Boundary strata and adjoint varieties\textsuperscript{1}

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\textsuperscript{1}report on recent work with C. Robles, based in part on earlier work with G. Pearlstein as well as P. Griffiths and M. Green.
Motivating principle: use Representation Theory to classify what is possible for VHS with given “symmetries”; use that in turn to decide what is geometrically possible or expected:

\[ \text{RT} \rightarrow \text{HT} \rightarrow \text{moduli} \]

\[ \mathbb{Q}-\text{algebraic group} \rightarrow \text{Hodge tensors} \rightarrow \text{algebraic cycles?} \]

GGK/Patrikis: poss. MTG

flag/Schubert varieties $\rightarrow$ maximal VHS $\rightarrow$ geom. realization?

nilpotent cones $\rightarrow$ bdry. components (LMHS) $\rightarrow$ degenerations

K-Pearlstein (appr. to Torelli?)

geom. of flag var. $\mathcal{D} \rightarrow$ diff’l. inv. of VHS $\rightarrow$ geom. realization?

(2nd FF)

smooth reps. of classes in $H^* (\mathcal{D}, \mathbb{Z})$ $\leftarrow$ enhanced $SL_2$-orbits
§1. Construction of Mumford-Tate domains

- $V = \text{vector space over } \mathbb{Q}$
- $Q : V \times V \to \mathbb{Q}$ nondegenerate symmetric bilinear form
- $\varphi : S^1 \to Aut(V_\mathbb{R}, Q)$, where $Q(v, \varphi(\sqrt{-1})\bar{v}) > 0 \ \forall v \in V_\mathbb{C}\{0\}$ (weight 0 PHS)
- $G \leq Aut(V, Q)$: $\mathbb{Q}$-algebraic closure of $\varphi(S^1)$
  $G = \text{subgroup fixing HTs pointwise (Chevalley’s thm.)}$
  Mumford-Tate (or Hodge) group
- $D := G(\mathbb{R}) \varphi \cong G(\mathbb{R})/H \subset G(\mathbb{C})/P =: \check{D}$
  Mumford-Tate domain $\subset$ compact dual

We think of M-T domains as parametrizing (a connected component of) all HS on $V$ polarized by $Q$, with the same Hodge numbers as $\varphi$, whose HTs include the fixed tensors of $G$. We shall loosely speak of $(V, Q, \varphi)$ as a “Hodge representation” of $G$. 
Problem [GGK]: How do we arrange for the M-T group to be a given (simple, adjoint) \( \mathbb{Q} \)-algebraic group \( G \)?

One way is to take \( V = \mathfrak{g} \). We need the crucial assumption that \( G(\mathbb{R}) \) contains a compact maximal torus \( T \).

Let \( \mathfrak{g}_\mathbb{R} = \mathfrak{k} \oplus \mathfrak{k}_\perp \) be a Cartan decomposition with \( \mathfrak{k} \subseteq \mathfrak{k} \) and involution \( \theta \). Write \( \Delta := \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) = \Delta_c \cup \Delta_n \), and \( \mathcal{R} \leq \Lambda := X^*(T_\mathbb{C}) \) for the lattice it generates.

- \( \theta \) Lie-alg. homom. \( \implies \exists \) homom. \( \mathcal{R} \to \mathbb{Z} \) sending \( \Delta_c \to 2\mathbb{Z}, \Delta_n \to 2\mathbb{Z} + 1 \).

- \( G \) adjoint \( \implies \mathcal{R} = \Lambda \implies \) this homom. is induced by a grading element \( E \in \sqrt{-1}\mathfrak{k} \).

- Set \( \varphi(z) := e^{2\log(z)E} \), so that \( (\text{Ad} \circ \varphi)(\sqrt{-1}) = \theta \).

- Since \( -B \) is \( > 0 \) on \( \mathfrak{k} \) and \( < 0 \) on \( \mathfrak{k}_\perp \), \( (\mathfrak{g}, \text{Ad} \circ \varphi, -B) \) is a PHS (of weight 0).
The Hodge decomposition takes the form \( \mathfrak{g}_\mathbb{C} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}^j \), where

\[
\mathfrak{g}^j := \mathfrak{g}_{\varphi}^{-j} = \begin{cases} 
\bigoplus_{\delta \in \Delta : E(\delta) = j} \mathfrak{g}_\delta, & j \neq 0 \\
\left( \bigoplus_{\delta \in \Delta : E(\delta) = 0} \mathfrak{g}_\delta \right) \oplus \mathfrak{t}_\mathbb{C}, & j = 0
\end{cases}
\]

write \( h^j := \dim_{\mathbb{C}} \mathfrak{g}^j \) for the Hodge numbers.

