B. The Bond regulator

1. K-theory of number fields

For a ring \( R \), \( K_0(R) \) is the Grothendieck group of (finitely generated) projective \( R \)-modules (with the usual relation). If \( R \) is a Dedekind domain, then any projective \( R \)-module is isomorphic to a sum of fractional ideals, and we have \( K_0(R) \cong \mathbb{Z} \oplus \frac{\{\text{fractional ideals}\}}{\{\text{principal fractional ideals}\}} \).

Recall also that \( K_1(R) \) is the unit group \( R^\times \).

Now let \( F \) be a number field; it is immediate that \( K_0(F) \cong \mathbb{Z} \) and \( K_1(F) \cong F^\times \). More interesting is its ring of integers \( \mathcal{O}_F \) (= roots of monic integer polynomials), which has:

- \( K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus \text{Cl}(F) \), \( \text{Cl}(F) \) the (abelian) ideal class group
  \( (h := |\text{Cl}(F)| < \infty \text{ by the Minkowski bound}) \)
- \( K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times \cong \mu(F) \oplus \mathbb{Z}d \) \( (\text{by Dirichlet's theorem on units}) \),
  where
  \[ d_n := \begin{cases} \frac{r_1 + r_2 - 1}{2}, & n = 0 \\ \frac{r_1 + r_2}{2}, & n \geq 2 \text{ even} \\ \frac{r_2}{2}, & n \text{ odd} \end{cases} \]

Write \( \alpha_1, \ldots, \alpha_k \) for a basis of \( \mathcal{O}_F^d \subset \mathcal{O}_F^\times \)
\( \beta_1, \ldots, \beta_d \) for a basis of \( \mathcal{O}_F \cong \mathbb{Z}^d \), and \( w := |\mu(F)| \).

\* We adopt the standard notation \( d = [F:Q] = r_1 + r_2 \), where
\( r_1 = \# \text{ of real embeddings} \ \sigma_1, \ldots, \sigma_{r_1} \)
\( r_2 = \# \text{ of conjugate pairs of complex embeddings} \ \sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}, \sigma_{r_1+r_2+1}, \ldots, \sigma_{r_1+2r_2} \).
The image of the (Dirichlet regulator) map
\[
\rho : \mathcal{O}_F^* \to \mathbb{R}^{r_1 + r_2}
\]
\[
\langle \alpha \rangle \mapsto (\log |\sigma_1(\alpha)|, \ldots, \log |\sigma_{r_1}(\alpha)|; \log |\sigma_{r_1+1}(\alpha)|^2, \ldots, \log |\sigma_{r_1+r_2}(\alpha)|^2)
\]
is a lattice of rank \(d_0\), lying in \(\ker(\varepsilon : (\mathbb{R}^{r_1 + r_2} \to \mathbb{R}) \simeq \mathbb{R}^{d_0})\) since
\[
\prod_{i=1}^d \sigma_i(\alpha) = 1 \quad \text{for any unit (it has to be a unit in } \mathbb{Z}) \text{). We define}
\]
- **discriminant** \(D_F := \left[\det(\sigma_i(\beta_j))\right]^2\)
- **regular** \(R_F := \frac{1}{\sqrt{2^{r_1} r_2}} \det(\text{any } d \times d \text{ minor of } \begin{bmatrix} r(\alpha) \\ r(\alpha, \alpha) \end{bmatrix})\)
- **Dedekind eta for** \(S_F(s) := \sum_{\mathfrak{a} \in \mathcal{O}_F^*} \frac{|\mathfrak{a}|^s}{\mathfrak{a}} \quad \text{for } S_\mathbb{Q}(s) = S(s); \text{see Riemann zeta}\)

**Theorem (Dirichlet):** \(S_F\) converges absolutely for \(\Re(s) > 1\) and analytically continues to a meromorphic function on \(\mathbb{C}\) with single (simple) pole at \(s = 1\), with residue
\[
\lim_{s \to 1} (s-1) S_F(s) = \frac{2^{r_1 + r_2} \pi^{r_2} R_F h}{|D_F|^{1/2} w}.
\]
Equivalently, using the functional equation of \(S_F\), one has
\[
\lim_{s \to 0} s^{-d_0} S_F(s) = -\frac{1}{w} R_F = -\frac{|K_0(\mathcal{O}_F)_{\text{tors}}|}{|K_1(\mathcal{O}_F)_{\text{tors}}|} R_F.
\]

