Boundary strata and adjoint varieties[1]

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[1] report on recent work with C. Robles, based in part on earlier work with G. Pearlstein as well as P. Griffiths and M. Green.

Motivating principle: use Representation Theory to classify what is possible for VHS with given "symmetries"; use that in turn to decide what is geometrically possible or expected:

$$\text{RT} \longrightarrow \text{HT} \longrightarrow \text{moduli}$$

$\mathbb{Q}$-algebraic group $\rightarrow$ Hodge tensors $\rightarrow$ algebraic cycles?

GGK/Patrikis: poss. MTG

flag/Schubert varieties $\xrightarrow{\text{Robles}}$ maximal VHS $\rightarrow$ geom. realization?

nilpotent cones $\rightarrow$ bdry. components (LMHS) $\rightarrow$ degenerations

K-Pearlstein (appr. to Torelli?)

geom. of flag var. $\check{D}$ $\rightarrow$ diff'l. inv. of VHS $\rightarrow$ geom. realization?

(2nd FF)

smooth reps. of classes in $H^*(\check{D},\mathbb{Z})$ $\dashleftarrow$ enhanced SL_2-orbits

§1. Construction of Mumford-Tate domains

- V = vector space over $\mathbb{Q}$
- $Q : V \times V \to \mathbb{Q}$ nondegenerate symmetric bilinear form
- $\varphi : S^1 \to Aut(V_{\mathbb{R}}, Q)$, where
  $Q(v, \varphi(\sqrt{-1})\bar{v}) > 0 \; \forall v \in V_{\mathbb{C}} \backslash \{0\}$ (weight 0 PHS)
- $G \leq Aut(V, Q)$: $\mathbb{Q}$-algebraic closure of $\varphi(S^1)$
  G = subgroup fixing HTs pointwise (Chevalley's thm.)
  Mumford-Tate (or Hodge) group
- $D := G(\mathbb{R}).\varphi \cong G(\mathbb{R})/H \underset{\text{open}}{\subset} G(\mathbb{C})/P =: \check{D}$
  Mumford-Tate domain $\subset$ compact dual

We think of M-T domains as parametrizing (a connected component of) all HS on V polarized by Q, with the same Hodge numbers as φ, whose HTs include the fixed tensors of G. We shall loosely speak of (V, Q, φ) as a "Hodge representation" of G.

Problem [GGK]: How do we arrange for the M-T group to be a given (simple, adjoint) $\mathbb{Q}$-algebraic group G?

One way is to take $V = \mathfrak{g}$. We need the crucial assumption that $G(\mathbb{R})$ contains a compact maximal torus T.

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{k}^{\perp}$ be a Cartan decomposition with $\mathfrak{t} \subseteq \mathfrak{k}$ and involution θ. Write $\Delta := \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \Delta_c \cup \Delta_n$, and $\mathcal{R} \leq \Lambda := X^*(T_{\mathbb{C}})$ for the lattice it generates.

- θ Lie-alg. homom. $\implies$ $\exists$ homom. $\mathcal{R} \to \mathbb{Z}$ sending $\Delta_c \to 2\mathbb{Z}$, $\Delta_n \to 2\mathbb{Z}+1$.
- G adjoint $\implies$ $\mathcal{R} = \Lambda$ $\implies$ this homom. is induced by a grading element $\mathtt{E} \in \sqrt{-1}\mathfrak{t}$.
- Set $\varphi(z) := e^{2\log(z)\mathtt{E}}$, so that $(\mathrm{Ad} \circ \varphi)(\sqrt{-1}) = \theta$.
- Since $-B$ is > 0 on $\mathfrak{k}$ and < 0 on $\mathfrak{k}^{\perp}$, $(\mathfrak{g}, \mathrm{Ad} \circ \varphi, -B)$ is a PHS (of weight 0).

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The Hodge decomposition takes the form $\mathfrak{g}_{\mathbb{C}} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}^j$, where

$$\mathfrak{g}^j := \mathfrak{g}_{\varphi}^{j,-j} = \begin{cases} \oplus_{\delta\in\Delta:\, E(\delta)=j}\mathfrak{g}_\delta, & j \neq 0 \\ \left(\oplus_{\delta\in\Delta:\, E(\delta)=0}\mathfrak{g}_\delta\right) \oplus \mathfrak{t}_{\mathbb{C}}, & j = 0 \end{cases};$$

write $h^j := \dim_{\mathbb{C}} \mathfrak{g}^j$ for the Hodge numbers.

We also claim that the M-T group of $\mathrm{Ad}\circ\varphi$ is G. Why?

