Problem Set 2

due Wednesday Sept. 30

(1) (a) Prove Proposition I.B.4.1; and (b) verify the first two bullets of Example I.B.4.1.
(2) Prove the “$\implies$” direction of Proposition I.B.4.5 (which should really be Prop. 4, not 5...).
(3) (a) Referring to the construction on p. 7 of I.B.4, determine the projections $\pi$ for $Sp_4$ which induce Hodge numbers $(2, 2)$ and $(1, 1, 1, 1)$ [weights 1 and 3 resp.] on the standard (4-diml) representation $V$.
[Note: the roots may be visualized as a three-by-three grid of dots,\footnote{The usual convention is to tilt this picture 45 degrees, but the “square” picture is a little easier to work with for our purposes.} with the center “dot” representing the 2-diml Cartan subalgebra $t$, and the other root-spaces 1-dimensional. Usually the root at $(1, -1)$ is called $\sigma_1$, and that at $(0, 1)$ is called $\sigma_2$; the others are $\sigma_1 + \sigma_2$, $\sigma_1 + 2\sigma_2$, and the negatives of all these. The compact roots are $\pm \sigma_2$. The weights of $V$ are $\pm \frac{1}{2} \sigma_1$ and $\pm (\frac{3}{2} \sigma_1 + \sigma_2)$.
(b) What are the dimensions of the Mumford-Tate domains so constructed? What is the rank of the horizontal distribution?
(c) Why doesn’t (a) work for Hodge numbers $(1, 2, 1)$ (weight 2)?
(4) Exercise I.B.6.1 (p. 13 of I.B.6)
(5) Exercise I.B.6.2 (same page)
(6) Find a VHS to which Prop. I.B.6.5 applies!