We also claim that the M-T group of \( \text{Ad} \circ \varphi \) is \( G \). Why?

Let \( M \leq G \) be (equivalently)

(a) the smallest \( \mathbb{Q} \)-algebraic group such that \( \text{Ad} \circ g \varphi g^{-1} \) factors thru \( M(\mathbb{R}) \) (\( \forall g \in G(\mathbb{R}) \))

(b) the M-T group of the family \( \{ \text{Ad} \circ g \varphi g^{-1} \}_{g \in G(\mathbb{R})} \) of polarized Hodge structures

(c) the M-T group of \( \text{Ad} \circ g_0 \varphi g_0^{-1} \) for all \( g_0 \in G(\mathbb{R}) \) in the complement of a meager set

Since \( \varphi \) is “sufficiently general”, we may take \( g_0 = 1 \) in (c).

By (a), \( M \trianglelefteq G \); so \( G \) simple \( \implies \) \( M = G \).
We remark that

\[ \dim_{\mathbb{C}}(\tilde{D}) = \sum_{j > 0} h^j \]

\[ \text{rank}(\mathcal{W}) = h^{-1} \] (horizontal distribution)

**Example:** \(( G = G_2 )\)

\[ U = 7 - \text{dim irrep} \]

\[ \begin{array}{cccc}
-2 & -1 & 0 & 1 \\
1, & 4, & 4, & 4, & 1 \\
\end{array} \]

\[ \begin{array}{cccc}
\text{dim}(\tilde{D}) = 5 \\
rk(W) = 4 \\
\end{array} \]

\[ g_{(U^p)} = U^{p+m} \]
Generalization to fundamental adjoint varieties

Recall: upon fixing $\Delta^+ \subset \Delta$, we have

- $\alpha_i$ simple roots $\leftrightarrow S^i$ simple grading elements $\delta^i_i$
- $\omega_i$ fundamental weights $\leftrightarrow H_j$ simple coroots $\delta^j_i$ $(H_j \in [g_{\alpha_i}, g_{-\alpha_i}], \alpha_i(H^i) = 2)$

Let $\mu = \sum \mu^i \omega_i$ be dominant ($\mu_i \geq 0$), assume $U = V^\mu$ real. Consider the assoc. parabolic subgroup $P \geq B \geq T$, where

- $\Delta(B) = \Delta^+$
- $I(p) := \{i | -\alpha_i \notin \Delta(p)\} = \{i | \mu^i \neq 0\}$

and the assoc. grading element $E := \sum_{i \in I} S^i$. Put $m := E(\mu)$.

- $U = U^{-m} \oplus \cdots \oplus U^m$ is the grading induced by $E$
- $U^m = U_\mu = \text{highest weight line}$
- the $G(\mathbb{C})$-orbit of $[U_\mu] \in \mathbb{P}U$ gives a homogeneous embedding of $G(\mathbb{C})/P$, minimal if $\mu^i \in \{0, 1\} \ (\forall i)$. 


The adjoint case is $U = \mathfrak{g}$. Root/weight computations show:

- $m = 2$, so $\{h^j\} = \{1, *, *, *, 1\}$

- unless $G$ is of type $E_8$, there exists a level 2 faithful Hodge representation (like $(2, 3, 2)$ in the $G_2$ example above)

- unless $G$ is of type $C$, the adjoint variety $\check{D} = G(\mathbb{C})/P \hookrightarrow \mathbb{P}\mathfrak{g}$ is minimally embedded

- unless $G$ is of type $A$ or $C$, the adjoint representation is fundamental ($\mathfrak{g} = V^{\omega_k}$), and the corresponding $\{\check{D}\}$ are the fundamental adjoint varieties.

Why study the adjoint varieties as Hodge-theoretic classifying spaces?

