Hodge Theory and Representation Theory

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*Kirk lecture at Washington University, April 14 (2011). Much of the talk based on joint work and correspondence with Mark Green and Matt Kerr*
I. Introduction

- talk based on topics at the interface of Hodge theory and representation theory
- much of the material is classical-fresh impetus from automorphic representation theory
- notable is the work of Carayol at the confluence of

(i) homogeneous complex manifolds (Mumford-Tate domains)
(ii) integral geometry (cycle and correspondence spaces associated to Mumford-Tate domains)
(iii) representation theory
Outline

I. Introduction

II. The classical case

III. Hodge structures and Mumford-Tate groups and domains

IV. Variation of Hodge structure and the structure theorem

V. Hodge representations and the classification theorem

VI. Discrete series - automorphic cohomology
I.2

(iii) representation theory (automorphic and Galois representations)

Today will talk on three particular aspects of Hodge theory and representation theory

(1) monodromy representations
(representations of $\pi_1(S)$)

(2) finite dimensional representations of reductive, $\mathfrak{g}$-algebraic groups (Hodge representations)

(3) infinite-dimensional of real, non-compact semi-simple Lie groups (discrete series and their limits)
I.3

Will only briefly mention the very interesting subject of automorphic representations and automorphic cohomology, partly because of time but mainly because someone more knowledgeable should give that talk.
II. The classical case

- began with elliptic functions (Count Fagnano, Euler, Legendre, and especially Abel)
- arising from mechanics and geometry, there was interest in integrals of algebraic functions

\[ \int r(x, y(x)) \, dx, \quad f(x, y'(x)) = 0 \]

- hyperelliptic integrals

\[ \int \frac{dx}{\sqrt{p(x)}}, \quad y^2 = p(x) \]

For degree \( p(x) = 2, 2 \) evaluate by "elementary functions"
II. 3.

The first non-elementary case is the arc length of the ellipse 
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad c < 0. \]

\[ \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad k = \frac{b}{a} \]

Analogous to using

\[ u = \int \frac{(\sin u, \cos u)}{\sqrt{1-x^2}} \]

\[ = \int \frac{(\sin u, \cos u)}{y} \]

to give the circle parametrically by arc length; the interest was in the properties of the functions \( x(u), y(u) \)
II.3

that do the same for the ellipse—e.g., are there formulas for doubling the length like the one for $\sin(2x)$?

Abel proved the following general result: Let $f(x, y) = 0$ define an algebraic curve

$$E = \quad \circ \quad \left\langle \quad \begin{array}{c}
y^2 = x^3 + ax + b
\end{array} \right. \quad \circ \quad \left\langle \quad \begin{array}{c}
y^2 = x^3 + ax + b
\end{array} \right.$$

Let $g(x, y, t) = 0$ define a family of curves $G_t$ rationally parametrized

Write $E_n G_t = \sum \left( x_i(t), y_i(t) \right)$
Abel's Theorem: For $w = \pi(x, y)dx \mid_{E}$

$$\sum \int_{\omega} x_i'(t), y_i'(t) \omega = \sum A(\log(t - t_2) + R(t))$$

Proof gave: For $\omega = \frac{q(x)dx}{\sqrt{p(x)}}$

$\text{deg } q(x) \leq \text{deg } p(x) - 3 \Rightarrow \text{RHS = constant}$

On $y^2 = x^3 + ax + b$, $w = \frac{dx}{y} \mid_{E}$ set

$$\mu = \int (p'(x), q'(x))$$

Then

$p(u_2 + u_3) = R(p(u_2), p(u_3), p'(u_1), p'(u_3))$

is the functional equation for the Weierstrass $p$-function.
Leads to (here replace $E$ by $X$)

\[ \mathbb{C}/\Lambda \rightarrow X, \quad \lambda \rightarrow (\rho(\lambda), \rho'(\lambda)) \]

- $X = 1$-dimensional compact, complex manifold
- $\pi^*(\omega)$ is holomorphic 1-form on $X$
- $V = H^1(X, \mathcal{O})$
- $V_{\mathcal{C}} = H^0_{\mathcal{C}}(X) \oplus H^{0,1}(X)$, $H^q = \overline{H^{q,0}}$
II. 6

