PART II: Higher Chow cycles and motivic cohomology

A. Bloch's construction

1. $K_0$ and algebraic cycles

Let $X$ be a quasi-projective variety over a field $k$,

\[
\begin{align*}
\text{Coh}(X) & \text{: category of coherent } \mathcal{O}_X \text{-modules} \\
\text{Vec}(X) & \text{: category of locally-free sheaves of finite rank (\textit{\Sigma} vector bundles)}.
\end{align*}
\]

Applying the Grothendieck-group construction yields abelian groups

\[
\begin{align*}
G_0(X) & \cong \mathbb{Z} [\text{Coh}(X)] \text{/ relations} \\
K_0(X) & \cong \mathbb{Z} [\text{Vec}(X)] \text{/ relations}
\end{align*}
\]

where the "relations" are

\[
[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}''] \quad \text{for any exact sequence } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0
\]

in the category. The Riemann-Roch theorem of Barten-Fulton-MacPherson states that there is an isomorphism of rational vector spaces

\[
(1) \quad G_0(X) \cong \bigoplus_p CH^p(X) \quad \mathbb{Q}.
\]

If we assume $X$ smooth, then this takes the form

\[
(2) \quad Gr_0 K_0(X) \cong CH^*(X) \quad \mathbb{Q}
\]

of an isomorphism of graded rings, due to Grothendieck. In subsequent sections we will pursue Bloch's generalization of these formulas to higher $K$-theory.
In this section I will briefly review the proof of (2), focusing on the role of the Chern classes which give the map from $K_0$ to $CH^*$ (or any other cohomology theory). First recall some properties of $K_0$ and $G_0$:

(3a) Given a morphism $f : X \to Y$, we have

- pullback map $f^* : E \to f^{-1} E$ on $K_0$ (or $G_0$ if $f$ flat)
- pushforward map $f_* : E \to E(-1)^* R^1f_*E$ on $G_0$, if $f$ proper

(3b) Localization sequence

$$G_0(W) \to G_0(X) \to G_0(X\setminus W) \to 0$$

(exact for any subvariety $W \subseteq X$)

(3c) Homotopy property

$$G_0(X \times A^1) \xrightarrow{\sim} G_0(X)$$

(same isomorphism for any $t$)

(3d) Ring structure on $K_0$ (or $K_0$-module structure on $G_0$)

$$[E_1] \cdot [E_2] := [E_1 \otimes E_2]$$

(3e) $X$ smooth $\Rightarrow$ $K_0 \cong G_0$ [we will still write one or the other below when it is conceptually more accurate]

Ideas: take projective resolution of $E \in Coh(X)$ by $\{E_i \in Vec(X) \}_{i \geq 1}$; then $[E] = \Sigma (-1)^i [E_i]$ (in $G_0$) shows the obvious map $K_0 \to G_0$ is surjective.

\[\text{Note: One of the motivations for higher Chow groups is that higher K-theory (or H-theory) extends this to the left, whereas we had (before this) no comparative extension for CH.}\]

\[\text{We will assume $X$ is smooth below, but more generally Chow (and higher Chow) groups turn out to be a "Borel-Moore homology theory", i.e. act like cohomology on smooth quasi-projective but not on a singular variety.}\]
If $X$ is smooth, we can directly define a ring structure on $G_0(X)$ by $[\mathcal{E}_1] \cdot [\mathcal{E}_2] := \sum_{k \geq 0} (-1)^k [\text{Tor}_k^G(\mathcal{E}_1, \mathcal{E}_2)]$. (This is consistent with (3d) and the map $G_0 \to \mathbb{K}_n$ in (3e).)

Now we turn to the Chern classes, henceforth assuming $X$ smooth.