Let $M \leq G$ be (equivalently)

(a) the smallest $\mathbb{Q}$-algebraic group such that $\mathrm{Ad}\circ g\varphi g^{-1}$ factors thru $M(\mathbb{R})$ $(\forall g \in G(\mathbb{R}))$

(b) the M-T group of the family $\{\mathrm{Ad}\circ g\varphi g^{-1}\}_{g\in G(\mathbb{R})}$ of polarized Hodge structures

(c) the M-T group of $\mathrm{Ad}\circ g_0\varphi g_0^{-1}$ for all $g_0 \in G(\mathbb{R})$ in the complement of a meager set

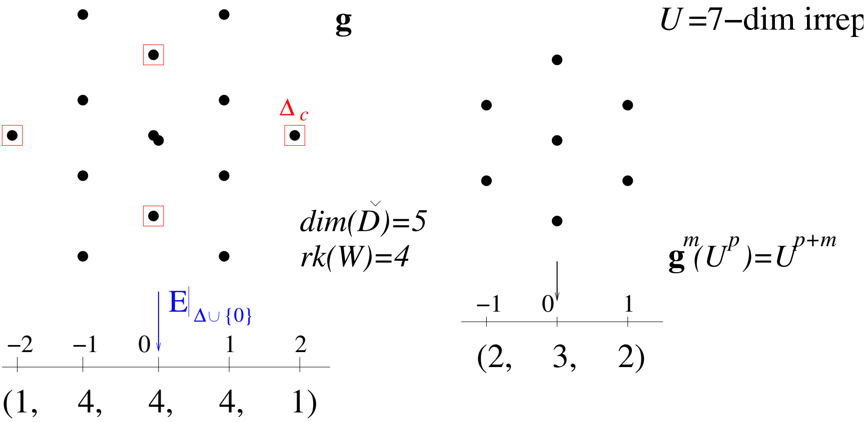
Since φ is "sufficiently general", we may take $g_0 = 1$ in (c).

By (a), $M \trianglelefteq G$; so G simple $\implies M = G$.

We remark that

- $\dim_{\mathbb{C}}(\check{D}) = \sum_{j>0} h^j$
- $\mathrm{rank}(\mathcal{W}) = h^{-1}$ (horizontal distribution)

Example: $(G = G_2)$



Generalization to fundamental adjoint varieties

Recall: upon fixing $\Delta^+ \subset \Delta$, we have

- α_i simple roots $\underset{\delta_i^j}{\longleftrightarrow}$ $\mathtt{S}^j$ simple grading elements
- ω_i fundamental weights $\underset{\delta_i^j}{\longleftrightarrow}$ $\underset{(H_j \in [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}],\, \alpha_i(H^i)=2)}{H_j}$ simple coroots

Let $\mu = \sum \mu^i \omega_i$ be dominant ($\mu_i \geq 0$), assume $U = V^\mu$ real. Consider the assoc. parabolic subgroup $P \geq B \geq T_{\mathbb{C}}$, where

- $\Delta(B) = \Delta^+$
- $I(\mathfrak{p}) := \{i \mid -\alpha_i \notin \Delta(\mathfrak{p})\} = \{i \mid \mu^i \neq 0\}$

and the assoc. grading element $\mathtt{E} := \sum_{i \in I} \mathtt{S}^i$. Put $m := \mathtt{E}(\mu)$.

- $U = U^{-m} \oplus \cdots \oplus U^m$ is the grading induced by $\mathtt{E}$
- $U^m = U_\mu =$ highest weight line
- the $G(\mathbb{C})$-orbit of $[U_\mu] \in \mathbb{P}U$ gives a homogeneous embedding of $G(\mathbb{C})/P$, minimal if $\mu^i \in \{0, 1\}$ $(\forall i)$.

The adjoint case is $U = \mathfrak{g}$. Root/weight computations show:

- $m = 2$, so $\{h^j\} = \{1, *, *, *, 1\}$
- unless G is of type E_8, there exists a level 2 faithful Hodge representation (like $(2, 3, 2)$ in the G_2 example above)
- unless G is of type C, the adjoint variety $\check{D} = G(\mathbb{C})/P \hookrightarrow \mathbb{P}\mathfrak{g}$ is minimally embedded
- unless G is of type A or C, the adjoint representation is fundamental ($\mathfrak{g} = V^{\omega_k}$), and the corresponding $\{\check{D}\}$ are the fundamental adjoint varieties.

Why study the adjoint varieties as Hodge-theoretic classifying spaces?
One reason: they are the "simplest" G/P with nontrivial IPR, in the sense of being precisely the cases where $\mathcal{W}$ is a contact distribution.

§2. Schubert VHS and classical subdomains

Let $D \subset \check{D}$ be a M-T domain with base point $F^\bullet = F^\bullet_\varphi \in D$, $\mathfrak{g} = \oplus \mathfrak{g}^j$ the corresponding Hodge decomposition, and T, Δ as before. Write $P \geq B \geq T_{\mathbb{C}}$ for the parabolic fixing $F^\bullet$, so that we have:

- $\Delta(B) =: \Delta^+ = \Delta(F^1\mathfrak{g}) \cup \Delta^+(\mathfrak{g}^0)$
- $\Delta(P) = \Delta(F^0\mathfrak{g})$.

