Homotopy obstructions for projective modules

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Abstract: The theory of vector bundles on compact hausdorff spaces X, guided the research on projective modules over noetherian commutative rings A. There has been a steady stream of results on projective modules over A, that were formulated by imitating existing results on vector bundles on X. The first part of this talk would be a review of this aspects of results on projective modules, leading up to some results on splitting projective A-modules P, as direct sum $P \cong Q \oplus A$. Our main interest in this talk is to define an obstruction class $\varepsilon(P)$ in a suitable obstruction set (preferably a group), to be denoted by $\pi_0(\mathcal{L}O(P))$. Under suitable smoothness and other conditions, we prove that

$$\varepsilon(P) \quad \text{is trivial} \iff P \cong Q \oplus A$$

Under similar conditions, we prove $\pi_0(\mathcal{L}O(P))$ has an additive structure, which is associative, commutative and has n unit (a "monoid"). In deed,

$$\mathcal{L}O(P) = \left\{ (I, \omega) : I \subseteq A \text{ is an ideal, and } \omega : P \twoheadrightarrow \frac{I}{I^2} \text{ is a surjective map.} \right\}$$

The two maps $\mathcal{L}O(P) \xleftarrow{T=0} \mathcal{L}O(P \otimes A[T]) \xrightarrow{T=1} \mathcal{L}O(P)$ induce a chain homotopy equivalence on $\mathcal{L}O(P)$, and the set of equivalence classes is defined to be $\pi_0(\mathcal{L}O(P))$. This theory emanates out of some germs of ideas given by Madhav V. Nori (around 1990).