One reason: they are the “simplest” $G/P$ with nontrivial IPR, in the sense of being precisely the cases where $\mathcal{W}$ is a contact distribution.
§2. Schubert VHS and classical subdomains

Let $D \subset \check{D}$ be a M-T domain with base point $F^\bullet = F^\bullet_\varphi \in D$, $\mathfrak{g} = \bigoplus \mathfrak{g}^j$ the corresponding Hodge decomposition, and $T, \Delta$ as before. Write $P \supset B \supset T_\mathbb{C}$ for the parabolic fixing $F^\bullet$, so that we have:

- $\Delta(B) =: \Delta^+ = \Delta(F^1 \mathfrak{g}) \cup \Delta^+(\mathfrak{g}^0)$
- $\Delta(P) = \Delta(F^0 \mathfrak{g})$.

Put $W^P := \{ w \in W | w \Delta^+ \supset \Delta^+(\mathfrak{g}^0) \}$, and note that

- $w \in W^P \implies \Delta_w := \Delta^- \cap w \Delta^+$ is closed in $\Delta$
- $\check{D} = \coprod_{w \in W^P} C_w := \coprod_{w \in W^P} Bw^{-1} F^\bullet$.

The Schubert varieties $X_w := \overline{wC_w}_{\text{Zar.}} \subset \check{D}$ satisfy

- $\dim_{\mathbb{C}} X_w = |\Delta_w| = \ell(w)$
- $\mathcal{T}_{F^\bullet} X_w = n_w := \bigoplus_{\alpha \in \Delta_w} \mathfrak{g}_\alpha$
- $X_w$ Schubert VHS (i.e. horizontal) $\iff n_w \subset \mathfrak{g}^{-1}$
  $\iff n_w$ abelian
Theorem (Robles)

\[
\max \text{dim}(\text{IVHS}) = \max \text{dim}(\text{SVHS}) = \\
\max \left\{ |\Delta_w| \mid w \in WP, \Delta_w \subset \Delta(g^{-1}) \right\}
\]

Example: \((G = G_2)\)

There exists only one \(X_w\) of dimension 2, and it is an SVHS (false for higher dim.)

\[\Downarrow\]

the maximal integral manifold of \(W\) has dimension 2

Is this \(X_w\) a M-T subdomain? If so, it would be a (smooth, Hermitian) homogeneous \(G'(\mathbb{R})\)-orbit, with \(g \supseteq g' = \text{Lie algebra closure of } n_w \oplus \overline{n_w}\). But this closure is all of \(g\). NO!
More generally, **what is the relationship between SVHS and horizontal (⇒ Hermitian) M-T domains?**

For any subdiagram \( \mathcal{D}' \subset \mathcal{D} \) of the Dynkin diagram of \( \mathfrak{g} \), have

- \( \mathfrak{g}' \subset \mathfrak{g} \) subalg. gen. by the root spaces \( \{\mathfrak{g}_\alpha| \alpha \in \mathcal{D}'\} \)
- \( X(\mathcal{D}') := G'(\mathbb{C}).F^\bullet \subset \check{\mathcal{D}} \) smooth Schubert variety

\( X(\mathcal{D}') \) is horizontal iff \( \mathfrak{g}' \subset \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \), in which case it is (the compact dual of) a homogeneously embedded Hermitian symmetric domain.

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**Theorem (K-R)**

Let \( X \subset \check{\mathcal{D}} \) be a SVHS. Then

\[
X \text{ smooth } \iff X = \prod_{i} X(\mathcal{D}_i) \text{ (homog. emb. HSD)}. 
\]

It is instructive to compare this with another recent result:
Theorem (Friedman-Laza)

- $X \subset \tilde{D}$ a smooth "VHS" (horizontal subvariety)
- $Y$ a (nonempty) connected component of $X \cap D$
  with strongly quasi-projective image in $\Gamma \backslash D$

$\implies Y$ is a (Hermitian) M-T subdomain.

On the other hand, with an arithmetic assumption on $\tilde{D}$, the [K-R] result has the following

Corollary

- $X \subset \tilde{D}$ a smooth Schubert VHS
- $Y$ a (nonempty) connected component of $X \cap D$

$\implies Y$ is a translate of a (Hermitian) M-T subdomain.

