2. Sketch of Borel's Theorem

(2) Sketch of proof of Theorem 2 (following Soulé)

\[ \text{Res}_{F/\mathbb{Q}} \text{SL}_N(F) =: G \quad \text{w} \quad G(\mathbb{Q}) = \text{SL}_N(\mathbb{Q}_F) =: \Gamma \]
\[ G(\mathbb{R}) = \text{SL}_N(\mathbb{R}) \]
\[ G(\mathbb{C}) = \prod_{\sigma \in \text{Hom}(F, \mathbb{C})} \text{SL}_N(\mathbb{C}) \]

(\( \mathbb{Q} \)-alg. group)

Prop. 1 (Borel): \( G \) a \( \mathbb{Q} \)-alg. group s.t. \( G(\mathbb{R}) \) connected
\[ \Gamma \leq G(\mathbb{R}) \text{ arithmetic} \]
\[ \implies H^q_{\text{cont}}(G) \cong H^q(\Gamma, \mathbb{R}) \quad \text{for } q \ll \text{rk}_\mathbb{Q} G. \]

Sketch: Put \( X = G/\Gamma \), \( X(\mathbb{R}) = G(\mathbb{R})/\Gamma \). Assume \( \Gamma \) torsion-free, so that
\[ \Gamma \subseteq G(\mathbb{R}) \text{ freely} \quad X \text{ contractible} \implies \]
\[ H^q(\Gamma, \mathbb{R}) = H^q(\pi_1(X), \mathbb{R}) = H^q(\pi_1(X(\mathbb{R}))) = H^q(G(\mathbb{R}), \mathbb{R}). \]

What is \( H^q_{\text{cont}} \)? Cohomology of complex
\[ \cdots \rightarrow C^0_{\text{cont}}(G) \rightarrow C^1_{\text{cont}}(G) \rightarrow \cdots \]
\[ \text{(continuous } \mathbb{R} \text{-valued fun on } G^{p+1}, \text{ d} p(\gamma_0, \gamma_1) = \sum_{i=0}^{p+1} \text{cont} (G^{p+1}, \mathbb{R}) \]
\[ X \text{ contractible} \implies [\mathbb{R} \rightarrow \mathcal{L}^*(X)] \implies \text{ horizontal diffeo in } E_0^{p+1} \subseteq \text{cont}(G^{p+1}, \mathcal{L}^*(X))^G \]
\[ \text{is exact off } E_0^{0,2}, \text{ so can replace dual complex by } \mathcal{L}^*(X)^G = \ker(\partial) \subseteq E_0^{0,2} \]
\[ \implies H^q_{\text{cont}}(G) \cong H^q(\mathcal{L}^*(X)^G). \]

Moreover, b/c Cartan involutions act by \((-1)\) on the cotangent space (\( X \) is a symmetric space) compatibly w/ differentials, it acts by \((-1)^2\) on forms \( \Omega^2 \) forcing
\[ \Delta: \Omega^2(X)^G \rightarrow \Omega^{2+}(X)^G \text{ to be 0}. \]

\( \blacklozenge \) this section is so far very rough
So we must show \( \Omega^q(x)^G = H^q(X^\omega) \to H^q(X^\omega)_G \) is \( \cong \).

\[
\begin{pmatrix}
H^q_{cont}(G) \\
H^q(G, \mathbb{R})
\end{pmatrix}
\]

Idea: do Hodge theory on \( X(\mathbb{R}) \). Fix \( h \) smooth \( G \)-equiv. metric on \( TX \), define volume form \( \rho \), \( \Delta = dd^* + d^*d \). Clearly \( R^*_G(X)^G \subset \ker \Delta \).