Given a locally free sheaf of $\mathcal{O}_X$-modules, we can form a projective bundle

$$E \longrightarrow \mathbb{P}(E) \overset{\pi}{\longrightarrow} X$$

with canonical line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$, restricting to $\mathcal{O}(1)$ on the fibers $\pi^{-1}(x) \cong \mathbb{P}^m$. Write $s \in CH^1(\mathbb{P}(E))$ for the class of the divisor associated to $\mathcal{O}_E(1)$.

**Proposition 1:** $CH^*(\mathbb{P}(E))$ is a free $(\pi^*)$ $CH^*(X)$-module with basis $1, s, \ldots, s^m$.

**Sketch:** Let $U \subset X$ be a Zariski open trivializing $\mathcal{E}$; that is, $\pi^{-1}(U) \cong U \times \mathbb{P}^m$. [Exercise: Show the Prop. holds in the case $X=U$, by stratifying $\mathbb{P}^m$ by affine spaces of decreasing dimension and using localization.]

By localization and induction on dimension we have (with $X\setminus U = \bigcup V_j$)

$$\phi \mathcal{H}^{*-i}(V_j) \langle 1, s, \ldots, s^m \rangle \longrightarrow CH^*(X) \langle 1, s, \ldots, s^m \rangle \longrightarrow CH^*(U) \langle 1, s, \ldots, s^m \rangle \longrightarrow 0$$

$$\phi \mathcal{H}^{*-i}(\pi^{-1}(V_j)) \longrightarrow CH^*(\mathbb{P}(E)) \longrightarrow CH^*(\pi^{-1}(U)) \longrightarrow 0$$

$\phi$ canonical = dual of tautological line bundles
from the first 2 rows of which \( a \) is surjective.

Next one verifies that \( \psi_i := \pi_* \circ (\xi^1) \circ \pi^* : CH^k(x) \to CH^k(x) \)
is 0 for \( i \neq m \), \( id \) for \( i = m \). [This is again by induction on dimension and localization, viz.]

\[
\begin{align*}
\oplus CH(V_i) & \to CH(x) \to CH(U) \to 0 \\
\downarrow & \downarrow \downarrow \\
\oplus CH(V_j) & \to CH(x) \to CH(U) \to 0
\end{align*}
\]

where the outer arrows are either \( id \) or 0.] But this implies that the vertical composition in (4) is an isomorphism, hence that \( \varphi \)
is injective.

\[\square\]

As an immediate corollary of Prop. 1, there is a \( (CH^*(x) -) \) linear relation on \( 1, s, ..., s^m, s^{m+1} \).

**Definition 1**: The Chern classes \( c_r(E) \in CH^r(x) \) are defined by the relation

\[
\sum_{r=0}^{m+1} \pi_i^*(c_r(E)) s^{m+1-r} = 0 \quad (m \in CH^{m+1}(IP(E)))
\]

and \( c_0(E) := 1 \). The total Chern class is \( c(E) := \sum_{r=0}^{m+1} c_r(E) \in CH^*(x) \).

One can show the following version of the "splitting principle":

\[
\exists f : x' \to x \text{ with } f^* : CH^*(x) \to CH^*(x') \text{ s.t.}
\]

\[
\begin{align*}
\forall i & : e' = f^*(e) \text{ has a filtration with } \text{Gr}(e') \subseteq \bigoplus_{i=1}^{m+1} L_i, \\
\end{align*}
\]

where \( L_i \) are invertible \((\leftrightarrow \text{ line bundles})\).