Put $W^P := \{w \in W |\, w\Delta^+ \supset \Delta^+(\mathfrak{g}^0)\}$, and note that

- $w \in W^P \implies \Delta_w := \Delta^- \cap w\Delta^+$ is closed in Δ
- $\check{D} = \coprod_{w \in W^P} C_w := \coprod_{w \in W^P} Bw^{-1}.F^\bullet$.

The Schubert varieties $X_w := \overline{wC_w}^{\text{Zar.}} \subset \check{D}$ satisfy

- $\dim_{\mathbb{C}} X_w = |\Delta_w| = \ell(w)$
- $T_{F^\bullet} X_w = \mathfrak{n}_w := \oplus_{\alpha \in \Delta_w} \mathfrak{g}_\alpha$
- X_w Schubert VHS (i.e. horizontal) $\iff$ $\mathfrak{n}_w \subset \mathfrak{g}^{-1}$ $(\implies \mathfrak{n}_w$ abelian$)$

Theorem (Robles)

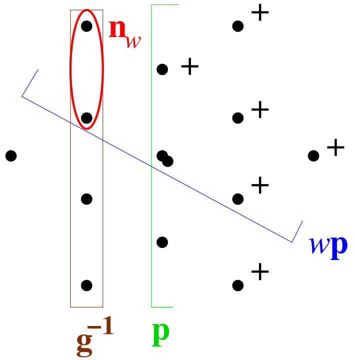
$$\max\dim(\mathrm{IVHS}) = \max\dim(\mathrm{SVHS}) = \max\left\{ |\Delta_w| \,\middle|\, w \in W^P,\ \Delta_w \subset \Delta(\mathfrak{g}^{-1}) \right\}$$

Example : $(G = G_2)$

There exists only one X_w of dimension 2, and it is an SVHS (false for higher dim.)

$\Downarrow$

the maximal integral manifold of $\mathcal{W}$ has dimension 2



Is this X_w a M-T subdomain? If so, it would be a (smooth, Hermitian) homogeneous $G'(\mathbb{R})$-orbit, with $\mathfrak{g} \supsetneq \mathfrak{g}' =$ Lie algebra closure of $\mathfrak{n}_w \oplus \overline{\mathfrak{n}_w}$. But this closure is all of $\mathfrak{g}$. NO!

More generally, **what is the relationship between SVHS and horizontal** ($\implies$ Hermitian) **M-T domains?**

For any subdiagram $\mathscr{D}' \subset \mathscr{D}$ of the Dynkin diagram of $\mathfrak{g}$, have

- $\mathfrak{g}' \subset \mathfrak{g}$ subalg. gen. by the root spaces $\{\mathfrak{g}_\alpha \,|\, \alpha \in \mathscr{D}'\}$
- $X(\mathscr{D}') := G'(\mathbb{C}).F^\bullet \subset \check{D}$ smooth Schubert variety

$X(\mathscr{D}')$ is horizontal iff $\mathfrak{g}' \subset \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1$, in which case it is (the compact dual of) a homogeneously embedded Hermitian symmetric domain.

Theorem (K-R)

Let $X \subset \check{D}$ be a SVHS. Then

$$X \text{ smooth} \iff X = \prod_i X(\mathscr{D}_i) \text{ (homog. emb. HSD).}$$

It is instructive to compare this with another recent result:

Theorem (Friedman-Laza)

- $X \subset \check{D}$ a smooth "VHS" (horizontal subvariety)
- Y a (nonempty) connected component of $X \cap D$ with strongly quasi-projective image in $\Gamma \backslash D$

$\implies$ Y is a (Hermitian) M-T subdomain.

On the other hand, with an arithmetic assumption on $\check{D}$, the [K-R] result has the following

Corollary

- $X \subset \check{D}$ a smooth Schubert VHS
- Y a (nonempty) connected component of $X \cap D$

$\implies$ Y is a translate of a (Hermitian) M-T subdomain.

The converse of the Corollary is false: there are plenty of non-Schubert, horizontal Hermitian M-T subdomains, and we will construct maximal integral ones later.