The following diagram is suggestive of the broader context (related to the Beilinson conjectures) into which the above story fits.
Boel's theorems

Boel generalized these results to higher K-theory. Recall that primitive cohomology $PH_*(X) := \{ x \in H_*(X) \mid \Delta x = x \otimes 1 + 1 \otimes x \}$ with dual $IH_*(X) := H_*(X)/\langle \text{up products} \rangle$ the indecomposables. For any $H$-space we have $\pi_*(X) \xrightarrow{\Sigma} PH_*(X, Q)$, and $BGL(Q)^+$ is an $H$-space. Therefore

$$K_m(Q) := \pi_*(BGL(Q)^+) \xrightarrow{\Sigma} PH_m(BGL(Q)^+) \cong \lim_{N \to \infty} PH_m(SL_N(Q))$$

$(Q=Q^b)$, we break "Boel's theorem" into 2 parts:

**Theorem 2 (Boel):** $K_{2n}(Q)$ is torsion; $K_{2n+1}(Q)$ has rank $d_n$.

**Theorem 3 (Boel):** There exists a natural "regulator map"

$$K_{2n+1}(Q) \to \lim_{N \to \infty} PH_{2n+1}(SL_N(Q), \mathbb{R}) \cong \lim_{N \to \infty} PH_{2n+1}(X_{N, x}, \mathbb{R})$$

whose image has corvolume (w.r.t. the $Q[b]$-structure on the RHS)

$$R_{F}^{(n+1)} \in \mathbb{Q}^{*}. \frac{|D_F|^{1/2}}{\pi^{(n+1)d - d_n}} \zeta_F(n+1).$$

Equivalently, using the functional equation gives

$$\lim_{s \to -n} (s+n)^{-d_n} \zeta_F(s) \in \mathbb{Q}^{*} \cdot R_{F}^{(n+1)}.$$ [more precisely,

$$= 2 \frac{|K_{2n}(Q)_{tor}|}{|K_{2n+1}(Q)_{tor}|} R_{F}^{(n+1)}]$$
Implications for higher Chow groups

In fact, the rank $d_n$ part of $K_{2n+1}(\mathbb{Q}_F)$ belongs to $K^{2n+1}_{2n+1}(\mathbb{Q}_F)$, and the other $C_{2n}^2$ are thus torsion; so for $n > 0$

**Corollary 1:**

(i) $\text{CH}^{n+1}(\mathbb{Q}_F, 2n+1)$ has rank $d_n$ \[\text{i.e. Spec } \mathbb{Q}_F\]

(ii) $\text{CH}^{n+1}(\mathbb{F}_q, 2n+1)$ has rank $d_n$

and these are the only exceptions to non-torsion higher Chow groups of $\mathbb{F}$ or $\overline{\mathbb{Q}}$.

**Heuristic argument for (ii):** Use localization

\[\Theta: \text{CH}^n(\mathbb{Q}_F, 2n+1) \rightarrow \text{CH}^{n+1}(\mathbb{Q}_F, 2n+1) \rightarrow \text{CH}^{n+1}(\mathbb{F}_q, 2n+1) \rightarrow \text{CH}^{n}(\mathbb{Q}_F/\mathbb{Q}, 2n)\]

and the fact that $K$-theory of a finite field (and form) is torsion.

The hint to the AF maps studied in the last section is given by

**Theorem 4 (Burgos):** The following diagram commutes:

\[\begin{array}{c}
\mathbb{Z} \longrightarrow (AJ(\mathbb{Q}(2)), \ldots, AJ(\mathbb{Q}(2))) \\
\text{CH}^{n+1}(\mathbb{F}, 2n+1) \rightarrow \text{AJ}(\mathbb{Q}(2)) \rightarrow (\mathbb{C}/\mathbb{Q}(n+1))^d \rightarrow K^M_n(\mathbb{Q}) \rightarrow \mathbb{R}(n)^d
\end{array}\]

\[\begin{array}{c}
\text{BGR} \rightarrow \text{BGR} \\
\text{cl. class} \rightarrow (0) \\
K_{2n+1}(\mathbb{Q}_F) \rightarrow (\mathbb{Z}/n)^d \rightarrow \mathbb{R}(n)^d
\end{array}\]

**Exercise:** Check that this gets with the dilogarithm examples at the end of the last section.

$\bigcirc$ in particular the Milnor $K$-theory $K_n^M(\mathbb{F}) = K_n^M(\mathbb{F})$ is torsion.