- \( \otimes \mathbb{Q} \otimes V \rightarrow \mathbb{Q} \) cup product

\[
0 = \sum \omega \otimes \omega \Rightarrow \mathbb{Q}(V^{\otimes 0}, V^{\otimes 0}) = 0
\]

\[
0 < \sum \omega \otimes \bar{\omega} \Rightarrow i \mathbb{Q}(V^{\otimes 0}, \overline{V^{\otimes 0}}) > 0
\]

- \( V^{\otimes 0} = \) line in \( \mathbb{C}^2 = \) point in \( \mathbb{P}^1 \)
  \[
  = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \text{Im} \eta > 0
  \]

Data \( (V, \mathbb{Q}, V^{'p,1}) \) is a **polarized Hodge structure** of weight \( n = 1 \) with Hodge numbers \( h^{0,1} = 1 \)

- Can give \( V = V^{\otimes 0} \otimes V^{1,2} \) by

\[
\mathbb{C}^* \rightarrow \text{Aut}(V^{'p,1})
\]

\[
\text{re} \theta \rightarrow \text{re} 2\pi i \theta \quad -2\pi i \theta
\]

\[
\text{re} \theta \rightarrow \text{re} \theta \otimes \text{re} \theta^\text{IR}
\]

\[
- \text{ r gives weight} \quad V^{'p,1} \rightarrow \mathbb{R}^{2\pi i(p-q) \theta}
\]

\[
- e^{2\pi i \theta} \text{ action gives} \quad V^{'p,1} \leftrightarrow e
\]
II 7

- set of elliptic curves $E_z$ is $\{ \text{Im} \tau > 0 \} = \mathcal{H} = \text{SL}_2(\mathbb{R})/ \text{SO}(2)$
- $E_z \cong E_{z'} \iff z' = \frac{ax + b}{cx + d}$
- $(a, b) \in \text{SL}_2(\mathbb{Z}) = \Gamma$
- $w \rightarrow \mathcal{M}_{z, z} \cong \mathbb{P} \setminus \mathcal{H}$

\[ \mathfrak{m} \rightarrow \mathfrak{m}' \rightarrow \mathcal{M}_{z, z} \rightarrow \mathbb{P} \rightarrow \mathbb{H} \]

- $y^2 = x(x-1)(x-t)$ gives for $S = \mathbb{P}^2 \setminus \{0, 1, t\}$
- $S \rightarrow \mathcal{M}_{z, z}$
- classically set $f(t) = \int \frac{dx}{\sqrt{x(x-1)(x-t)}}$

\[ \tau = \int_{\mathcal{M}} \frac{dw}{\sqrt{w^2 + \mathcal{H}}}, \text{ up to}
\]

action of monodromy - image is finite index in $\text{SL}_2(\mathbb{Z})$, determines $f(t)$
Hodge representation

\[ \phi : \text{SL}_2(\mathbb{Q}) \to \text{Aut}(V, \mathbb{Q}) \]

\[ \phi : S^2 \to \text{Aut}(V_{\text{IR}}, \mathbb{Q}) \]

\[ e^{2\pi i \theta} \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

Discrete series. Over \( \mathcal{H} \) we have the Hodge bundle \( V^{1,0} \to \mathcal{H} \). Then

\[ V^{1,0} \cong \text{SL}_2(\mathbb{R}) \times \mathbb{C} \cong \text{SO}(2) \times \mathbb{C} \]

\[ \omega_{\mathcal{H}} \cong (V^{1,0})^{\otimes 2} \text{, sections } s(t) dt \]

\( V^{1,0}, \mathcal{H} \) have metrics

\[ H^0(\mathcal{H}, V^{1,0}) = \left\{ \frac{1}{2} \int_{\mathcal{H}} |f(z)|^2 \text{Im}^2 \right\}^{\mathbb{R}_+} dxdy \]

\[ V^{1,0} = (V^{1,0})^{\otimes 2} \]
II.9

\( w \rightarrow \text{discrete series } H_k, \ k \in \mathbb{Z} \)

irreducible unitary matrix coeff. in \( L^2 \)