§3. Lines on $\check{D}$ and a differential invariant of VHS

The Corollary suggests that there might be lots of singular SVHS, in view of the G_2 example. A systematic construction of Schubert varieties is given by incidence correspondences:

$$
\begin{tikzcd}
 & G(\mathbb{C})/\{P \cap Q\} \arrow[dl] \arrow[dr] & & \\
G(\mathbb{C})/Q & & G(\mathbb{C})/P \arrow[r, equal] & \check{D} \\
\Sigma \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, shift right, "\mathrm{X}"'] & & \mathrm{X}(\Sigma) \arrow[u, phantom, sloped, "\subseteq"] \arrow[ll, shift right, "\mathrm{X}^{-1}"'] &
\end{tikzcd}
$$

where $P, Q \geq B$. Note that:

- $\mathrm{X}, \mathrm{X}^{-1}$ preserve Schubert varieties
- a point is a Schubert variety (e.g. $P/P = F^{\bullet}$)
- $\mathrm{X}(Q/Q) = X(\mathscr{D}')$, where $\mathscr{D}' = \mathscr{D} \setminus (I(\mathfrak{q}) \setminus I(\mathfrak{p}))$

Case of P maximal

- P maximal $\implies I(\mathfrak{p}) = \{k\}$. If $I(\mathfrak{q})$ contains the nodes adjacent to $\{k\}$, then $X(\mathscr{D}')$ is a $\mathbb{P}^1$ thru $F^\bullet$.

In this case:

- $X_0 := \mathrm{X}(\mathrm{X}^{-1}(F^\bullet))$ is a Schubert variety consisting of all $\mathbb{P}^1$'s thru $F^\bullet$ on $\check{D}$ (in its minimal embedding).

Specializing to the case where $\check{D}(\subset \mathbb{P}\mathfrak{g})$ is a fundamental adjoint variety, let

- $\mathcal{C}_0 :=$ the $G^0(\mathbb{C})$-orbit of the highest weight line in $\mathbb{P}\mathfrak{g}^{-1}$.

Then

- $\mathcal{C}_0 \cong G^0(\mathbb{C})/\{P \cap G^0(\mathbb{C})\}$ is a homogeneous Legendrian variety
- $X_0 \cong \mathrm{Cone}(\mathcal{C}_0)$ is a singular Schubert VHS

Some data for the fundamental adjoint varieties $\check{D}$ and their associated "subadjoint" varieties $\mathcal{C}_0$ of lines through a point:

$\mathfrak{g}_{\mathbb{C}}$	$\check{D}$	$\mathfrak{g}_{\mathbb{C}}^{0,ss}$	$\mathcal{C}_0$
$\mathfrak{so}(n)$	$OG(2,\mathbb{C}^n)$	$\mathfrak{so}(n-4)\oplus\mathfrak{sl}(2)$	$\mathbb{P}^1\times\mathcal{Q}^{n-6}$
$\mathfrak{e}_6$	E_6/P_2	$\mathfrak{sl}(6)$	$Gr(3,\mathbb{C}^6)$
$\mathfrak{e}_7$	E_7/P_1	$\mathfrak{so}(12)$	$\mathcal{S}_6$
$\mathfrak{e}_8$	E_8/P_8	$\mathfrak{e}_7$	E_7/P_7
$\mathfrak{f}_4$	F_4/P_1	$\mathfrak{sp}(6)$	$LG(3,\mathbb{C}^6)$
$\mathfrak{g}_2$	G_2/P_2	$\mathfrak{sl}(2)$	$\nu_3(\mathbb{P}^1)$

We now relate these varieties of lines to the Griffiths-Yukawa kernel. Let $\mathcal{V} = \oplus_{j=0}^{n} \mathcal{V}^{n-j,j}$ be a VHS over $\mathcal{S}$, with associated period map $\Phi : \mathcal{S} \to \Gamma \backslash D$ ($D =$ M-T domain). Denote by $\mathcal{D}$ a holomorphic differential operator on $U \subset \mathcal{S}$ of order n.

The composition

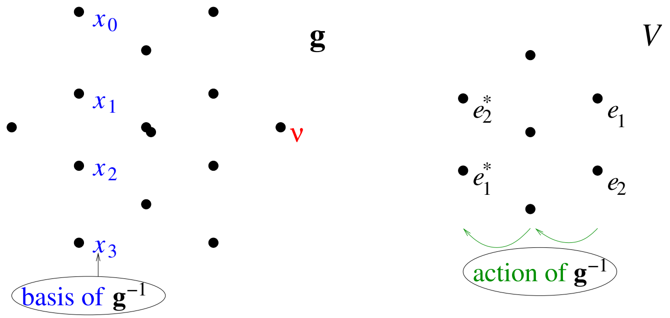
$$\mathcal{O}_U(\mathcal{V}^{n,0}) \hookrightarrow \mathcal{O}_U(\mathcal{V}) \xrightarrow{\mathcal{D}} \mathcal{O}_U(\mathcal{V}) \twoheadrightarrow \mathcal{O}_U(\mathcal{V}^{0,n})$$

depends only on $\sigma(\mathcal{D})$, giving rise to the G-Y coupling

$$\begin{array}{ccccc} \mathrm{Sym}^n T_s\mathcal{S} & \to & \mathrm{Hom}(V_s^{n,0}, V_s^{0,n}) & = & (V_s^{0,n})^{\otimes 2} \\ d\Phi \downarrow & \nearrow & \uparrow (*) & & \\ \mathrm{Sym}^n \mathfrak{g}^{-1} & \underset{(\cdot)^n}{\leftarrow} & \mathfrak{g}^{-1} & & \end{array}$$

Write $\mathcal{Y} \subset \mathbb{P}\mathfrak{g}^{-1}$ for the kernel of $(*)$ (at $F^\bullet \in D$).

