D. Higher Abel–Jacobi maps and filtrations on Chow groups.

1. Mumford’s Theorem on 0-cycles

We have seen so far that, for cycles of codimension at least 2 on a smooth projective variety \( X/\mathbb{C} \), \( \text{AJ} \) is not in general surjective. Still, if \( H^0(X_0) \) injects into \( \mathbb{I}^0(x_0) \), we would know a lot of the structure. The fundamental result we’ll explain in this section is that this is not the case for “most” varieties, so that further Hodge–theoretic invariants are needed to understand the Abel–Jacobi kernel. We’ll prove the result for 0-cycles on a surface, and discuss generalizations and related results and conjectures.

Lemma 1: Let \( F = \overline{F} \subseteq \mathbb{C} \), with \( \mathbb{C} \) finitely generated (e.g.) \( /\overline{F} \). Then \( \overline{F}/\overline{F} \) smooth, projective & a point \( p \in \mathbb{Y}(\mathbb{C}) \) s.t. \( \overline{F}/\overline{F} \). Proof: \( \mathbb{E} = \overline{F} / (\overline{F}(\mathbb{Y}(\mathbb{C})) \rightarrow \text{homomorphism} \ p : \overline{F}[x_1, \ldots, x_5] \rightarrow \mathbb{E} \),

with \( R = \mathbb{C}(x) \) a domain ring \( \Rightarrow \quad I = \ker(\varphi) \) a prime ideal \( \Rightarrow \quad Y = \text{Var}(I) \subseteq \mathbb{A}^{t+5} \) reducible, of dimension \( t = \text{dim}(\mathbb{E}) \). So \( \overline{Y} : \overline{F}[x_1, \ldots, x_5] \rightarrow \mathbb{E} \) induces (by taking fraction fields) \( \overline{F}(\overline{Y}) \rightarrow \mathbb{E} \), which is evaluation at \( p = (\overline{y}_1, \ldots, \overline{y}_5) \in Y(\mathbb{C}) \).

Now take \( \overline{y} \in Y \) distinguished & \( \overline{Y} \sim \overline{Y} \) a good compactification, done. \( \square \)

Remark: We will often mean by “very general point” a point of
maximal transcendence degree, i.e., \( p = \infty \) in the clear point. \( \Box \)

Now let \( \pi : X \to S \) be a morphism of smooth proj. varieties over \( \mathbf{K} \), and set \( L = K(S) \). A choice of hyperplane section (given by a linear form \( H \)) induces a \( \mathbf{K} \)-algebra homomorphism

\[
\begin{array}{ccc}
K[S] & \xrightarrow{\phi} & L \\
\text{(homog.)} & \xrightarrow{\pi} & \text{(coord.)} \\
\text{(refined)} & \xrightarrow{\deg} & \mathbf{F}/H^d \\
\end{array}
\]

Let \( \eta : \text{Spec}(L) \to S \) be the generic point induced by \( \eta \), and set

\[
X_\eta := X \times_\eta \text{Spec}(L).
\]

**Lemma 2:** \( \text{CH}^0(X_\eta) \cong \varprojlim \text{CH}^0(\pi^{-1}(U)) \).

**Proof:** The restriction \( r : \text{CH}^0(S) \to \text{CH}^0(X_\eta) \) is surjective since we can clear denominators of equations for any cycle on \( X_\eta \). Now, an irreducible \( Z \in \mathcal{Z}^0(S) \) vanishing in \( \mathcal{Z}^0(X_\eta) \)

\[
\mathcal{Z} \in \pi^{-1}(E), \text{ for some } E \subset S \text{ of codim. } 1.
\]

In the localization sequence (with \( \mathcal{S} = S \setminus \text{codim. } 1 \))

* There is, we can replace \( L \)-cohom. by \( K[S] \)-ones.
21 \rightarrow CH^p_\psi (\pi^{-1}(E)) \rightarrow CH^p(X) \rightarrow CH^p(\pi^{-1}(U)) \rightarrow 0,

It clearly factors through the right-hand term. Finally, if \( W \in \mathcal{E}(X_\psi, \phi') \) is a rational equivalence inducing \( r(3) \equiv 0 \), clearly denominator is its cuspforms yields \( \tilde{W} \in \mathcal{E}(X_\psi, \phi') \) inducing \( \tilde{\gamma} \equiv 0 \), where \( \tilde{\gamma} \) vanishes in \( \mathcal{E}(X_\psi) \). So the map \( \phi \) is injective in the limit.