- weight picture \( (\text{weights} = \mathbb{Z}, \ \text{roots} = 2\mathbb{Z}) \)

\[ \begin{align*}
-2 & \quad 0 \quad \text{deg root} \\
H_k & \leftrightarrow \text{character } e^{-2\pi i \theta} \\
\theta = -1 & \rightarrow \text{limit of discrete series}
\end{align*} \]

\( \text{tempered, Maass form} \)
III. Hodge structures; Mumford–Tate groups and domains

Notations:
- $V = \mathbb{Q}$-vector space
- $Q : V \otimes V \to \mathbb{Q}$, $Q(v \otimes w) = (-1)^m Q(w \otimes v)$
- $C^* = \text{real Lie group } \mathbb{R}^+ \times S^1 \to \mathbb{C} = \text{re}^{	ext{reim} \theta}$

Definition: Hodge structure is $(V, \hat{\omega})$
- $\hat{\omega} : C^* \to \text{GL}(V_\mathbb{R})$
- $\hat{\omega} |_{\mathbb{R}^+}$ defined over $\mathbb{Q}$

Weight decomposition: $V = \bigoplus V_m$ over $\mathbb{Q}$
where $\hat{\omega}(r) = r^m$ on $V_m$

Hodge decomposition:
- $V \mathbb{C} = \bigoplus V^{p,q}$
- $V^{p,q} = \bigoplus V^{p,q}$
- $V^{p,0} = \{ v \in V_{\mathbb{C}} : \hat{\omega}(z)v = z^p \bar{z}^q v \}$
III.2

Hodge filtration \( F^p = \bigoplus_{p' \geq p} V^{p', q} \)

\( V = V_m \Rightarrow \) pure Hodge structure of weight \( m \) - for these suffices to have

\[ \begin{align*}
\varphi &: S^2 \to GL(V_{\mathbb{R}}) \\
\varphi(e^{2\pi i \theta}) &= e^{2\pi i (p-q) \theta} \\
\varphi(v), v &\in V_{p,q}
\end{align*} \]

Defn: A polarized Hodge structure of weight \( m \) is \((V, Q, \varphi)\) where \((V, \varphi)\) is as above with

\[ \varphi : S^2 \to \text{Aut} (V_{\mathbb{R}}, Q) \]

and

\[ \begin{align*}
Q(V^{p,q}, V^{p', q'}) &= 0 & p' \neq m-p \\
\omega_{p-q} Q(V^{p,q}, V^{p', q'}) &> 0
\end{align*} \]

Hodge–Riemann bilinear relations
Setting $C = \varphi(u)$ (Weil operator) these are the same as
\[
\begin{cases}
Q(F^p, F^{n-p+2}) = 0 \\
Q(v, C\overline{u}) > 0 \quad v \neq 0
\end{cases}
\]
This uses
\[
\begin{cases}
F^p \cap \overline{F^q} = V^{p,q} \\
F^p \oplus \overline{F^{n-p+2}} \cong V_C
\end{cases}
\]

Example: $H^n(X, \mathcal{O})$ where $X$ is a smooth projective variety / $C$.

Def.: When $n = 2p$, $H^{p,q}_f = V \cap V^{p,q}$ are the Hodge classes.

Hodge conjecture: For $V = H^n(X, \mathcal{O})$, Hodge classes are represented by algebraic cycles $\Rightarrow H^n(X, \mathcal{O})$ determines $X$ as a motive.
The set of PHG's \((V, Q, \varphi)\) with given \(h^p,q's\) is a **period domain**

\[
D = \frac{G_{\mathbb{R}}}{H_\varphi}
\]

where \[
\begin{aligned}
G_{\mathbb{R}} &= \text{Aut}(V_{\mathbb{R}}, \mathbb{Q}) \\
H_\varphi &= \text{compact} = Z(\varphi(S^2)), \ H > T
\end{aligned}
\]

**Ex** \(n = 2\), \(D = \mathfrak{h}_g = \text{Sp}(2g, \mathbb{R})/U(\mathbb{R})\)

is **Siegel upper-half-plane** given by \(\{Z : Z = ^tZ, \text{Im} Z > 0\}\), \(\mathfrak{Q} = (0 \ I)\)

\[
Z \leftrightarrow V^{3,0} = \text{span}\left(\begin{pmatrix} Z \\ I \end{pmatrix}\right)
\]

