Hodge Theory Conference Exercises and Question

Ex. Compute the curvature of Z and D.

Ex. I discussed the following example in the talk. Fix a basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$. Given $u^a e_a$ and $v^b e_b$ in $\mathbb{C}^3$, define Hermtian product of signature $(1,2)$ by

$$\mathcal{H}(u,v) \overset{\text{dfn}}{=} -\overline{u}^1 v^1 + \overline{u}^2 v^2 + \overline{u}^3 v^3 \,.$$

Let

$$G \overset{\text{dfn}}{=} \mathrm{SU}(1,2) = \{g \in \mathrm{SL}_3\mathbb{C} \mid g^*\mathcal{H} = \mathcal{H}\} \,.$$

Let $V = T = \mathrm{U}(1) \times \mathrm{U}(1)$ be the diagonal matrices in G. Then $G_{\mathbb{C}} = \mathrm{SL}_3\mathbb{C}$, $P = B$ is a Borel subgroup and the compact dual $\check{D} = \mathrm{Flag}_{1,2}\mathbb{C}^3 = \mathrm{SL}_3\mathbb{C}/B$ is the (full) flag variety. An element of $\check{D}$ is expressed as $F^\bullet = (F^1 \subset F^2)$. We noted that there are three open $\mathrm{SU}(2,1)$–orbits, one of which is non-classical

$$D^{+,-} \overset{\text{dfn}}{=} \left\{ F^\bullet \in \check{D} \mid \mathcal{H}|_{F^1} \text{ is pos. def.}, \ \mathcal{H}|_{F^2} \text{ is def. of sig } (1,1) \right\}$$

and the K–orbit in this domain D is $\mathbb{P}^1$. Identify the set $\{g \in \mathrm{SL}_3\mathbb{C} \mid gZ \subset D\}$.

Ex. This exercise illustrates an application of the Z–connectedness of a non-classical domain D to identify some line bundles on D with no holomorphic sections. We continue with the example above. Let ω_1, ω_2 denote the fundamental weights of $\mathfrak{g}_{\mathbb{C}}$. A weight $\mu = m^i\omega_i \in \mathrm{span}_{\mathbb{Z}}\{\omega_1, \omega_2\}$ determines a $G_{\mathbb{C}}$–homogeneous line bundle $L_\mu \to \check{D}$ on the compact dual. Then L_μ restricts to a $G_{\mathbb{R}}$–homogeneous line bundle the flag domain D and compact subvariety Z.

(a) Let $\mathfrak{k}_{\mathbb{C}}$ denote the Lie algebra of $K_{\mathbb{C}}$, and show that the restriction of μ to the semisimple part $\mathfrak{sl}_2\mathbb{C} = [\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}] \subset \mathfrak{k}_{\mathbb{C}}$ is the weight $\mu' = (m^1 + m^2)\eta$, where η is the fundamental weight of $\mathfrak{sl}_2\mathbb{C}$.

(b) If $m_1 + m_2 \geq 1$, then the Bott–Borel–Weyl Theorem implies $H^0(Z, L_\mu) = H^0(\mathbb{P}^1, L_\mu) = 0$. Conclude that $H^0(D, L_\mu) = 0$.

Q. Given D, does there exist $n(D) < \infty$ such that *any* two points $x, y \in D$ can be joined by a chain of $n(D)$ varieties Z_u?