One defines \( R^*_G(X)_\log \), shows \( \subseteq R^*_G(X)_L^2 \left[ \|1\|_{L^2}^2 = \int_X h(\omega, \omega) \rho < \infty \right] \),

and that \( \text{any } L^2 \text{ cohom. class has a harmonic } L^2 \text{ representative } + \text{ harmonic } L^2 \text{ exact } \Rightarrow 0 \)

\[
\Rightarrow H^q(R^*_G(X)) = H^q(R^*_G(X)_\log) = \ker \Delta \cap R^*_G(X)_L^2 \cong R^*_G(X)^G.
\]

if \( \delta \) small

Now write \( \delta y = \text{Lie } G(\mathbb{R}) = \delta \theta \otimes \rho \) (Cartan decmp.)

\[
\delta y = \delta \theta \otimes \rho = \text{Lie } G_u(\mathbb{R}) \quad \text{, } G_u(\mathbb{R}) \text{ compact.}
\]

We have \( R^*_G(X)^G = \text{Hom}_G(\Lambda^q(\delta y), \mathbb{R}) = \text{Hom}_u(\Lambda^q(y_w), \mathbb{R}) \in R^*_u(X_u)^G \)

where \( X_u = G_u(\mathbb{R})/K \) (compact).

The \( \delta y \) maps on \( H^q_{cont}(G) \) are constant. Since \( G_u \) compact, we may arrange in cohomology class \( \Omega^*G_u(\mathbb{R})/K) \Rightarrow \Omega^*(G_u(\mathbb{R})/K)^G \). So clearly we get

\[
\Omega^*G_u(\mathbb{R})/K) \Rightarrow \Omega^*(G_u(\mathbb{R})/K)^G = H^q(X_u, \mathbb{R}).
\]

Prop. 2 (Borel):

\[
H^q_{cont}(SL_N(\mathbb{R})) \cong H^q(SO(N)/SU(N), \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R})
\]

\[
H^q_{cont}(SL_N(\mathbb{R})) \cong H^q(SU(N)/SU(N), \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R})
\]

where \( \varepsilon_1, \varepsilon_2 \in \Lambda^*(\mathbb{R}^N, \mathbb{R}) \).

So

\[
\Lambda^*(\mathbb{R}^N, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^N, \mathbb{R})
\]

where \( \varepsilon_1, \varepsilon_2 \in \Lambda^*(\mathbb{R}^N, \mathbb{R}) \).

Therefore,

\[
K_m(\mathbb{R}) \cong \mathbb{R} \cong \mathbb{R} \cong \mathbb{R}
\]
\[
\frac{\pi}{2} \left\{ I \left( (\lambda^* e_i s)^{\otimes r_1} \otimes (\lambda^* e_j s)^{\otimes r_2} \right) \right\}^2 \cong \left\{ R \left( e_{m-1, \ldots, e_m} \right) \right\}^2
\]

Prop. 2, in dimension

\[
= R \left( e_{m-1, \ldots, e_m} \right) \cong \left\{ R \left( e_{m-1, \ldots, e_m} \right) \right\}
\]

\( m = 2n+1 \n \)

\( n \) even.

\[
\begin{array}{l}
\text{(ii) Sketch of proof of Theorem 3 (following Bloch)} \\
\end{array}
\]

Let's start with a computation: how many pts. on \( SL_n \) over a twin field \( F_q, q = p^m \)?

\( SL_2 \) for 1st vector orbit \( q(q^2 - 1) \)

\( SL_3 \) for 1st vector orbit \( q^2(q^3 - 1) \)

\( SL_n \) for 1st vector orbit \( q^2(q^3 - 1)^n - q(q^2 - 1)^n \)

\[
p = \prod_{j=2}^{n} q^{j-1} (q^j - 1) = \prod_{j=2}^{n} (1 - q^{-j}) \prod_{j=2}^{n} q^{j-1} q^{-j} = q^{n^2 - 1} \prod_{j=2}^{n} (1 - q^{-j})
\]

Now there is something called \( p \)-adic integration:

\( V = \text{smooth dim.} \text{n}/\text{Fricke F} \Longrightarrow V(\Omega_p) \text{ "invertible model" } \subseteq V(\Omega_p) \text{ "p-adic vector space"} \)