Remark 2: I'll sometimes write \( n \) to indicate when \( n \) is a generic point of.

Lemma 3: Let \( F \in \mathcal{E} \in \mathbb{C} \) with no other assumptions, and \( Y/F \) smooth projective. Then \( \ker (CH^p(Y/F) \rightarrow CH^p(Y/F)) \) is torsion.

Proof: Suppose first that \( F \in \mathcal{F} \), and \( Z \in \text{kernel} \). The rational equivalence (of \( Z \) to 0) is given by a cycle \( W \in \mathcal{E}(Y/F, \phi) \). Collecting coefficients of its defining equations yields an algebraic extension \( \tilde{F} \) of \( F \). But then \( Z \) is defined over \( \tilde{F} \), and gives a rational equivalence of \( Z \) to 0.

Next we can let \( F = \mathcal{F} \) and \( E = \mathbb{C} \) (all other cases follow from this and the last one). Since \( E \) is a limit of \( f \)-g. extensions of \( F \), by Lemmas 1 and 2 we have

\[
(3) \quad CH^p(Y_E) = \lim_{V/F} CH^p(Y_E \times V).
\]
If \( \pi \in \text{CH}^0(Y_F) \) goes to 0 in \( \text{CH}^0(Y_C) \), it does so via
\[
\text{CH}^0(Y_F) \overset{\pi^*}{\rightarrow} \text{CH}^0(Y_F \times V).
\]
But since \( F = \overline{F} \), \( \exists \rho \in V(\overline{F}) \)

hence a retraction \( \pi : \text{CH}^0(Y_F \times V) \rightarrow \text{CH}^0(Y_F) \), \( \rho \in V(\overline{F}) \)
with \( \rho \circ \pi^* = \rho \).
So \( \pi^* \) is injective, done.

\[ \square \]

Remark 3:
We make the elementary observation that any variety \( \overline{C} \) is actually the base change to \( C \) of a variety \( X \) defined over a f.g. extension \( K \) of \( \overline{C} \) (or \( \overline{Q} \)). If \( \rho \in X(C) \) is very general, then \( \rho \) presents \( K(X) \) as a subfield of \( C \).

In accord with Remark 3, we now let \( X \) be a smooth projective surface over a field \( K \) f.g. over \( \overline{Q} \). The following is the simplest case of the Bloch-Srinivas "decomposition of the diagonal".

Theorem 1 (Bloch-Srinivas): Suppose \( C \) smooth proper curve (not necessarily irreducible), and cycle \( z \in \text{CH}^2(C \times X) \), s.t.
\[
z^* : \text{CH}^0_0(C_C) \rightarrow \text{CH}^0_0(X_C) \text{ is surjective}.
\]
Then \( \mathbb{F} \in \text{NE} \) \( K \), codimension 1 alg. subset \( D, E \subset X \), and cycles \( \Gamma_1 \in \mathbb{Z}^1(ExX), \Gamma_2 \in \mathbb{Z}^1(ExD) \) such that

\[ \text{w/o loss \,} C \text{ is det'd over } K \] (otherwise enlarge \( K \)).
\( (4) \quad N \cdot \Delta_x \equiv \Gamma_1 + \Gamma_2 \). \\

Proof: Set \( D = \{(x) \times \mathbb{Z}\} \), so that \( CH_0(D \cap C) \to CH_0(X \cap C) \) hence \( CH_0((X \times D) \cap C) = 0 \) by the assumption. Now take \( T = T_1 : X \times X \to X \) in Lemma 2, so that "\( x \cap x \)" = \( x \times X \) and \( L = K(X) \). Write \( \Delta_x(y) \) for the restriction of \( \Delta_x \) to \( y \times X \cong X \).