The converse of the Corollary is false: there are plenty of non-Schubert, horizontal Hermitian M-T subdomains, and we will construct maximal integral ones later.
§3. Lines on $\mathcal{D}$ and a differential invariant of VHS

The Corollary suggests that there might be lots of singular SVHS, in view of the $G_2$ example. A systematic construction of Schubert varieties is given by incidence correspondences:

$$
\xymatrix{ & G(\mathbb{C})/\{P \cap Q\} \\
G(\mathbb{C})/Q \ar@<.5ex>[ur] \ar@<-.5ex>[dr] & & G(\mathbb{C})/P \ar@<.5ex>[ul] \ar@<-.5ex>[dl] \\
\Sigma & & X(\Sigma) \\
& X^{-1} \\
& X \\
& \Sigma \\
}$

where $P, Q \geq B$. Note that:

- $X, X^{-1}$ preserve Schubert varieties
- A point is a Schubert variety (e.g. $P/P = F^\bullet$)
- $X(Q/Q) = X(\mathcal{D}')$, where $\mathcal{D}' = \mathcal{D} \setminus (l(q) \setminus l(p))$
Case of $P$ maximal

- $P$ maximal $\implies I(p) = \{k\}$. If $I(q)$ contains the nodes adjacent to $\{k\}$, then $X(D')$ is a $\mathbb{P}^1$ thru $F^\bullet$.

In this case:

- $X_0 := X(X^{-1}(F^\bullet))$ is a Schubert variety consisting of all $\mathbb{P}^1$'s thru $F^\bullet$ on $\mathcal{D}$ (in its minimal embedding).

Specializing to the case where $\mathcal{D}(\subset \mathbb{P}g)$ is a fundamental adjoint variety, let

- $C_0 :=$ the $G^0(\mathbb{C})$-orbit of the highest weight line in $\mathbb{P}g^{-1}$.

Then

- $C_0 \cong G^0(\mathbb{C})/\{P \cap G^0(\mathbb{C})\}$ is a homogeneous Legendrian variety

- $X_0 \cong \text{Cone}(C_0)$ is a singular Schubert VHS
Some data for the fundamental adjoint varieties $\tilde{D}$ and their associated “subadjoint” varieties $C_0$ of lines through a point:

<table>
<thead>
<tr>
<th>$\mathfrak{g}_C$</th>
<th>$\tilde{D}$</th>
<th>$\mathfrak{g}_C^{0,ss}$</th>
<th>$C_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{so}(n)$</td>
<td>$OG(2, \mathbb{C}^n)$</td>
<td>$\mathfrak{so}(n-4) \oplus \mathfrak{sl}(2)$</td>
<td>$\mathbb{P}^1 \times Q^{n-6}$</td>
</tr>
<tr>
<td>$e_6$</td>
<td>$E_6/P_2$</td>
<td>$\mathfrak{sl}(6)$</td>
<td>$Gr(3, \mathbb{C}^6)$</td>
</tr>
<tr>
<td>$e_7$</td>
<td>$E_7/P_1$</td>
<td>$\mathfrak{so}(12)$</td>
<td>$S_6$</td>
</tr>
<tr>
<td>$e_8$</td>
<td>$E_8/P_8$</td>
<td>$\mathfrak{e}_7$</td>
<td>$E_7/P_7$</td>
</tr>
<tr>
<td>$f_4$</td>
<td>$F_4/P_1$</td>
<td>$\mathfrak{sp}(6)$</td>
<td>$LG(3, \mathbb{C}^6)$</td>
</tr>
<tr>
<td>$g_2$</td>
<td>$G_2/P_2$</td>
<td>$\mathfrak{sl}(2)$</td>
<td>$\nu_3(\mathbb{P}^1)$</td>
</tr>
</tbody>
</table>
We now relate these varieties of lines to the Griffiths-Yukawa kernel. Let $\mathcal{V} = \bigoplus_{j=0}^{n} \mathcal{V}^{n-j,j}$ be a VHS over $S$, with associated period map $\Phi : S \to \Gamma \backslash D$ ($D = \text{M-T domain}$). Denote by $\mathcal{D}$ a holomorphic differential operator on $U \subset S$ of order $n$.