Moreover, each \( L_i \cong O(D_i) \) for some divisor \( D_i \), and \( e_i(L_i) = CH(D_i) =: x_i \).
One readily checks that \( \alpha_i \) (resp. \( \Pi \alpha_i \)) is the pullback of the zero-section via a section of \( 2; i \) (resp. \( E \')). Writing \( \Pi': \mathcal{P}(E') \to X' \), we have \( \mathcal{O}(-1) \to (\Pi')^* E' \) hence a vanishing section \( 1 \in \mathcal{O} \to (\Pi')^* E' \langle 1 \rangle \). Since the \( c_k \)'s of the graded pieces of \((\Pi')^* E' \langle 1 \rangle \) are \( \delta + \alpha_i \), we get \( \Pi'(\delta + \alpha_i) = 0 \implies c_r(E) = \sigma_r(\delta + \alpha_i) \) (where \( \sigma_r \) is \( r \)-th elementary symmetric function) \( \implies c(E) = \Pi(1 + \delta_i) \). From this the Whitney sum formula

\[
(6b) \quad 0 \to E' \to E \to E'' \to 0 \implies c(E) = c(E') \cdot c(E'')
\]

\[= \implies c : K_0(X) \to CH^*(X) \text{ well-defined, sending addition to multiplication.} \]

follows at once. We also note that functoriality

\[
(6c) \quad f : Y \to X \implies c(f^{-1} E) = f^* (c(E))
\]

follows easily from (5), and hence checking

\[
(6d) \quad E = 0 \to 0 \implies c(E) = 1 + c_1(D)
\]

as an Exercise. \( \text{[Hint: } \mathcal{P}(E) = X \text{ and } \mathcal{O}_{\mathcal{P}(E)}(-1) = E \text{ is the tautological line.]} \]

Now (6c) is actually a terrible defect: a sum in \( K_0 \) should not map to a product in \( CH^* \). To correct this, introduce

**Definition 2**: The Chern character \( ch(E) \in CH^*(X) \) is defined by

\[
(7) \quad f^* (ch(E)) = \sum_{i=0}^{n} e^{-i}.
\]

(\( \text{Note the } \mathbb{Q}-\text{coefficient. This is why we have to } \otimes \mathbb{Q} \text{ in } (1) \delta(2) \).

Obviously, they come from the denominators in \( e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \).

\( \text{Yes, we used this above; and no, there is no circularity here.} \)

\( \text{ } \)

\( \text{ } \)
Using (7) one can check the properties
\[(8c)\] \quad 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \Rightarrow \text{ch}(E) = \text{ch}(E') + \text{ch}(E'')
and
\[(8b)\] \quad \text{ch}(E_1 \otimes E_2) = \text{ch}(E_1) \cdot \text{ch}(E_2),

as well as functoriality and $\text{ch}(O(O)) = e^{\varepsilon(O)}$. Finally, we have

**Definition 3:** The Todd class $\text{td}(E) \in CH^*(X)_q$ is defined by
\[(9)\] \quad $f^*(\text{td}(E)) = \prod_{i=1}^{\text{rank}(E)} \frac{e^i}{1 - e^{-x_i}}$. /\*

From (7) and (9) one may compute
\[(10a)\] \quad $\text{ch}(E) = (m+1) + c_1(E) + \frac{1}{2} (c_1(E)^2 - 2c_2(E)) + \frac{i}{3!} (c_1^3 - 3c_1c_2 + 3c_3) + \cdots$
\[(10b)\] \quad $\text{td}(E) = 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1(E)^2 + c_2(E)) + \frac{1}{72} c_1c_2 + \cdots$

Note that the Todd class is invertible; it also satisfies (6b) and (6c) (replacing $c$ by $td$) [⇒ $\text{td} : k_0(X) \rightarrow CH^*(X)_q$ well-defined, surjective, addition → mult.].

Now let $F^iG_0(X)$ denote the subgroup of $G_0(X) (\cong k_0(X))$ generated by $[E]$ with $c_0(E) \geq i$ ($|E|$ = support of $E$).

