# AN EXERCISE AND A PROBLEM

**Exercise.** The goal of this exercise is to verify the theorem of Cattani, Deligne, and Kaplan, as well as the main technical result from my talk, in the special case of a nilpotent orbit on the punctured disk $\Delta^*$.

You can do the exercise without knowing much about degenerations of variations of Hodge structure, but let me first say a few words about that. Let $\Delta$ denote the unit disk, with coordinate $s$, and let $\mathbb{H}$ be the upper half-plane, with coordinate $z = x + iy$; sending $z$ to $s = e^{2\pi i z}$ makes $\mathbb{H}$ into the universal covering space of $\Delta^*$. A nilpotent orbit is a special kind of variation of Hodge structure on $\Delta^*$; it can be described by giving a mixed Hodge structure $(W, F)$ and a nilpotent operator $N$.

Let $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of finite rank. For the mixed Hodge structure, we take one of the following type:

$$\begin{array}{ccccc} & & I^{1,1} & & \\ & I^{1,0} & & I^{0,1} & \\ I^{1,-1} & & I^{0,0} & & I^{-1,1} \\ & I^{0,-1} & & I^{-1,0} & \\ & & I^{-1,-1} & & \end{array}$$

In other words, $H_{\mathbb{C}}$ is the direct sum of the nine subspaces $I^{p,q}$, and we set

$$W_\ell = \bigoplus_{p+q \leq \ell} I^{p,q} \quad \text{and} \quad F^k = \bigoplus_{p \geq k} I^{p,q}.$$

For simplicity, we also assume that $I^{q,p} = \overline{I^{p,q}}$; then $(W, F)$ is an $\mathbb{R}$-split mixed Hodge structure. Let $N$ be a nilpotent endomorphism of $H_{\mathbb{Q}}$ that satisfies $N(I^{p,q}) \subseteq I^{p-1,q-1}$. Let $Q \colon H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$ be a symmetric bilinear form such that $(W, F)$ is polarized by the pair $(Q, N)$. This means that

$$Q(Nv, w) + Q(v, Nw) = 0$$

for every $v, w \in H_{\mathbb{C}}$; that $I^{p,q}$ and $I^{p',q'}$ are orthogonal under $Q$ unless $p + p' = q + q' = 0$; that $W = W(N)$ is the weight filtration for $N$; and that for $p + q \geq 0$, one has

$$i^{p-q} Q(v, N^{p+q} \overline{v}) > 0$$

for every nonzero vector $v \in I^{p,q} \cap \ker N^{p+q+1}$. Under these assumptions, it is known (by a result of Cattani, Kaplan, and Schmid) that the mapping

$$\Phi \colon \mathbb{H} \to D, \quad \Phi(z) = e^{zN} F,$$

is the period mapping for a polarized variation of Hodge structure of weight zero. In fact, this period mapping descends to a polarized variation of Hodge structure of weight zero on $\Delta^*$.

Having introduced all the notation, here is the exercise: Suppose that we have a sequence of integral classes $h_m \in H_{\mathbb{Z}}$ such that $Q(h_m, h_m)$ is bounded. Suppose that $z_m = x_m + iy_m \in \mathbb{H}$ is a sequence of points with $x_m$ bounded and $y_m \to \infty$.

(1) Suppose that every $h_m$ is a Hodge class, meaning that $h_m \in \Phi^0(z_m)$. Show that after passing to a subsequence, $h_m$ is constant and satisfies $Nh_m = 0$. This is a special case of the theorem by Cattani, Deligne, and Kaplan.
(2) Prove the same result under the weaker assumption that the sequence
$$Q\big(h_m, e^{z_m N} v\big)$$
remains bounded for every choice of $v \in F^1$. This is a special case of the theorem from my talk.

**Problem.** Let $\mathcal{H}$ be a polarized variation of $\mathbb{Z}$-Hodge structure of weight zero, defined on a Zariski-open subset $X_0$ of a projective complex manifold $X$. The locus of Hodge classes (and its compactification as defined in my talk) gives rise to variations of Hodge structure (more precisely, Hodge modules) on certain subsets of $X$. The problem is to describe these more directly in terms of $\mathcal{H}$ and its extension $M = j_{!*}\mathcal{H}[\dim X]$ to a polarized Hodge module on $X$.

Let me briefly review the setup. Let $F^p\mathcal{H}$ denote the Hodge bundles, and $\mathcal{H}_{\mathbb{Z}}$ the local system. As in my talk, we consider the covering space
$$\pi \colon T_{\mathbb{Z}} \to X_0$$
whose sheaf of holomorphic sections is $\mathcal{H}_{\mathbb{Z}}$. A point of $T_{\mathbb{Z}}$ can be thought of as a pair $(x, h)$ with $x \in X_0$ and $h \in \mathcal{H}_{\mathbb{Z},x}$. The *locus of Hodge classes* is the subset
$$\operatorname{Hdg}(\mathcal{H}) = \big\{ (x,h) \in T_{\mathbb{Z}} \,\big|\, h \in F^0\mathcal{H}_x \big\} \subseteq T_{\mathbb{Z}}.$$
Now suppose that $W \subseteq \operatorname{Hdg}(\mathcal{H})$ is an irreducible component. By the theorem of Cattani, Deligne, and Kaplan, $W$ is an algebraic variety, finite and proper over its image $Z = \pi(W)$. By construction, the restriction of the local system $\pi^{-1}\mathcal{H}_{\mathbb{Z}}$ to $W$ has a section that gives a Hodge class at every point. At least generically, the pushforward of $\mathbb{Z}_W$ to $Z$ is a variation of Hodge structure of weight zero on $Z$; it is somehow related to $\mathcal{H}$, although I do not know the precise relationship. Similarly, we can consider the holomorphic mapping
$$\varepsilon \colon T_{\mathbb{Z}}(K) \to T(F_{-1}\mathcal{M});$$
recall that, for any integer $K \in \mathbb{Z}$, we defined
$$T_{\mathbb{Z}}(K) = \big\{ (x,h) \in T_{\mathbb{Z}} \,\big|\, |Q_x(h,h)| \leq K \big\} \subseteq T_{\mathbb{Z}},$$
where $Q$ denotes the polarization on $\mathcal{H}$. If we compactify $\operatorname{Hdg}(\mathcal{H}) \cap T_{\mathbb{Z}}(K)$ by taking the closure of $\varepsilon\big(T_{\mathbb{Z}}(K)\big)$, normalizing it, and then considering the preimage of the zero section from $T(F_{-1}\mathcal{M})$, we obtain a finite union of projective algebraic varieties $W_j \subseteq \operatorname{Hdg}(\mathcal{H})$. By projecting down to $X$, we get a finite number of polarizable Hodge modules $M_j$, supported on the subvarieties $Z_j = \pi(W_j)$.

The problem is to obtain the $M_j$ more directly from $\mathcal{H}$ or $M$, without going through the rather complicated construction above. A solution would be especially useful in the case where $\mathcal{H}$ is the variation of Hodge structure on the second cohomology of the family of hyperplane sections of a Calabi-Yau threefold.