The composition

$$O_U(\mathcal{V}^{n,0}) \hookrightarrow O_U(\mathcal{V}) \xrightarrow{\mathcal{D}} O_U(\mathcal{V}) \twoheadrightarrow O_U(\mathcal{V}^{0,n})$$

depends only on $\sigma(\mathcal{D})$, giving rise to the G-Y coupling

$$\text{Sym}^n T_s S \quad \rightarrow \quad \text{Hom}(\mathcal{V}_s^{n,0}, \mathcal{V}_s^{0,n}) = (\mathcal{V}_s^{0,n}) \otimes 2$$

Write $\mathcal{Y} \subset \mathbb{P} \mathfrak{g}^{-1}$ for the kernel of $(\ast)$ (at $F^\cdot \in D$).
Example: \((G = G_2)\) \(V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}\) 7-diml irrep

\[
\begin{pmatrix}
-x_0 \\
-x_1 \\
-x_2 \\
-x_3
\end{pmatrix}
\]

basis of \(g^{-1}\)

\[
\begin{pmatrix}
e_1^* \\
e_2^*
\end{pmatrix}
\]

action of \(g^{-1}\)

Given \(\xi := \sum \xi_i x_i \in g^{-1}\), one computes

\[
e^* [\xi^2] e = \begin{pmatrix}
-2\xi_1\xi_2 + 2\xi_2^2 & \xi_1\xi_2 - \xi_0\xi_3 \\
\xi_1\xi_2 - \xi_0\xi_3 & -2\xi_0\xi_2 + 2\xi_1^2
\end{pmatrix}
\]

whose vanishing defines the twisted cubic \(\nu_3(\mathbb{P}^1) \subset \mathbb{P}g^{-1}\).
This example is generalized to $E_6$, $E_7$, $F_4$ by the

**Theorem (K-R)**

For $\tilde{D} = G(\mathbb{C})/P$ a fundamental adjoint variety, and $V$ a level 2 Hodge representation such that $V^{2,0}$ is a faithful representation of $\mathfrak{g}^0$, we have $\mathcal{Y} = C_0$. (Much more generally, $\mathcal{Y}$ contains the horizontal lines through $F^\bullet$.)

**Sketch:** Given $[\xi] \in \mathcal{Y}$, $\xi^2(u) = 0 \ \forall u \in V^{2,0}$. Fix $v \in \mathfrak{g}^2 \setminus \{0\}$, so $\text{ad}_\xi^2 v \in \mathfrak{g}^0$. Then $(\text{ad}_\xi^2 v)u = v\xi\xi u = 0$ implies $\text{ad}_\xi^2 v = 0$ second fundamental form vanishes at $\xi \implies [\xi] \in C_0$. □

When the conclusion of the Theorem holds,

- $\mathcal{Y} = \ker(G-Y)$ gives Hodge-theoretic meaning to $C_0$
- $C_0 \cong G^0(\mathbb{C})/\cdots$ gives a homogeneous description of $\mathcal{Y}$
- $\text{ad}_\xi^2 v = 0$ produces explicit projective homogeneous equations for both.
§4. \(G(\mathbb{R})\)-orbits in \(\mathring{D}\) and asymptotics of VHS

Given the input:

- \(D \subset \mathring{D}\) M-T domain (parametrizing wt. 0 HS on \(V\))
- \(\Gamma \leq G(\mathbb{Q})\) neat arithmetic
- \(\sigma^0 \subset \sigma = \mathbb{Q}_{\geq 0}\langle N_1, \ldots, N_m \rangle \subset g_{\mathbb{Q}}\) abelian nilpotent

we define (and assume nonempty):

\[
\tilde{B}(\sigma) := \left\{ F^\bullet \in \mathring{D} \mid e^{\sum t_i N_i} F^\bullet \in D \text{ for } \text{Im}(\tau_i) \gg 0 \right\}
\]

which parametrizes LMHS \((F^\bullet, W(\sigma)_\bullet), 2\)

\[
\mathcal{B}(\sigma) := e^{i\sigma} \setminus \tilde{B}(\sigma) = \text{boundary component} \text{ assoc. to } \sigma
\]

which parametrizes \(\sigma\)-nilpotent orbits \((\sigma, e^{i\sigma} F^\bullet), \) and

\[
\overline{B}(\sigma) = \Gamma_\sigma \setminus \mathcal{B}(\sigma), \text{ where } \Gamma_\sigma := \text{stab}_\Gamma(\sigma).
\]