Hodge structures admit operations of linear algebra - \(\oplus\), \(\otimes\), \(\text{Hom}\), .. Also sub-Hodge structures. In polarized case a sub-HS is polarized and we have \(\oplus\) using \(\text{...}\).
Example: \( n = 2, \ h^{1,0} = 2 \) and \( h^{1,1} = 1 \)

\[ D = \text{SO}(4,1)/U(2) \]

non-classical object

\[ \langle 0 \rangle \]

Mumford-Tate groups are the basic symmetry groups of Hodge theory. They encode the \( \mathbb{Q} \)-structure and the circle action.

Definitions:

(i) Given \((V, \tilde{\varphi})\) the Mumford-Tate group \( G_{\tilde{\varphi}} \) is the smallest \( \mathbb{Q} \)-algebraic subgroup of \( \text{GL}(V) \) such that

\[ \tilde{\varphi}(C^\mathbb{Q}) \subset G_{\tilde{\varphi}}(\mathbb{R}) \]

(ii) Given \((V, Q, \varphi)\) the Mumford-Tate group \( G_{\varphi} \) is the smallest \( \mathbb{Q} \)-algebraic subgroup of \( \text{Aut}(V, Q) \) such that \( \varphi(S^2) \subset G_{\varphi}(\mathbb{R}) \)
For \((V, \varphi)\) of pure weight
\[ G_{\varphi} = G_{\text{aut}} \times G_{\varphi} \]

**Basic property**: Set \( T'' = \oplus V \otimes V \)

Then \( H_{\varphi}'' \subset T''' \) is a sub-algebra

\[ G_{\varphi} = \{ \varphi \in \text{Aut}(V, \varphi) ; \varphi \text{ fixes } H_{\varphi}'' \} \]

Here "fixes" is pointwise. Note \( \varphi \in H_{\varphi}'' \).

\[ \text{[\#]} \]

are the \( \mathbb{Q} \)-algebraic equations
that define \( G_{\varphi} \).

Let \( \varphi \in D \). Then the associated
Mumford-Tate domain is \( D_{\varphi} = G_{\varphi, \mathbb{R}} \)-orbit of \( \varphi \).
Then
\[ D_\varphi = \begin{cases} 
\text{set of PHS's } \psi \text{ with } \\
G_\psi \leq G_\varphi \quad (= \text{ for generic } \psi) \\
H^*_{\psi} \geq H^*_{\varphi} \quad (= \text{ for generic } \psi)
\end{cases} \]

Example: \((V, Q, \mathbb{IF} = \mathbb{Q}(\sqrt{-1}))\), \(\dim V = 6\)
Then set of PHS's with \(n=3\),
\(h^3,0 = 1\), \(h^{2,1} = 2\) and an \(\mathbb{IF}\)-action
is Mumford-Tate domain. We can draw the picture of this one. Write

- \(V_{\mathbb{IF}} = V_+ \oplus V_-\), \(Q(V_+, V_+) = 0\)
- \(1h(u, v) = i \cdot Q(u, v)\) \(\forall u, v \in V_+, \mathbb{C}\)
- \(U_{1h} \cong U(2, 1) \otimes \text{Res} (\text{Aut}(V, Q, \mathbb{IF})_{\mathbb{IF}/\mathbb{Q}})\)

Picture:

\[
\begin{array}{cccc}
\vee & \vDash & \vee & V_+ \cdot \mathbb{C} \\
\vDash & \vee & \vee & V_- \cdot \mathbb{C} \\
(3,0) & (2,1) & (1,2) & (0,3)
\end{array}
\]
PHS is determined by top row which is a flag in \( \mathbb{P}^4 \), \( c = \mathbb{P}^2 \). Hodge–Riemann bilinear relations give

\[ \Delta = \text{unit ball}, \quad h(\varphi, \varphi) \neq 0 \]

(think of \( 1z_0^2 - 1z_1^2 - 1z_2^2 > 0 \) in \( \mathbb{P}^2 \))

Then generic \( G_{2,1} = U(2,1) \),

\[ D \subset (\text{incidence}) \subset \mathbb{P}^2 \times \mathbb{P}^2 \]

(\( p, \ell \))
IV. Variation of Hodge structure and the structure theorem

Just as the Hodge structure reflects the structure of algebraic varieties, variations of Hodge structure reflect the structure of families of algebraic varieties (think of algebraic equations with parameters, which is how they usually occur). There are two situations for a family $\mathcal{X} \to S$, $\pi^{-1}(s) = \mathcal{X}_s$.