Example : $(G = G_2)$ $V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ 7-diml irrep



Given $\xi := \sum \xi_i x_i \in \mathfrak{g}^{-1}$, one computes

$$ {}_{e^*}[\xi^2]_e = \begin{pmatrix} -2\xi_1\xi_2 + 2\xi_2^2 & \xi_1\xi_2 - \xi_0\xi_3 \\ \xi_1\xi_2 - \xi_0\xi_3 & -2\xi_0\xi_2 + 2\xi_1^2 \end{pmatrix} $$

whose vanishing defines the twisted cubic $\nu_3(\mathbb{P}^1) \subset \mathbb{P}\mathfrak{g}^{-1}$.

This example is generalized to E_6, E_7, F_4 by the

Theorem (K-R)

For $\check{D} = G(\mathbb{C})/P$ a fundamental adjoint variety, and V a level 2 Hodge representation such that $V^{2,0}$ is a faithful representation of $\mathfrak{g}^0$, we have $\mathcal{Y} = \mathcal{C}_0$. (Much more generally, $\mathcal{Y}$ contains the horizontal lines through $F^\bullet$.)

Sketch: Given $[\xi] \in \mathcal{Y}$, $\xi^2(u) = 0\ \forall u \in V^{2,0}$. Fix $v \in \mathfrak{g}^2 \setminus \{0\}$, so $\mathrm{ad}_\xi^2 v \in \mathfrak{g}^0$. Then $(\mathrm{ad}_\xi^2 v)u = v\xi\xi u = 0 \overset{\text{faithful}}{\Longrightarrow} \mathrm{ad}_\xi^2 v = 0 \Longrightarrow$ second fundamental form vanishes at $\xi \implies [\xi] \in \mathcal{C}_0$. $\square$

When the conclusion of the Theorem holds,

- $\mathcal{Y} = \ker(\text{G-Y})$ gives Hodge-theoretic meaning to $\mathcal{C}_0$
- $\mathcal{C}_0 \cong G^0(\mathbb{C})/\cdots$ gives a homogeneous description of $\mathcal{Y}$
- $\mathrm{ad}_\xi^2 v = 0$ produces explicit projective homogeneous equations for both.

§4. $G(\mathbb{R})$-orbits in $\check{D}$ and asymptotics of VHS

Given the input:

- $D \subset \check{D}$ M-T domain (parametrizing wt. 0 HS on V)
- $\Gamma \leq G(\mathbb{Q})$ neat arithmetic
- $\sigma^\circ \subset \sigma = \mathbb{Q}_{\geq 0}\langle N_1, \ldots, N_m \rangle \subset \mathfrak{g}_\mathbb{Q}$ abelian nilpotent

we define (and assume nonempty):

- $\tilde{B}(\sigma) := \left\{ F^\bullet \in \check{D} \,\middle|\, \begin{array}{l} e^{\sum t_i N_i} F^\bullet \in D \quad \text{for } Im(\tau_i) \gg 0 \\ N_i F^\bullet \subset F^{\bullet-1} \end{array} \right\}$

which parametrizes LMHS $(F^\bullet, W(\sigma)_\bullet)$,[2]

- $B(\sigma) := e^{\mathbb{C}\sigma} \backslash \tilde{B}(\sigma) =$ **boundary component** assoc. to σ

which parametrizes σ-nilpotent orbits $(\sigma, e^{\mathbb{C}\sigma} F^\bullet)$, and

- $\bar{B}(\sigma) = \Gamma_\sigma \backslash B(\sigma)$, where $\Gamma_\sigma := \mathrm{stab}_\Gamma(\sigma)$.

One may "partially compactify" $\Gamma \backslash D$ by $\bar{B}(\sigma)$s (log manifold).

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[2] $N(W(\sigma)_\bullet) \subset W(\sigma)_{\bullet-2}$ and $N^k : Gr_k^{W(\sigma)} \xrightarrow{\cong} Gr_{-k}^{W(\sigma)}$ $(\forall N \in \sigma^\circ)$.