\( \omega = \text{nonsingular n-form on} \ V/F \text{ measure } \omega_p \text{ on} \ V(\Omega_p) \)

\[
\begin{align*}
\int_{V(\Omega_p)} dx &= 1 \\
\int_{\Omega_p/\theta_p} dx &= q^{-1} \\
\int_{(p)} dx &= q^n \quad \Rightarrow \int_{V(\Omega_p)} \omega_p = q^{-n} |V(F_q)| \quad \Rightarrow \int_{V(F_q)} \omega_p = \prod \int_{V(\Omega_p)} \omega_p \\
\Rightarrow \int_{SL_n(A_{F,F})} \omega_p &= \frac{1}{n} \prod_{j=2}^{n} q^{-(n^2 - 1)} q^{-n} \prod_{j=2}^{n} (1 - q^{-j}) = \prod_{j=2}^{n} \frac{1}{p} \prod_{j=2}^{n} S_F(j)^{-1}.
\end{align*}
\]
Next recall two from above \( \bigcup \mathcal{X} = G(\mathbb{R})/K \) 
compact\[ X_n = G_n(\mathbb{R})/K \] 
regular\[ H^m_{conn}(G_n(\mathbb{R}),, H^m_{luc}(G_n(\mathbb{R})/K)) \Rightarrow H^{m+1}(\mathcal{X}_n, K; \mathbb{R}) = H^{m+1}_{conn}(\mathcal{X}_n, \mathbb{R}) \]
\( \lambda^* \) and \( \varphi \).

Put \( G_n = \text{Res}_{F/\mathbb{Q}} SL_n(F) \). Then
\[ I^{2n+1}(H^m_{conn}(G_n, \mathbb{R}),, H^m_{luc}(X_n, \mathbb{R})) = I^{2n+1}(H^m_{luc}(X_n, \mathbb{R}),, \mathbb{R}) \]
\( \lambda^* \) and \( \varphi \).

Set \( Y_N(\Gamma) = \Gamma \setminus G_n(\mathbb{R}), X_n(\Gamma) = Y_N(\Gamma)/K_N \), where \( \Gamma = SL_n(\mathbb{Q}_F) \) etc.

\[ H^m_{luc}(K_n, \mathbb{C}) \otimes H^m(\mathcal{X}_n, \mathbb{C}) \overset{\lambda^*}{\approx} H^m_{luc}(G_n, \mathbb{C}) \]
\( \lambda \overset{\lambda}{\to} \lambda \overset{\lambda}{\to} \lambda \)

\[ H^m_{luc}(K_n, \mathbb{C}) \otimes H^m(\mathcal{X}_n, \mathbb{C}) \overset{\lambda^*}{\approx} H^m_{luc}(G_n, \mathbb{C}) \]
\( \lambda \overset{\lambda}{\to} \lambda \overset{\lambda}{\to} \lambda \)

View \( \mu \) as map from \( H^m_{luc}(X_n, \mathbb{C}) \to H^m(\mathcal{X}_n, \mathbb{C}) \).

Now \( H^{2n+1}(G_n, \mathbb{Q}),, H^{2n+1}(G_n, \mathbb{Q}) \)
\[ I^{2n+1}(\mathcal{X}_n, \mathbb{Q}) \]
\( \mathcal{X}_n \)

Write
\[ L_n(a_{\mathbb{Q}} T, \mathbb{Q}) = \bigwedge^d \bigotimes a T \]
\[ L_n(G_{n}, \mathbb{Q}) = \bigwedge^d \bigotimes G_{n} \]
\[ L_n(X_{n}, \mathbb{Q}) = \bigwedge^d \bigotimes X_{n} \]

Claim 1:
\[ \lambda^* \quad L_n(a_{\mathbb{Q}} T, \mathbb{Q}) = (\bigotimes a T)^{m+1} \mathcal{X}_n = L_n(G_{n}, \mathbb{Q}) \]
Claim 2:
\[ \beta^* \quad L_n(a_{\mathbb{Q}} T, \mathbb{Q}) = i^{r_2} S_{\mathcal{X}_n} \quad L_n(Y_{n}, \mathbb{Q}) \]

Claims 1 & 2 \( \Rightarrow \)
\[ L_n(G_{n}, \mathbb{Q}),, L_n(X_{n}, \mathbb{Q}) = \bigwedge^d \bigotimes G_{n} \]
\[ \bigwedge^d \bigotimes X_{n} \]

Claim 1 & 2 \( \Rightarrow \)
\[ L_n(\mathcal{X}_n, \mathbb{Q}) = L_n(G_{n}, \mathbb{Q}) \]