By Lemma 3 (taking \( F = L \) and \( E = C \)), \( CH^2((x \times D) \cap C) \) = torsion. So in the localization sequence

\( (5) \quad CH^1(D \cap C) \to CH^2(X \cap C) \to CH^2((x \times D) \cap C) \)

we have

\[
N \cdot \Delta_x(y) \rightarrow 0 \quad \text{for some } N.
\]

Since \( CH^1(x \times D) \rightarrow CH^1(y \times D) \), \exists \( \Gamma_1 \in CH^1(x \times D) \)

so \( (N \Delta_x - \Gamma_1) \mid_{x \times x} \equiv 0 \) \( \Rightarrow \) \( (N \Delta_x - \Gamma_1) \mid_{U \times X} \equiv 0 \) \( \text{Lemma 2} \) \( U \subset X \) \( \text{tors. q. (K)} \).

Writing \( E := X \setminus U \) we are now done (by localization again).

\[ \square \]

Corollary 1: In the situation of the Theorem, \( H^{y,0}_1(x \cap C) = \{0\} \).

Proof: By (4) we have

\[ H^2(x \cap C) = N \left[ d_{\alpha} \right] \times H^2(x \cap C) = \left[ \Gamma_1 \right] \times H^2(x \cap C) + \left[ \Gamma_2 \right] \times H^2(x \cap C), \]

with \( \Gamma_1, \Gamma_2 \) regarded as cycles in \( x \times x \).

Let \( \tilde{\Gamma}_1 \) resp. \( \tilde{\Gamma}_2 \) be cycles on \( \tilde{E} \times \tilde{X} \) resp. \( \tilde{X} \times \tilde{D} \) \( (\tilde{E} \to E, \tilde{D} \to D \text{ desingularizations}) \) pushing forward to \( \Gamma_1, \Gamma_2 \).
Then writing \( f^* : E \to X, \quad g^* : B \to X, \) we have
\[
H^2(X, \mathbb{C}) = \left[ \tilde{f}^* \right]_x f^* \left( H^2(X, \mathbb{C}) \right) + \left( \tilde{g}^* \right)_x g^* \left( H^2(X, \mathbb{C}) \right)
\]
\[
\cong \left[ \tilde{f}^* \right]_x H^2(E, \mathbb{C}) + \left( \tilde{g}^* \right)_x H^2(B, \mathbb{C})
\]
\[
\cong H^1(E^\vee) \times H^1(B^\vee)
\]
\[
\cong H^1(X)
\]
and so \( H^2(X) = \mathbb{C} \).

Now let \( \overline{X}/\mathbb{C} \) be a smooth projective surface. Define
\[
(\mathbf{6}) \quad \Gamma_N : \mathbb{X}^{(N)} \times \mathbb{X}^{(N)} \to CH_0^{\text{hom}}(\overline{X})
\]
\[
(\xi_{i_1}, \xi_{i_2}) \mapsto \xi(\overline{x_{i_1} - x_{i_2}})
\]

By an argument with Hilbert (or Chow) schemes, the fibers of \( \Gamma_N \) are countable unions of closed subvarieties of \( \mathbb{X}^{(N)} \times \mathbb{X}^{(N)} \), of maximal dimension \( r_N \) (which is achieved for a very general fiber).