One may “partially compactify” \(\Gamma \setminus D\) by \(\overline{B}(\sigma)\)s (log manifold).

\[N(W(\sigma)_\bullet) \subset W(\sigma)_{\bullet - 2} \text{ and } N^k : Gr_k^{W(\sigma)} \xrightarrow{\cong} Gr_{-k}^{W(\sigma)} \text{ for } (\forall N \in \sigma^0).\]
Structure of $B(\sigma)$

Write

- $M_\sigma = \exp \{ \text{im}(\sum N_i) \cap (\cap \ker(N_i)) \}$
- $Z(\sigma) = Z_0(\sigma) \cdot M_\sigma$ for the centralizer of $\sigma$ in $G$
- $G_\sigma \leq Z_0(\sigma)$ for the M-T group of generic $Gr^W(F^\bullet, W(\sigma)_\bullet)$

Then we have

- fibration $B(\sigma) \rightarrow D(\sigma) = \text{M-T domain of generic } Gr^W(F^\bullet, W(\sigma)_\bullet)$

- $B(\sigma) = \{ G^{ss}_\sigma(\mathbb{R}) \ltimes M_\sigma(\mathbb{C}) \}.F^\bullet_0$ and
  $D(\sigma) = G^{ss}_\sigma(\mathbb{R}).Gr^W F^\bullet_0$

- naive limit map
  $\Phi^\sigma_\infty : B(\sigma) \rightarrow \partial D \subset \check{D}$
  $F^\bullet \mapsto \lim_{\text{Im}(\tau) \rightarrow \infty} e^{\tau N} F^\bullet$ (any $N \in \sigma^\circ$).
[K-R] contains a general prescription for using a set
\[ \mathcal{B} = \{ \beta_1, \ldots, \beta_s \} \subset \Delta(\mathfrak{g}^1) \]
of strongly orthogonal roots to explicitly construct \( \mathbb{Q}\)-split \((\sigma, F_0^*)\). The motivation is to parametrize \( G(\mathbb{R})\)-orbits in \( \partial D \) in the image of \( \Phi^{\sigma}_\infty \). I will discuss only \( s = 1 \). Fix a base point \( o \) (\( \longleftrightarrow F^* \)) in \( D \).

Let \( \beta \in \Delta(\mathfrak{g}^1) \), with associated \( \mathfrak{sl}_2^\beta = \langle N, Y, N^+ \rangle \) \((N \in \mathfrak{g}_{-\beta})\). Apply the \textbf{Cayley transform} \( c_\beta = \text{Ad} \left( e^{\frac{\pi}{4}(X_{-\beta} - X_\beta)} \right) \) to

- \( t_\mathbb{C} \rightsquigarrow \mathfrak{h} \)
- \( \mathfrak{g}_\alpha \rightsquigarrow '{\mathfrak{g}}_\alpha \)
- \( E \rightsquigarrow '{E} \)
- \( o \rightsquigarrow '{o} \)
- \( (F^* \rightsquigarrow '{F}^*) \)
Then

1. $F^\bullet \in \hat{B}(N)$, and $g_N^{ss} = \ker \{ \beta |_h \} \oplus \bigoplus_{\alpha \parallel \beta} 'g_{\alpha}$

2. 'E, Y give a (Deligne) bigrading $g_{p,q} = g_p \cap g_{p+q}$ of $g_C$

whose dimensions $h_{p,q}$ are the Hodge-Deligne numbers of the (limit) MHS $(F^\bullet, W(N)_.)$ associated to 'o.

**Remark:** We can use this to construct non-Schubert M-T subdomains. Define the “enhanced $SL_2$-orbit”

$$X(N) := e^{\mathbb{C}N} G_N^{ss} \cdot \text{Zar} \cdot 'o = G_N^{ss} \times SL_2^\beta \cdot 'o \subset \hat{D};$$

then (with an arithmetic assumption on o)

1. $Y(N) := X(N) \cap D$ is a M-T domain

2. $X(N) = \hat{D}(N) \times \mathbb{P}^1 \supset D(N) \times S_5 = Y(N)$

3. If $E(\alpha) \in \{-1, 0, 1\} \ \forall \alpha \parallel \beta$, then $Y(N)$ is a HSD.

4. If $\hat{D} = G(\mathbb{C})/P$ ($P$ maximal) and $\dim X(N) \geq 2$, then $X(N)$ is not Schubert.
### §5. “Minimal” boundary of adjoint varieties

Let $\tilde{D}$ be a fundamental adjoint variety (note $E = S^i$).

