilde{V}^\lambda) = ANF_{\Sigma}(\tilde{V}^\lambda) \) if \( \Gamma \) is torsion-free, so the topological invariant gives

\[
ANF_U(\tilde{V}^\lambda) \to H^1(\Sigma, \tilde{V}^\lambda) \quad \text{for any } U \subseteq \Sigma.
\]

Hence \( ANF_U(\tilde{V}^\lambda) \) vanishes, and we are forced to look for an
normal function on \( \text{étale neighborhood} \ T \) (finite cases of \( U \)).

\[\textbf{Theorem 2 (Kest-K):} \quad \text{For } D \text{ of "tube type", the only pairs } \]
\[(D, \lambda) \text{ s.t. } \tilde{V}^\lambda \to \mathbb{P}^D \text{ has the possibility of admitting a } \]
nontrivial bound function over an \( \text{étale neighborhood, are } \]

\[
\left\{
\begin{array}{c}
(I_2, 2, \frac{w}{6} + aw_2), (I_3, 3, w_3), (II_4, a_1 + a \{w_1\}), (II_6, w_6), \\
(III, a_2), (III_2, w_1 + aw_2), (III_3, w_3), (III_{2n}, a_1 + aw_2), (III_{2n-2}, a_1 + \{a_1\}),
\end{array}
\right.
\]

and \( (EVI, w_7) \).

We were able, so far, to construct one new normal function
for a case on this list: \((I_3, 3, w_3)\), which is the
irreducible VHS of type \((1, 9, 9, 1)\) contained in \( H^3 \) of the
universal Weil oblian 6-fold and $\pi \backslash SU(3,3) / K = B_{3,3}$. 

As in the setting of Schen's proof of the Hodge conjecture for certain Weil 4-folds, a moduli space of generalized Prym varieties dominates $B_{3,3}$. That is, the general Weil 6-fold parametrized by $B_{3,3}$ arises as the quotient

$$A = J(C^\sim) / J(C)$$

where $C^\sim \to C$ is an unramified 3:1 cover. Now (gss 10) (gss 4)

one defines the Prym - Ceresa cycle $Z_{\mathbb{C}/C} \in \text{Griff}_1(A)_q$

to be the push-forward of the Ceresa cycle $Z_\mathbb{C} \in \text{Griff}_1(J(C))_q$
to $A$. By degenerating C

$$\text{gss 9} \quad \rightarrow \quad \text{gss 3}$$

as in Ceresa, we reduce nonvanishing of $\overline{AT}^5(Z_{\mathbb{C}/C})$ to that of the Ceresa normal function, thereby producing the desired normal function over a finite cover of $B_{3,3}$.

$B_{3,3}$ These are Weil 6-folds with multiplication by $\mathbb{Q}(S_3)$ and a certain polarization invariant.