Structure of $B(\sigma)$

Write

- $M_\sigma = \exp\{\mathrm{im}(\sum N_i) \cap (\cap \ker(N_i))\}$
- $Z(\sigma) = Z_0(\sigma) \cdot M_\sigma$ for the centralizer of σ in G
- $G_\sigma \leq Z_0(\sigma)$ for the M-T group of generic $Gr^W(F^\bullet, W(\sigma)_\bullet)$

Then we have

- fibration $\begin{tikzcd} B(\sigma) \arrow[r, two heads] & D(\sigma) \arrow[l, bend left, dashed, "\mathbb{Q}\text{-split}(F_0^\bullet, W(\sigma)_\bullet)"] \end{tikzcd} =$ M-T domain of generic $Gr^W(F^\bullet, W(\sigma)_\bullet)$
- $B(\sigma) = \{G_\sigma^{ss}(\mathbb{R}) \ltimes M_\sigma(\mathbb{C})\}.F_0^\bullet$ and

  $D(\sigma) = G_\sigma^{ss}(\mathbb{R}).Gr^W F_0^\bullet$
- naive limit map

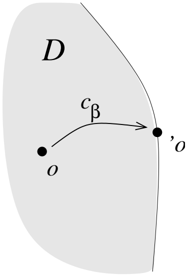
$$\begin{array}{rccc} \Phi_\infty^\sigma : & B(\sigma) & \to & \partial D \subset \check{D} \\ & F^\bullet & \mapsto & \lim\limits_{Im(\tau)\to\infty} e^{\tau N} F^\bullet \quad (\text{any } N \in \sigma^\circ). \end{array}$$

[K-R] contains a general prescription for using a set $\mathfrak{B} = \{\beta_1, \ldots, \beta_s\} \subset \Delta(\mathfrak{g}^1)$ of strongly orthogonal roots to explicitly construct $\mathbb{Q}$-split $(\sigma, F_0^\bullet)$. The motivation is to parametrize $G(\mathbb{R})$-orbits in ∂D in the image of Φ_∞^σ. I will discuss only $s = 1$. Fix a base point $o\,(\longleftrightarrow F^\bullet)$ in D.

Let $\beta \in \Delta(\mathfrak{g}^1)$, with associated $\mathfrak{sl}_2^\beta = \langle N, Y, N^+\rangle$ ($N \in \mathfrak{g}_{-\beta}$). Apply the Cayley transform $\mathbf{c}_\beta = \mathrm{Ad}\left(e^{\frac{\pi}{4}(X_{-\beta} - X_\beta)}\right)$ to

- $\mathfrak{t}_\mathbb{C} \rightsquigarrow \mathfrak{h}$
- $\mathfrak{g}_\alpha \rightsquigarrow {}'\mathfrak{g}_\alpha$
- $\mathtt{E} \rightsquigarrow {}'\mathtt{E}$
- $o \rightsquigarrow {}'o$

$(F^\bullet \rightsquigarrow {}'F^\bullet)$



Then

- ${}'F^{\bullet} \in \tilde{B}(N)$, and $\mathfrak{g}_N^{ss} = \ker\{\beta|_{\mathfrak{h}}\} \oplus \bigoplus_{\alpha \perp\!\!\!\perp \beta} {}'\mathfrak{g}_{\alpha}$
- ${}'\mathtt{E}$, Y give a (Deligne) bigrading $\mathfrak{g}^{p,q} = \mathfrak{g}^{p}_{{}'\mathtt{E}} \cap \mathfrak{g}_{p+q}^{Y}$ of $\mathfrak{g}_{\mathbb{C}}$

whose dimensions $h^{p,q}$ are the Hodge-Deligne numbers of the (limit) MHS $({}'F^{\bullet}, W(N)_{\bullet})$ associated to ${}'o$.

Remark : We can use this to construct non-Schubert M-T subdomains. Define the "enhanced SL_2-orbit"

$$X(N) := \overline{e^{\mathbb{C}N} G_N^{ss}.{}'o}^{\mathrm{Zar}} = G_N^{ss} \times SL_2^{\beta}.{}'o \subset \check{D};$$

then (with an arithmetic assumption on o)

- $Y(N) := X(N) \cap D$ is a M-T domain
- $X(N) = \check{D}(N) \times \mathbb{P}^1 \supset D(N) \times \mathfrak{H} = Y(N)$
- If $\mathtt{E}(\alpha) \in \{-1, 0, 1\}$ $\forall \alpha \perp\!\!\!\perp \beta$, then $Y(N)$ is a HSD.
- If $\check{D} = G(\mathbb{C})/P$ (P maximal) and $\dim X(N) \geq 2$, then $X(N)$ is not Schubert.

§5. "Minimal" boundary of adjoint varieties

Let $\check{D}$ be a fundamental adjoint variety (note $\mathtt{E} = \mathtt{S}^i$).

Proposition (K-R)

There is a unique codimension-1 $G(\mathbb{R})$-orbit in ∂D.

Sketch:

- [K-P] $\mathbb{R}$-codim. of orbit $\ni {}'o$ is given by $\sum_{p,q>0} h^{p,q}$;
- [KP] codim.-1 orbits are of the form $G(\mathbb{R}).\mathbf{c}_\beta o$, $\beta \in \Delta(\mathfrak{g}^1)$.

Acting by $W(\mathfrak{g}^0)$, wma $(\beta, \alpha_j) \leq 0 \; \forall j \neq i$, i.e. $\alpha_j(H^\beta) \leq 0$.