**Proposition (K-R)**

There is a unique codimension-1 $G(\mathbb{R})$-orbit in $\partial D$.

**Sketch:**

1. [K-P] $\mathbb{R}$-codim. of orbit $\ni o$ is given by $\sum_{p,q>0} h^{p,q}$;
2. [KP] codim.-1 orbits are of the form $G(\mathbb{R}).c_\beta o$, $\beta \in \Delta(g^1)$.

Acting by $W(g^0)$, wma $(\beta, \alpha_j) \leq 0 \forall j \neq i$, i.e. $\alpha_j(H^\beta) \leq 0$.

In the bigrading defined by $c_\beta$, $g^{1,1} \supset 'g^1 \supset 'g^\beta$

$$\implies \beta = \alpha_i + \sum_{j \neq i} m_j \alpha_j \ (m_j \geq 0)$$

$$\implies p(\alpha_i) + q(\alpha_i) = \alpha_i(H^\beta) = \beta(H^\beta) - \sum_{j \neq i} m_j \alpha_j(H^\beta) \geq 2$$

$$\implies q(\alpha_i) \geq 1 \implies \text{codim} > 1 \text{ unless } \beta = \alpha_i. \ \Box$$
Let \( o (= c_{\alpha_i} o) \) belong to this real codimension-1 orbit, with associated MHS \( (F^\bullet, W(N)_\bullet) \) and bigrading \( g^{p, q} \).

**Proposition (K-R)**

The \( h^{p, q} = \dim_{\mathbb{C}} g^{p, q} \) are

\[
\begin{align*}
\text{(e.g. for } G_2, \ a = b = 1 \\
\text{for } F_4, \ a = 6 \text{ and } b = 10)\
\end{align*}
\]

**Sketch:**

We know \( \dim g^2 = 1, \ g^{>2} = \{0\}, \dim g_2 = 1, \ g^{>2} = \{0\} \). Now \( \alpha_i, \tilde{\alpha} \) are non-short roots, and

- \( H^i \in [g_{\alpha_i}, g_{-\alpha_i}] \) s.t. \( \alpha_i(H^i) = 2 \)
- \( S^i \in [g_{\tilde{\alpha}}, g_{-\tilde{\alpha}}] \) s.t. \( \tilde{\alpha}(S^i) = 2 \).

\[
\begin{align*}
&\implies \exists w \in W \text{ sending } H^i \mapsto -S^i \\
&\implies w(g^{p, q}) = w(g^p \cap g_{p+q}) = g_{-p} \cap g^{-(p+q)} = g^{-(p+q)}, q. \quad \square
\end{align*}
\]
Note that \( \nu \) identifies the (faithful) representations of \( g_0 \) on \( g_1 \) and \( g^0 \) on \( g^{-1} \). Moreover, \( D(N) \subset \mathbb{P}g_1 \) is the M-T domain for the Hodge representation of \( G_0 \) on \( g_1 \), which leads to:

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**Theorem (K-R)**

(a) \( \check{D}(N) \subset \mathbb{P}g_1 \) is \( \cong C_0 \subset \mathbb{P}g^{-1} \).

(b) \( X(N) \cong \mathbb{P}^1 \times C_0 \) (cylinder on \( C_0 \))

(c) \( \check{B}(N) \rightarrow \Gamma_N \backslash D(N) \) is a family of intermediate Jacobians associated to a VHS (with Hodge numbers \( (1, a, a, 1) \)) over a Shimura variety.

(d) Over \( D(N) \), these VHS recover the Friedman-Laza list of maximal weight 3 Hermitian VHS of CY type.

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Applications? Automorphic cohomology; geometric realizations; cohomology of \( \check{D} \).
\( H^*(\tilde{D}, \mathbb{Z}) \) is generated by Schubert varieties, and the “horizontal” part (invariant characteristic cohomology) by Schubert VHS. Do the subadjoint cylinder classes \([X(N)]\) yield smooth representatives of the subadjoint cone classes \([X_w]\)?