**Proposition 2:** \[ Y : Z^*(X) \rightarrow G_0(X) \] is surjective.
\[ \mathbb{Z} \text{irred} \mapsto [0_2] \]

**Sketch:** Suff. to show $\mathbb{Z}^i(X) \xrightarrow{Y^i} F^iG_0(X)$. \quad (Proof: claim)

Given $E$ with $c_0(E) = i$, \[ Y(E) := \sum_{w \in X} \left[ \sum_{x \in \mathbb{X}_w (E_w)} \right] \quad \text{we have } Y(0) \equiv [E] \mod F^{i+1}
\]

(Codim, point) (length) (orst $a \in w$)

Here are some properties of $Y$:

\[(11)\] \quad If $E_1 \cap E_2$ is proper, then $Y(E_1) \cdot Y(E_2) \equiv X(E_1 \cdot E_2) \mod F^{i+1}$.
\[ (E_1, E_2) \in \mathbb{Z}^i(X), (E_1 \cdot E_2) \in \mathbb{Z}^i(X)) \]
\[ \text{Sketch :} \quad Y(z_1), Y(z_2) = \left[ \text{Tor}_0^{O_x}(O_{z_1}, O_{z_2}) \right] + \varepsilon(-1)^q \left[ \text{Tor}_0^{O_x}(O_{z_1}, O_{z_2}) \right] \\
\quad \quad = \left[ O_{z_1}, O_{z_2} \right] + \mathbb{F}_{i+j+1} \]

(12) If \( f: Y \to X \) has \( f^* \) defined, then \( Y_f(f^*Z) = f^*(Y_xZ) \mod \mathbb{F}_{i+j+1} \).

[follows from (11) \( \ell \) intersecting with the graph of \( f \)]

(13) \( z_1 \equiv z_2 \in \mathbb{Z}_F(X) \implies Y(z_1) = Y(z_2) \mod \mathbb{F}_{i+j+1} \).

\[ \text{Sketch :} \quad \text{If } W \in \mathbb{Z}_F((X \times X)) \text{ is the normal equivalence (w.r.t. \text{adm.})} \]

\[ \quad \text{then } Y_{x}(x) - Y_{x^*}(x) = \gamma_{x_x}(x_{x^*} W) = \gamma_{x^*}(x_{x^*} W) \equiv (x_{x^*} - x_{x^*}) Y_{x^*}(W) = 0 \] \[ \text{equiv. by (3c)} \]

(14) \( z_1 \in F_{i+1}^{G_0}(X), z_2 \in F_{i+1}^{G_0}(X) \implies z_1 \cdot z_2 \in F_{i+j+2} \).

\[ \text{Sketch :} \quad \text{By Prop. 2, w.m.a. } z_j = y_x(z_j), \text{ and by (13) w.m.a.} \]

\( z_1 \cdot z_2 \) is proper of codim. \( i+j+2 \) \( \text{done by (11).} \)

[Chern moving lemma]

(15) \( \text{Oh Y : } \oplus C^i_0(X) \to \oplus G^{i*}_{F_0}(X) \) is a surjective homomorphism of graded rings (w.r.t. pullback).

[follows from Prop 2 + (11)-(14) at once]

To get the graded homomorphism in the opposite direction, we will need

**Proposition 3** : Given a closed immersion \( i: Y \subset X \) of codim. \( i \),

\[ \text{ch} \left( [O_Y] \right) \equiv [Y] \mod \oplus_{j \geq 0} \chi^{j}(Y) \mathbb{Q} \]

The proof uses

Grotendieck - Riemann - Roch (GRR) Theorem : Given a proper morphism \( f: Y \to X \) of smooth quasi-projective varieties, and

\[ \text{Grotendieck - Riemann - Roch (GRR) Theorem : Given a proper morphism } f: Y \to X \text{ of smooth quasi-projective varieties, and} \]
$s \in K_0(Y)$, we have

$$\text{(17)} \quad \text{ch} \left( f_* s \right) \cdot \text{td} (T_X) = f_* \left\{ \chi (s) \cdot \text{td} (T_Y) \right\}$$

in $\text{CH}^*_X$. \vspace{0.5em}

\textit{Exercise:} Recover Riemann-Roch (Curves) & Noether's Theorem (Surface) from the case of $X$ a point. \vspace{0.5em}