Setting
\[
(\mathbf{7}) \quad d_N = \dim \text{Im}(\Gamma_N) = 4N - r_N
\]
we have the

**Definition:**

(i) \( CH_0^{\text{hom}}(\overline{X}) \) is finite-dimensional \( \iff \{d_N\} \) is bounded

(ii) \( CH_0^{\text{hom}}(\overline{X}) \) is representable \( \iff \Gamma_N \) is surjective for \( N > 0 \)

(\( \leftarrow \)) \( CH_0^{\text{hom}}(\overline{X}) \) can be parameterized by an algebraic variety of finite dimension.)
Another nice property which, if it holds, implies (ii) is

\[
\text{triviality of the kernel of the Albanese map}
\]

8) \[ AJ^2_0 : (H^2_{\text{hom}}(X)) = (H^0_0(X)) \rightarrow J^2_0(X) = \frac{\text{Sel}^1(X)}{H^2(X, \mathbb{Z})} \]

\[ \text{or } "\text{Ab}^1(X)" \]

Given that Severi had claimed in 1934 that (iii) always held, you can imagine that the following would cause a sensation:

**Theorem 2 (Mumford):** If \( h^{1,0}(X) \neq 0 \), then (i), (ii), and (iii) fail. (Indeed, \( \ker (AJ^2_0) \) itself is non-representable!)

**Proof:** Since \( J^2(X) \) is a finite-dimensional algebraic variety, (iii) follows from (i), and we begin by proving (i).

If \( d_N \) is bounded, then the dimension of the image of \( C^0 : X^{(N)} \rightarrow (H^2_{\text{hom}}(X)) \) stabilizes at some \( K \)

\[ \Sigma_k : i \rightarrow \Sigma_k - Nk \]

\[ \Rightarrow \text{a maximal dimensional component } Z_N \text{ of a general fiber has dim. } 2N-K. \]

\[ \Rightarrow \text{we cannot have } Z_N \subset X^{(N+i)} + W \text{ for } W \subset X^{(i)} \text{ of dim } < i. \]

(Otherwise, for \( w \in W(1) \) v.g., \( Z_N w = Z_N \cap (X^{(N+i)} + \{w\}) \subset Z_N \); has dim. \( \leq 2(N-i) - k \) on the one hand, and dim > \( 2N-K-i \) on the other, a contradiction.) Moreover, assuming \( N \geq K \), we have dim \( Z_N \geq N \), and we may assume dim \( Z_N = N \) by replacing \( Z_N \) by its intersection with some hypersurfaces.
While \( R : \bar{X}^N \to \bar{X}(N) \), \( P_i : \bar{X}^N \to \bar{X}^{2i} \), and \( Z \subset \bar{X}^{2i} \) for any irreducible component dominating \( \bar{X}^N \), with defining subvarieties \( \tau : \bar{Z} \to \bar{Z} \).

Then \( \dim(P_i(Z)) \geq \dim(Z) + 2i \). Given an ample curve \( C \subset \bar{X} \), consider the (effective) divisors \( D_i := (P_i \circ \tau)^{-1}(C) \) \((i = 1, \ldots, N)\).

Since their sum is ample (as \( Z \)), the Hodge index theorem implies \( D_1 \cdot \ldots \cdot D_N \neq 0 \) on \( \bar{Z} \). [Exercise] Thus \( Z \) intersects \( C^N \), and so \( Z \) meets \( C(N) \).

We have shown that a very general fiber of \( \sigma_N^0 \) meets \( C(N) \) for \( N \gg 0 \). But then \( \text{Im}(\sigma_N^0) = \text{Im}(\sigma_N^0 / C(N)) \) for \( N \gg 0 \), and so \( C^{\text{hom}}(C) \to C^{\text{hom}}(\bar{X}) \), which contradicts Corollary 1.

It remains to show (ii). Suppose \( \sigma_n \) is surjective for some \( n \). For each \( N \in \mathbb{N} \), let

\[
R_N = \{ (z_1, z_2, w_1, w_2) \mid \sigma_n(z_1, z_2) = \sigma_n(w_1, w_2) \} \subset \bar{X}(N) \times \bar{X}(N) \times \bar{X}(n) \times \bar{X}(n);
\]

then \( R_N \to \bar{X}(N) \times \bar{X}(n) \). In particular, \( R_N \) is a component of \( R_N^0 \) of dimension \( \geq 4N \), and its intersection with \( \bar{X}(N) \times \bar{X}(n) \times \{ (w_1, w_2) \} \) belongs to a (very general) fiber of \( \sigma_n \) and has dimension \( \geq 4N - 4n \). So \( d_n \leq 4n \) is bounded, contradicting (i).