- **Local**
  $$\begin{array}{c}
  \mathcal{X} \\
  \downarrow \\
  \Delta \mathcal{X} = S
  \end{array}$$

- **Global**
  $$\begin{array}{c}
  \mathcal{O}(1,1,1) \\
  \downarrow \\
  \mathcal{X} = \mathcal{O} \cap \pi^{-2}(D) \\
  \downarrow \\
  \mathcal{S} = \mathcal{S} \cap D
  \end{array}$$
In both cases the critical topological invariant is the monodromy representation

$$
\pi_1(S) \to \text{Aut}(H^n(B_{c_0}, G))
$$

Before we can define the Hodge-theoretic analogue, remark that period domains $D$ have a canonical distribution

$$I \subset TD$$

given by the condition

$$\frac{dF^t_p}{dt} \leq F^t_{p-2}$$

on a curve $t \to F^t_0 \in D$.

Let $D = G_m/H$ and $\Gamma \subset G$ be
a discrete subgroup (usually an arithmetic one)

**Defn:** A variation of Hodge structure (VHS) is given by

\[ \Phi : S \to \Gamma \]  

satisfying local liftability and

\[ \Phi_\ast : TS \to I. \]

The monodromy group is

\[ \Gamma_\Phi = \Phi_\ast (\pi_2(S)) \]

local case: \[ \Gamma_\Phi = \{ T^k \}_{k \in \mathbb{Z}}. \] Then

\[ (T^d - I)^{n+1} = 0 \]
global case - let $G_{\overline{\mathbb{F}}}$ be the Mumford-Tate of $\Phi(s)$ for a generic $s \in S$ - after possibly passing to a finite covering it is the subgroup of $\text{Aut}(V, \mathbb{Q})$

leaving invariant the space of invariant Hodge tensors. Then first

$$\Gamma_{\overline{\mathbb{F}}} \subset G_{\overline{\mathbb{F}}}$$

Theorem: $\Gamma_{\overline{\mathbb{F}}}^{\mathbb{Q}}$ is a semi-simple $\mathbb{Q}$-algebraic subgroup of $\text{Aut}(V, \mathbb{Q})$

In fact, $\Gamma_{\overline{\mathbb{F}}}^{\mathbb{Q}}$ is a normal subgroup of $G_{\overline{\mathbb{F}}}^{\text{Der}}$. Now $G_{\overline{\mathbb{F}}} \sim G_{\mathbb{F}} \times \cdots \times G_{\mathbb{F}}$
Let $D_{\omega} = G_{\omega}(\mathbb{R})$-orbit of $\tilde{H}(s) (= \text{lift of } H(5) \text{ to } D)$. Then

(i) $\Gamma_{\tilde{H}} = \prod_{i=1}^{n} \Gamma_i$, $\Gamma_i \subset \Gamma \cap G_{\omega}$

(ii) $G_{\tilde{H}} = \prod_{i=1}^{n} G_{\omega_i} \times G'$

$\Phi : S \rightarrow \prod_{i=1}^{n} \frac{\Gamma_i \backslash D_i}{\Gamma_i \Theta} \times D' \times \prod_{i=1}^{n} \frac{D_i}{\Gamma_i \Theta}$

where $\prod_{i=1}^{n} \Gamma_i \Theta = G_{\omega}$

Conclusion: $\Gamma_i$ may not be an arithmetic group (examples of Picard- Deligne- Mostow), but it has the same tensor invariants as an arithmetic group.
V. Hodge representations and the classification theorem

Historically, the main use of Mumford-Tate groups and domains has been in the weight $n=2$ case (families of abelian varieties). In particular the general question: Which reductive, $\mathbb{Q}$-algebraic groups $G$ are Mumford-Tate groups of a PHS?

has been outstanding. Traditionally one took a particular type of PHS and looked for the possible MT groups
I.2