Write \( K_{2n+1}(T, \mathbb{Q}) \)
\[ \bigwedge \bigotimes T \]
\[ P_{2n+1}(T, \mathbb{Q}) = \bigwedge \bigotimes T \]
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We will only say something about Claim 2. Suppose 
\[ \beta^* L_j(y_{N}, q) = \sum_{l} L_j(y_{N(l)}, q) \text{ for some } s \in \mathbb{C}, 1 \leq j \leq n. \] We have generators \[ \mathcal{R}(q_1, q_{2j+1}) \] for \[ L_s(y_{N}, q). \] Define \[ q_j = \prod_{s=1}^{n} R_{F/q_{2j+1}}. \]

Then we get \[ \beta^*(q_j) = \prod_{i=1}^{j} \mathcal{H}(q_j^{2i+2j})(y_{N(r)}, q). \]

If we can exhibit compact subvarieties \[ Z_j \subset Y_M(r) \] with \[ d(q_j^{2i+2j}) = d(z_j) \]

\[ 0 \neq \sum_{j=1}^{n} \beta^*(q_j) \in \prod_{i=1}^{n} \mathcal{F}(q_i) \cdot q_j, \quad 1 \leq j \leq n, \]

then we get (up to factors) \[ s_j \in \mathcal{F}(q_j) \cdot q_j \text{ as desired.} \]

We now construct the \( Z_j. \)

- Let \( D = \text{ division algebra of dim. } (j+1)^2/F, \) tensor over archimedean place \( \mathcal{H} \subset D^* \) s.t. \( \mathcal{H} \) is an order of \( \mathcal{H} \).

\[ H = R_{F/q} \mathcal{H} \]

\[ H'(A_F)/H'(F) \text{ is compact; let } U \subset H'(A_F) \text{ be compact open} \]

strong oppn. \( \Rightarrow H'(A_F) = H'(k_{F(R)}). U \cdot H'(F) \)

Set \( T = (H'(k_{F(R)}). U) \cap H'(F); \text{ then} \)

\[ U \to H'(A_F)/H'(F) = H(A_F)/H(F) \]

(\( \star \star \))

\[ H'(k_{F(R)}/T) = H(k_{F(R)}/T) \text{ is a fibration w/compact fibers.} \]

- Embed \( H' \hookrightarrow SL_N, \) hence \( H \hookrightarrow G_N. \)

(by regular rep. of \( H' \subset D \subset F^{(j+1)^2} \))

Take subvariety \( \mathcal{F}_j = H(R)/(H(R) \cap M) \) of \( \mathcal{F}_M(R)/M \)

New \( \beta^*(q_j)|_{\mathcal{F}_j} \) is a form of max. degree (Borel proves \( \neq 0 \)), so we can integrate.

In fact, we can replace \( H \) by any invariant volume form defined on \( H \):
\[ \omega = \sqrt{10} \omega^{(j+4j)/2} \wedge \omega \]

(\( \star \star \))

In the fibration (\( \star \star \)), let \( \omega_{\mathcal{F}_j} \) denote the measure on \( U, H(k_{F(R)})/T \) induced by \( \omega \). We have...
$$Q = \int_{\mathcal{R}} \frac{H(A_\mu)}{H(\mathcal{Q})} \cdot \int_{U_\mu} w_{\mu} \cdot \int_{H(\mathcal{R})/T} w_{\mathcal{R}} \ .$$

given worst on Tamagawa's of
gradients at $D^\mu/T^\mu$

The first factor is (up to $Q$)
$$\int_{H(\mathcal{R},f)} w_{\mathcal{R}} \cdot \varepsilon\left(\prod_{k=2}^{j=1} S_{\mathcal{R}}(h)^{-1}\right) \cdot Q \ .$$

$$\Rightarrow \int_{H(\mathcal{R})/T} w_{\mathcal{R}} \cdot \varepsilon\left(\prod_{k=2}^{j=1} S_{\mathcal{R}}(h)^{-1}\right) \cdot Q \ .$$

$$\Rightarrow \int_{\mathcal{R}} \beta \eta_j \cdot \left(\prod_{k=2}^{j=1} S_{\mathcal{R}}(h)^{-1}\right) \cdot Q \ .$$