\textit{Proof of Prop. 3:} By a localization argument (remove singular locus of $Y$) one reduces to the case of $Y$ smooth. Then GRR applies with $Z = \{Y\}$ (or $Y$), and using $0 \to T_Y \to \Omega^1 X \to N_{Y/X} \to 0$, we get

$$\text{ch} \left( \{Y\} \right) \cdot \text{td} (T_X) = f_* \left\{ \text{ch} \left( \Omega^1 X \right) \cdot \text{td} (T_Y) \right\} = f_* \left\{ \text{td} N_{Y/X} \right\} \cdot f_* \text{td} T_X$$

$$= f_* \left\{ \text{td} N_{Y/X} \right\} \cdot \text{td} T_X$$

$$\Rightarrow \quad \text{ch} \left( \{Y\} \right) = f_* \left\{ \text{td} N_{Y/X} \right\} = f_* \left\{ 1 + \ldots \right\} = [Y] + \ldots$$

(7) \text{ case } K_0 = G_0 \text{ for } X \text{ smooth, so we now use } K_0. \text{ We can now prove}

\textit{Theorem 2 (Grothendieck):} We have isomorphism of graded rings

$$\text{Gr} \text{CH} (X)_{\mathbb{Q}} \overset{\text{Gr} Y}{\longrightarrow} \text{Gr}_F K_0 (X)_{\mathbb{Q}} \overset{\text{Gr} (\chi)}{\longrightarrow} \text{Gr} \text{CH} (X)_{\mathbb{Q}}.$$  

\textit{Proof:} By Prop. 3, $\text{ch} \left( \Omega^1 K_0 (X)_{\mathbb{Q}} \right) \subset \text{CH} (X)_{\mathbb{Q}}$, with

$$[O_Y] \mapsto [Z] \text{ for } Z \in \text{CH} (X). \text{ By (15), } \text{Gr} Y \text{ is surjective, which proves } \text{Gr} \text{ch} \text{ injective. \vspace{0.5em}}$$

In fact, Grothendieck's original version of (18) was a little different — specifically, the filtration $F$ (by codimension) was replaced by the "$Y$-filtration".
To define it, we first describe the $\lambda$-operations. Since for $0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$ we have $[\lambda^i \mathcal{E}] = \sum_{j=0}^{i} \lambda^j \mathcal{E}' \cdot (\lambda^{i-j} \mathcal{E}'')$, $\lambda(\mathcal{E}) := \sum_{i \geq 0} [\lambda^i \mathcal{E}] t^i$ induces a well-defined map $K_0(X) \rightarrow K_0(X)[t]$ (sending addition to multiplication); in particular,

$$\lambda_i : K_0(X) \rightarrow K_0(X)$$

$$[\mathcal{E}] \mapsto [\lambda^i \mathcal{E}]$$

is well-defined (but non-linear). Now set

$$\nu_i : K_0(X) \rightarrow K_0(X)$$

$$x \mapsto \lambda_i(x + (i-1)[0_X])$$

and define $F^i_\nu K_0(X)$ by

$$F^i_\nu := F^i_{\text{cod}} \cap \{ \text{this is the F above} \}$$

$$x \in F^i_\nu \Rightarrow \nu^{-i}(x) \in F^i_\nu$$

$$F^i_\nu \cdot F^j_\nu \subseteq F^{i+j} \nu.$$

Then (18) holds with $F^i_{\text{cod}}$ replaced by $F^i_\nu$, and

$$CH^p(X)_Q \cong Gr^p_{\nu} K_0(X)_Q \cong Gr^p_{\text{cod}} K_0(X)_Q \cong K^{(p)}_0(X)_Q$$

(standard notation) $\mapsto K^{(p)}_0(X)_Q$.

$$\lambda_i(x + y) = \sum_{j=0}^{i} (-1)^j \lambda_j(x) \lambda^{i-j}(y)$$