$\bigcirc$ with earlier work by Beilinson, Rapoport, Dupont-Hain-Edixhoven.
Finally, we mention an explicit connection between homology of the general linear group and higher Chow groups, which in particular is believed to reverse $\beta$ above. Briefly, each group (co)homology $H^\cdot (G, M) = \text{Ext}^\cdot (\mathbb{Z}, M)$ is isomorphic to $H^\cdot (\text{Hom}_{\mathbb{Z}[G]} (F, M))$ where $F$ is a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules (e.g. $F_n = \mathbb{Z} [G^{n+1}]$ with "differences" $(g_0, \ldots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n))$, so

- $H^\cdot (G, M) = \text{Ext}^\cdot (\mathbb{Z}, M) = H^\cdot (\text{Hom}_{\mathbb{Z}[G]} (F, M))$
- $H^\cdot (C^\cdot (G, M))$, $C^\cdot (G, M) = \{ \text{func} \circ \phi : G^{n+1} \to M \mid \phi (g_0, \ldots, g_n) = g \phi (g_0, \ldots, g_n) \}$
- $H^\cdot (G, M) = \text{Tor}^\cdot (\mathbb{Z}[G], M) = H^\cdot (F \otimes_{\mathbb{Z}[G]} M)$

We are interested in the case $G = GL_n(F)$, $M = \mathbb{Z} = \text{trivial } \mathbb{Z}[G]$-module.

Let $v \in F^P$ be a "general" vector, and write for any $(n+1)$-tuple $(g_0, \ldots, g_n) \in G^{n+1}$ the system of $p$ linear equations

$$\sum_{j=0}^n x_j g_j v = 0.$$ 

This defines a codim. $p$ linear subspace of $\Delta_n^F$, and one can check that the construction even yields a map of complexes

$$C_\cdot (GL_n(F), \mathbb{Z}) \to \mathbb{Z}^p (F, \cdot)$$

(at least, when it is well-defined). So we get a map

$$H^\cdot (GL_n(F), \mathbb{Z}) \to \text{CH}^p_{\Delta} (F, n)$$

which is believed to be the isomorphism $\beta$ in the case $H_{2n+1} (G_{2n+1}(F))$.

In particular, this should give $\text{LCH}^p = \text{CH}^p$, i.e. all cycles have a "linear" representative.
Finally, we mention some concrete examples of the classes $\varepsilon_i$:

- $\overline{H}^1_{\text{cont}}(\text{SL}_1(\mathbb{C}), \mathbb{R}) = \mathbb{R} \langle e_1 \rangle$, \quad $\varepsilon_1(g_0, g_1) = \log |\frac{g_1}{g_0}|$

- $\overline{H}^3_{\text{cont}}(\text{SL}_2(\mathbb{C}), \mathbb{R}) = \mathbb{R} \langle e_3 \rangle$, \quad $\varepsilon_3(g_0, g_1, g_2, g_3) \sim D_2(\text{CR}(g_0^t, g_1, g_2, g_3, g_4))$

  where $D_2(x) = \text{Li}_2(x) + \log(1-x) \log |x|$, note that this is invariant by all $g_i \mapsto g_i g$, and restricts to 0 on $\text{SL}_2(\mathbb{R})$. Finally, it is a cusp form by the 5-term relation $\varepsilon(-1) \varepsilon_3(g_0, g_1, g_2, g_3, g_4)$ on $D_2$.

- The maps $H_{2n-1}(\text{SL}_n(\mathbb{C}), \mathbb{Q}) \to (\mathbb{L}) \mathcal{H}_{2n}^n(\mathbb{Q}, 2n-1)_\mathbb{Q}$ should give (up to a multiplier $i$)

$$\tilde{\varepsilon}_{2n-1} \in \overline{H}^{2n-1}_{\text{cont}}(\text{SL}_n(\mathbb{C}), \mathbb{Q}/\mathbb{Q}(n))$$

of $\varepsilon_{2n-1}$. 