For example, when $n=1$ for the Mumford-Tate domain one has

$$D \subset \mathfrak{h}_g$$

Thus, $D$ is an equivariantly embedded Hermitian symmetric sub-domain of $\mathfrak{h}_g$. Over $\mathbb{R}$, E. Cartan classified these - it's a short list, essentially (in the irreducible case)

$$\begin{align*}
\{ & \text{su}(p,q) \\
\text{so}(a,q) \\
\text{sp}(p,q) \}
\end{align*}$$

no exceptional groups. This leads, with some work, to a reasonable understanding in the $n=1$ case.
5.3

In general, it turns out that one should invert the question and ask the better question: For a given \( G \), in how many ways can \( G \) be realized as a Mumford-Ta"{e}t group?

**Defn:** A Hodge representation \((G, \varphi, \varphi)\) is given by \((V, Q)\) and \(\varphi: G \to \text{Aut}(V, Q)\), defined over \( \mathbb{Q} \), such that \((V, Q, \varphi \circ \varphi)\) is a polarized Hodge structure.

**Defn:** A representation \( \varphi: G \to \text{GL}(V) \) leads to a Hodge representation if there is \( \varphi \), as above.
From now on we will deal with semi-simple groups. For Lie algebras we have for the simple ones

complex   real forms  $\mathcal{O}$-forms

We will state a result that gives the irreducible representations of the real forms that lead to Hodge representations. These all come from

* The example of $\mathfrak{u}(2,1)$ above shows that the reductive case is the most natural one.
representations of \(\mathbb{Q}\)-forms. Given an irreducible representation of a \(\mathbb{Q}\)-form, one may go to the \(\mathbb{R}\)-form and check to see if it leads to a Hodge representation.

A first general result is

**If** \(G\) **has a** Hodge representation, **then** \(G^R\) **contains a compact maximal torus** \(T\)

- This rules out \(SL(n)\) for \(n \geq 3\)
- The converse holds for \(G^{ad}\)

So up to isogeny, the first question is resolved in the semi-simple case.
To state the classification result we have

- $\mathfrak{g}_C$ = complex, simple Lie algebra
- $\mathfrak{t} \subseteq \mathfrak{g}_C$ = Lie algebra complexified of $\mathfrak{T}$
  = Cartan sub-algebra
- $\mathfrak{P} \subseteq \mathfrak{t}^*$ = weight lattice
  $\mathfrak{U}$
- $\mathfrak{R} = \text{root lattice}$

Then

\[
\begin{cases}
\{ \text{connected real Lie groups with Lie algebra a real form } \mathfrak{g}_M \text{ of } \mathfrak{g}_C \} \\
\{ \text{lattices } \mathfrak{P}' \text{ with } \mathfrak{P} \supseteq \mathfrak{P}' \supseteq \mathfrak{R} \}
\end{cases}
\]

Denote the groups by $G_{M, \mathfrak{P}}$. 
$\mathfrak{sl}_2$ - non-compact real form

\[
\begin{align*}
\mathfrak{h}, \mathfrak{z} & : 0 \quad \text{d} = 2 \\
[\mathfrak{h}, \mathfrak{z}] & = -2 \mathfrak{z} \\
[\mathfrak{z}, \mathfrak{z}] & = \mathfrak{h}
\end{align*}
\]

$\rightarrow$

\[
\mathbb{P} = \mathbb{Z}
\]

$\rightarrow$

$SL_2(\mathbb{R})$, $SL_2(\mathbb{R})/\pm I = \mathbb{PSL}_2(\mathbb{R})$

$T_{\mathfrak{p}'} = \mathfrak{z} / \Lambda$, $\Lambda = \text{Hom}(\mathfrak{p}', \mathbb{Z})$

$Z(G_{\mathbb{R}, \mathfrak{p}'}) = \mathfrak{p}' / \mathbb{R}$, $\pi_2(G_{\mathbb{R}, \mathfrak{p}'}) = \mathfrak{p}'$
II.8

Cartan decomposition

\[ \mathfrak{g} \rightarrow k \oplus \mathfrak{p} \]