In the bigrading defined by $\mathbf{c}_\beta$, $\mathfrak{g}^{1,1} \supset {}'\mathfrak{g}^1 \supset {}'\mathfrak{g}^\beta$

$\implies \beta = \alpha_i + \sum_{j \neq i} m_j \alpha_j \;\; (m_j \geq 0)$

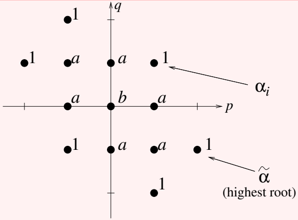
$\implies p(\alpha_i) + q(\alpha_i) = \alpha_i(H^\beta) = \beta(H^\beta) - \sum_{j \neq i} m_j \alpha_j(H^\beta) \geq 2$

$\implies q(\alpha_i) \geq 1 \implies$ codim > 1 unless $\beta = \alpha_i$. $\square$

Let $'o\,(=\mathbf{c}_{\alpha_i}o)$ belong to this real codimension-1 orbit, with associated MHS $('F^\bullet, W(N)_\bullet)$ and bigrading $\mathfrak{g}^{p,q}$.

Proposition (K-R)

The $h^{p,q}=\dim_{\mathbb{C}}\mathfrak{g}^{p,q}$ are



(e.g. for G_2, $a=b=1$
for F_4, $a=6$ and $b=10$)

Sketch:

We know $\dim\mathfrak{g}^2=1$, $\mathfrak{g}^{>2}=\{0\}$, $\dim\mathfrak{g}_2=1$, $\mathfrak{g}_{>2}=\{0\}$.
Now $\alpha_i,\tilde{\alpha}$ are non-short roots, and

- $H^i\in[\mathfrak{g}_{\alpha_i},\mathfrak{g}_{-\alpha_i}]$ s.t. $\alpha_i(H^i)=2$
- $\mathtt{S}^{\mathtt{i}}\in[\mathfrak{g}_{\alpha},\mathfrak{g}_{-\alpha}]$ s.t. $\tilde{\alpha}(\mathtt{S}^{\mathtt{i}})=2$.

$\implies \exists w\in W$ sending $H^i\mapsto -\mathtt{S}^{\mathtt{i}}$

$\implies w(\mathfrak{g}^{p,q})=w(\mathfrak{g}^p\cap\mathfrak{g}_{p+q})=\mathfrak{g}_{-p}\cap\mathfrak{g}^{-(p+q)}=\mathfrak{g}^{-p-q,q}$. $\square$

Note that w identifies the (faithful) representations of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ and $\mathfrak{g}^0$ on $\mathfrak{g}^{-1}$. Moreover, $D(N)\,(\subset \mathbb{P}\mathfrak{g}_1)$ is the M-T domain for the Hodge representation of G_0 on $\mathfrak{g}_1$, which leads to:

Theorem (K-R)

(a) $\check{D}(N) \subset \mathbb{P}\mathfrak{g}_1$ is $\cong \mathcal{C}_0 \subset \mathbb{P}\mathfrak{g}^{-1}$.

(b) $X(N) \cong \mathbb{P}^1 \times \mathcal{C}_0$ (*cylinder* on $\mathcal{C}_0$)

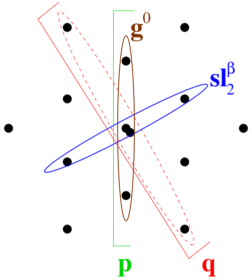
(c) $\bar{B}(N) \twoheadrightarrow \Gamma_N \backslash D(N)$ is a family of intermediate Jacobians associated to a VHS (with Hodge numbers $(1, a, a, 1)$) over a Shimura variety.

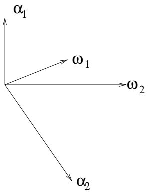
(d) Over $D(N)$, these VHS recover the Friedman-Laza list of maximal weight 3 Hermitian VHS of CY type.

Applications? Automorphic cohomology; geometric realizations; cohomology of $\check{D}$.

$H^*(\check{D}, \mathbb{Z})$ is generated by Schubert vareities, and the "horizontal" part (invariant characteristic cohomology) by Schubert VHS. Do the subadjoint cylinder classes $[X(N)]$ yield smooth representatives of the subadjoint cone classes $[X_w]$?