\( \square \)

Mumford's theorem was generalized and amplified by other results, for example:
Theorem 3 (Rothmaler): Let $X/C$ be a smooth proj. variety with $H^i(X) \neq 0$ for some $i \geq 2$. Then the Albanese kernel $\ker(AJ^X_0) \subset CH_0^{hm}(X)$ is non-representable.

Theorem 4 (Voisin): Let $X/C \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 7$. Then no 2 points are rationally equivalent! (This is particularly dramatic since $\text{Alb}(X) = \mathbb{P}^1$.) So no Hodge-theoretic maps we have defined so far can "detect" $p \neq q$.

What do 0-cycles in the Albanese kernel look like?

If $A$ is an abelian variety / $C$, the addition law yields the so-called Pantrayagin product $\times$ on cycles. We have

$$(CH_0^{hm}(A))^2 \subset \ker(A1b)$$

Since in the picture

$$g) \quad \left( \begin{array}{cc} \cdot & \cdot \\ -p \cdot & + \end{array} \right) \times \left( \begin{array}{cc} \cdot & \cdot \\ -q \cdot & + \end{array} \right) = \left( \begin{array}{cc} \cdot & \cdot \\ -p \cdot \cdot & +q \cdot \\ \cdot & -q \cdot \cdot \end{array} \right)$$

we have $\int p \cdot w = 0$ for $w \in \mathcal{R}(A)$ by cancellation. As we shall see, such cycles are in general NOT $\equiv 0$, and are closely tied to the spirit of Mumford's original proof.

Last but certainly not least, here are two conjectures that attempt to place some order on the apparent chaos just unleashed:
Conjecture 1 (Bloch): For $\mathbb{F}/\mathbb{F}$ smooth projective surface with $h^{2,0}(\mathbb{F}) = 0, \ker (A/b) = 0$.

(This "curve to Mumford" is known for surfaces of Kodaira dimension $\geq 2$, projective complete intersections, and many other cases, but not in generality.)

Conjecture 2 (Bloch-Beilinson): For $\mathbb{F}/\mathbb{F}$ smooth projective,

\[ (10) \quad C_\mathcal{F} : \text{CH}^n(\mathbb{F}) \otimes \mathbb{Q} \to H^{2n}(\mathcal{F}_\mathbb{Q}, \mathbb{Q}(n)) \]

is injective. (In particular, for $\mathbb{F}/\overline{\mathbb{F}}$ a smooth projective surface, $\text{AT} : \text{CH}^n(\mathbb{F}) \to \mathbb{T}^2(\mathcal{F}_{\mathbb{Q}})$ is injective.)

Note: This conjecture is NOT saying that $C_{\mathcal{F}}$ is injective on \( \text{CH}^n(\mathcal{F}_{\mathbb{Q}}) \). It is simply saying that $\text{CH}^{\mathcal{F}} \& \text{AT}$ suffice to detect (modulo torsion) the cycles defined over $\mathbb{Q}$.

Example: $\mathbb{F}$ is surface / $\overline{\mathbb{Q}}$. Then Conj. 2 would imply that for any 2 points $p, q \in \mathbb{F}(\overline{\mathbb{Q}})$, $\exists N \in \mathbb{N}$ s.t. $N(p-q) \in \mathcal{F}_{\mathbb{Q}}$ (Why?). On the other hand, Mumford's theorem says that differences of $p, q \in \mathbb{F}(\overline{\mathbb{Q}})$ generate an "co-dimension 1" group modulo $\mathcal{F}_{\mathbb{Q}}$.

For surfaces with $h^{2,0}(\mathbb{F}) \neq 0$, I don't know of a case when Conj. 2 has been proved. However, if we extend the conjecture to certain degree of relative cases, or to higher Chow groups, it turns out that the fact that $\text{ker}(\text{Ch}^{\mathcal{F}})$ is finite...