\( \rightarrow \) roots separate into compact and non-compact ones

\[ \text{Ex} \quad \text{SU}(2,1), \; K = \mathbb{U}(2), \; \mathfrak{p}/\mathfrak{k} \cong \mathbb{Z}/3\mathbb{Z} \]

Define

\( \psi : \mathbb{R} \rightarrow \mathbb{Z}/2\mathbb{Z} \)

\( \psi(a) = \begin{cases} 0 & \text{a compact} \\ 1 & \text{a non-compact} \end{cases} \)

and \( \varphi = \psi \circ \mathbb{Z}/4\mathbb{Z} \) where
V. 9

\[ \Phi(\alpha) = \begin{cases} \\ 0 & \alpha \text{ compact} \\ 2 & \alpha \text{ non-compact} \end{cases} \]

Theorem: Let \( G \) be a simple \( \mathbb{Q} \)-algebraic group, and
\[ \sigma: G \to GL(V) \]
an irreducible representation with highest weight \( \lambda \in \mathcal{P} \). Let \( \delta \) be the minimal integer such that \( \delta \lambda \in \mathbb{R} \).
Then \( \sigma \) leads to a Hodge representation if, and only if, there is a positive integer \( m \) such that
\[ \Phi(\delta \lambda) = \delta m \quad (\text{mod} \, 4) \]

Note: Need to be careful about the definition of since \( \delta \sigma \) may not be irreducible
\[ \begin{align*}
A_r &\quad \text{su}(A_q) \quad \text{sl}_2 \\
B_r &\quad \text{so}(2p, 2q + 1) \\
C_r &\quad \text{sp}(p, q), \ \text{sp}(n) \\
D_r &\quad \text{so}(2p, 2q), \ \text{so}^*(2n) \\
E_6 &\quad E_{III}, E_{III} \\
E_7 &\quad E_{V}, E_{VI}, E_{VII} \\
E_8 &\quad E_{VIII}, E_{IX} \\
F_4 &\quad F_{I}, F_{II} \\
G_2 &\quad G \\
\end{align*} \]

\[ \begin{array}{c}
\text{Much shorter list for odd weight} \\
\text{Result about faithful Hodge representations}
\end{array} \]
The roles of $\gamma$ and $\Phi$ enter as follows:

1. $Q(m,v) = (-1)^n Q(v,m) \rightarrow n (\mod 2) \leftrightarrow \gamma$

2. $p-q \rightarrow p-q (\mod 4) \leftrightarrow \Phi$

The key computation is this:

For $T = \pm 1/A$, the circle is given by $\lambda q \in A$. Then let $\mathbb{Z} \lambda + R \rightarrow \mathbb{Z}$ project to $\tilde{\mathbb{Z}} \lambda + R \rightarrow \mathbb{Z}/4 \mathbb{Z}$. Then $\lambda q$ gives a PHS for $\pm Q$ if, and only if,

\[
\begin{cases}
\tilde{\lambda} q | R = \Phi \\
\tilde{\lambda} q (\lambda) \text{ is even/odd if } Q \text{ is symmetric alternating}
\end{cases}
\]
V. 9.6

Where does the $Q$ come from?

Given $\phi : G_{\mathbb{R}} \to GL(V_{\mathbb{R}})$ we have

$$\text{End} (V_{\mathbb{R}}) = \left\{ \begin{array}{ll} \mathbb{R} \\ \mathbb{A} \\ \mathbb{H} \end{array} \right. \quad G_{\mathbb{R}}$$

In the real case there is, up to scaling, a unique $Q$. In the complex and quaternionic cases

$$V_{\mathbb{C}} = \left\{ \begin{array}{ll} u \omega \bar{u}, & u \neq \bar{u} \\ u \omega \bar{u}, & u = \bar{u} \end{array} \right. \quad \mathbb{C}$$

It follows that there are unique, again up to scaling, invariant $Q$'s that are either symmetric or alternating. The list of the real Lie algebras of simple MT groups is
Example: In his 1905 paper E. Cartan studied the geometry of \((M, Ω)\) where \(\dim M = 5\) and \(Ι ⊂ Τ M\) is a non-degenerate rank 2 distribution. To this date he associated two "geometries" = Cartan connection plus invariants (think curvature). Each has a "flat" model with symmetry group \(G_2, R\) (think constant curvature - only 2 here). Each of these is a Mumford-Tate domain