Example :
$(G = G_2)$



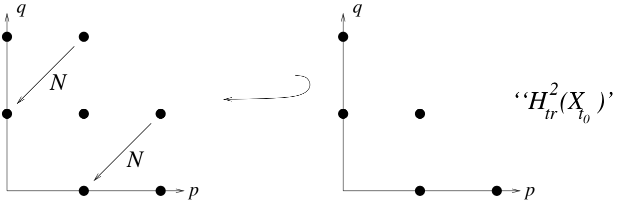


$$X_w \cong X(\underbrace{G^0.Q/Q}_{\Sigma}) \;,\quad X(N) \cong (\underbrace{SL_2^\beta.Q/Q}_{\Sigma(N)})$$

$$(\omega_1, \omega_1) = 2(\omega_1, \alpha_1) \quad \Longrightarrow \quad [\Sigma(N)] = 2[\Sigma]$$
$$\Longrightarrow \quad [X(N)] = 2[X_w].$$

§6. A (partial) geometric realization

We look for degenerations of varieties predicted by the "codim. 1" boundary components of adjoint domains.
For G_2, this should take the form of a 1-parameter family of surfaces $\{X_t\}$ with H^2_{tr} Hodge numbers $(2, 3, 2)$, M-T group G_2, and LMHS of the form



where N is the monodromy logarithm and bullets denote 1-dimensional spaces. In fact, just such a family has been constructed by N. Katz using elliptic fibrations; the M-T group is determined by a moment computation using elliptic convolution over finite fields. We shall describe a special case.

Begin with the rational elliptic surface

$$\mathcal{E} \to \mathbb{P}^1_z : \ y^2 = x(1-x)(x-z^2)$$

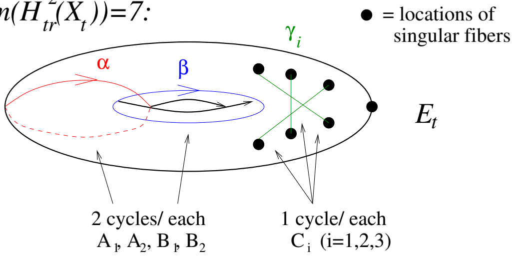
with singular fibers $(2\,I_4,\ 2\,I_2)$ at $z = -1, 0, 1, \infty$.
For any $t \neq 0, \frac{\pm 2}{3\sqrt{3}}, \infty$, base change by

$$E_t \to \mathbb{P}^1_z : \ w^2 = tz(z-1)(z+1) + t^2$$

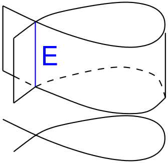
to obtain an elliptic surface $X_t \to E_t$ with 7 singular fibers,

$$\Omega^2(X_t) = \mathbb{C}\langle \omega_1(t), \omega_2(t) \rangle = \mathbb{C}\left\langle \tfrac{dx}{y} \wedge \tfrac{dz}{w}, \tfrac{dx}{y} \wedge \tfrac{zdz}{w} \right\rangle, \text{ and}$$

$dim(H^2_{tr}(X_t))$=7:



Degenerating X_t as $t \to t_0 = \frac{2}{3\sqrt{3}}$ yields



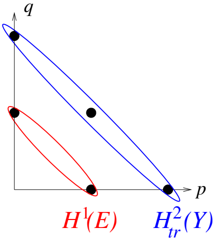
$$\begin{tikzcd} X_{t_0} \arrow[d] & Y \arrow[l, "\text{desing.}"'] \\ E_{t_0} & \end{tikzcd}$$

K3 surface
Picard rank 19

with

$$\mathsf{E} = \{y^2 = x(1-x)(x - \tfrac{1}{3})\}.$$

The part of $H^2(X_{t_0})$ not coming from the 19 algebraic classes on Y indeed takes the form



In fact, we can "determine" the limiting period in

$$\bar{B}(N) \to \Gamma\backslash\mathfrak{H}.$$

- Since $G_N \cong SL_2$ and $j(\mathrm{E}) \notin \mathbb{Z}$,

  $$H^2_{tr}(Y) \cong Sym^2 H^1(\mathrm{E})$$

  and the point in the base is determined by the (non-CM) Hodge structure $H^1(\mathrm{E})$.

- The point in the fiber

  $$\mathbb{C}^2/\mathbb{Z}\langle \tbinom{1}{0}, \tbinom{2\tau/3}{1}, \tbinom{\tau^2/3}{2\tau}, \tbinom{0}{3\tau^2} \rangle \cong J(Sym^3 H^1(\mathrm{E})) \subset J(H^2_{tr}(Y)^\vee \otimes H^1(E))$$

  is given by $\int_{B_1} \omega$, $\int_{B_2} \omega$ $(\omega \in \Omega^2(Y))$.

The image of the period map into $\Gamma\backslash D$ is contained (at least locally) in 2-dimensional integral manifolds. Does X_t belong to a 2-parameter family?

(For F_4, one expects a 7-parameter family of surfaces with H^2_{tr} Hodge numbers $(6, 14, 6)$!)

To determine which deformations of X_t "preserve G_2", it may be necessary to "see" the cubic Hodge tensor geometrically: we need $\mathfrak{Z} \in CH^3(X_t \times X_t \times X_t)$ inducing an "octonionic cross-product" on $H^2_{tr}(X_t)$.

– Thank You –