**Example:** 
\( (G = G_2) \)

\[
X_w \cong X(G^0 \cdot \mathbb{Q}/\mathbb{Q}) \cdot \Sigma, \quad X(N) \cong (SL^\beta_2 \cdot \mathbb{Q}/\mathbb{Q}) \cdot \Sigma(\mathbb{N})
\]

\[
(\omega_1, \omega_1) = 2(\omega_1, \alpha_1) \quad \Rightarrow \quad [\Sigma(N)] = 2[\Sigma]
\]

\[
\Rightarrow \quad [X(N)] = 2[X_w].
\]
§6. A (partial) geometric realization

We look for degenerations of varieties predicted by the “codim. 1” boundary components of adjoint domains. For $G_2$, this should take the form of a 1-parameter family of surfaces $\{X_t\}$ with $H^2_{tr}$ Hodge numbers $(2, 3, 2)$, M-T group $G_2$, and LMHS of the form

$$H^2_{tr}(X_t)$$

where $N$ is the monodromy logarithm and bullets denote 1-dimensional spaces. In fact, just such a family has been constructed by N. Katz using elliptic fibrations; the M-T group is determined by a moment computation using elliptic convolution over finite fields. We shall describe a special case. 
Begin with the rational elliptic surface

$$\mathcal{E} \to \mathbb{P}^1_z : y^2 = x(1-x)(x-z^2)$$

with singular fibers \((2\,I_4, 2\,I_2)\) at \(z = -1, 0, 1, \infty\).

For any \(t \neq 0, \frac{\pm 2}{3\sqrt{3}}, \infty\), base change by

$$E_t \to \mathbb{P}^1_z : w^2 = tz(z-1)(z+1) + t^2$$

to obtain an elliptic surface \(X_t \to E_t\) with 7 singular fibers,

$$\Omega^2(X_t) = \mathbb{C}\langle \omega_1(t), \omega_2(t) \rangle = \mathbb{C}\langle \frac{dx}{y} \wedge \frac{dz}{w}, \frac{dx}{y} \wedge zdz \rangle,$$

and

$$\dim(H^2_{tr}(X_t)) = 7:$$

- 2 cycles/ each \(A_1, A_2, B_1, B_2\)
- 1 cycle/ each \(C_i \) \((i=1,2,3)\)

\(\bullet\) = locations of singular fibers
Degenerating $X_t$ as $t \to t_0 = \frac{2}{3\sqrt{3}}$ yields

$$E = \{y^2 = x(1 - x)(x - \frac{1}{3})\}.$$  

The part of $H^2(X_{t_0})$ not coming from the 19 algebraic classes on $Y$ indeed takes the form
In fact, we can “determine” the limiting period in

\[ \tilde{\mathcal{B}}(N) \to \Gamma \backslash \mathfrak{H}. \]

- Since \( G_N \cong SL_2 \) and \( j(E) \notin \mathbb{Z} \),

\[ H^2_{tr}(Y) \cong \text{Sym}^2 H^1(E) \]

and the point in the base is determined by the (non-CM) Hodge structure \( H^1(E) \).

- The point in the fiber

\[ \mathbb{C}^2 / \mathbb{Z} \langle \left( \frac{1}{0}, \frac{2\tau}{1}, \frac{\tau^2}{2\tau}, \frac{0}{3\tau^2} \right) \rangle \cong J(\text{Sym}^3 H^1(E)) \]

\[ \subset J(H^2_{tr}(Y)^\vee \otimes H^1(E)) \]

is given by \( \int_{B_1} \omega, \int_{B_2} \omega \) (\( \omega \in \Omega^2(Y) \)).
The image of the period map into $\Gamma \backslash D$ is contained (at least locally) in 2-dimensional integral manifolds. Does $X_t$ belong to a 2-parameter family?

(For $F_4$, one expects a 7-parameter family of surfaces with $H_{tr}^2$ Hodge numbers $(6, 14, 6)$!)

To determine which deformations of $X_t$ “preserve $G_2$”, it may be necessary to “see” the cubic Hodge tensor geometrically: we need $3 \in CH^3(X_t \times X_t \times X_t)$ inducing an “octonionic cross-product” on $H_{tr}^2(X_t)$. 
– Thank You –