- \(D_1 \leftrightarrow n = 2, \ h_0^2 = 2, \ h^1_1 = 3\)
- \(D_a \leftrightarrow n = 6, \ all \ h^1_1 = 1\)

The \(Ι\) is the infinitesimal period relation
VI. Discrete series, automorphic representations

. For a Mumford–Tate domain $D = G_{\mathbb{R}} / H$, it turns out to be better to work on $G_{\mathbb{R}} / T$, thought of as Hodge flags

$$\text{Ex SU}(2,1)/T \leftrightarrow \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \quad V_{+1, \ell} \quad V_{-1, \ell}$$

. It is known (Harish-Chandra) that the $G_{\mathbb{R}}$ that have discrete series representations are exactly those with a compact maximal torus $T$. These are the real forms of Mumford–Tate groups
The geometric realization of the discrete series representations is due to Schmid. It goes as follows:

- Complex structures on $D = G_{\text{max}}/T$ are parametrized by Weyl chambers $C = \text{choice of positive roots}$
- The characters $\chi: T \to S^1$ are given by $\text{Hom}(\Lambda, \mathbb{T}) \cong \mathbb{P}$, where $T = \mathbb{Z}/\Lambda$
- Given $\lambda$ with unitary character $\chi_\lambda$, there is a homogeneous, holomorphic line bundle $L_\lambda \to D$
\[ \nabla \nu \]

- there are canonical, $G_{\mathfrak{M}}$-invariant metrics in $L_\lambda$, $D \mapsto H^\&_{(a)} (D, L_\lambda)$

- $\nu = \frac{1}{2} \left( \Sigma d \right)_{d > 0}

Theorem: Given $\lambda$ such that $\lambda + \varphi$ is regular, choose $C$ so that $\lambda + \varphi$ is anti-dominant. Let $d = \dim C K/\Gamma$.

Then $H^\&_{(a)} (D, L_\lambda) = 0$ for $\varphi \neq d$ and

$H^d_{(a)} (D, L_\lambda)$ is the discrete series representation with Harish-Chandra character $\Theta_{\lambda + \varphi}$
\[ \text{Ex } \text{SL}_2(\mathbb{R}) \]

\[ \begin{array}{c}
  * \\
  * \\
  \vdots \\
  o \\
  \vdots \\
  * \\
\end{array} \]

\[ \lambda + \gamma \text{ anti-dominant } \iff \lambda \preceq -\gamma \]

\[ \text{Ex } \text{SU}(a,1) \]

\[ \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
  + \\
  + \\
  + \\
\end{array} \]

\[ \leftrightarrow \text{ discrete series realized as } \mathcal{H}^{a}_{(2)}(D, \mathbb{L})(L_{\lambda})'s \]
What does this have to do with Hodge theory?

For $D = G_{\mathfrak{m}}/H$ a Mumford-Tate domain, we have

$$X = \Gamma \backslash D = \left\{ \begin{array}{l}
\text{moduli space of equivalence classes of PHS's with} \\
\text{generic MT group } G
\end{array} \right\}$$

Classically, automorphic forms are in

$$H^0(X, L^{\otimes k}_{(a)})$$

However, if say $\Gamma$ is co-compact then except in the rare classical cases we have

$$H^0(X, L_\lambda) = 0 \text{ for all } \lambda$$
for \( k \geq k_0 \), \( H^0(X, L^{-k}) = 0 \)
for \( g \neq d \), and
\[
\dim H^d(X, L^{-g}) = C \operatorname{vol}(X) k^N + \ldots
\]
where \( \dim D = N \)

Conclusion: Except in special cases, no automorphic forms. Always, lots of automorphic cohomology.

Classically, the discrete parts of
\[
L^2(G_\mathbb{R}), \quad L^2(\Gamma \backslash G_{\mathbb{R}})
\]
\[
\uparrow
\]
discrete series \hspace{1cm} automorphic representation
are closely related, as in the very classical $\text{SL}_2(\mathbb{R})$ case. The RHS is in turn closely related to $H^0(X, L_X)$'s. The geometric and arithmetic of the RHS in non-classical cases is in its earliest stages of investigation.