Open problem
Recent Advances in Hodge Theory
PIMS, Vancouver, June 10–20, 2013

Burt Totaro

Problem. Find the smallest positive integer d you can such that there is a smooth complex projective 3-fold X with a very ample divisor A and a positive integer b not dividing 6 such that $A^3 = d$ and every complex curve C on X has $A \cdot C \equiv 0 \pmod{b}$. For example, b could be 4.

Kollár showed that if a positive integer d has this property, then the integral Hodge conjecture fails for very general hypersurfaces of degree d in $\mathbf{P}^4$ [3]. In some cases, one can push the method to disprove the integral Hodge conjecture among hypersurfaces of degree d in $\mathbf{P}^4$ over $\mathbf{Q}$; this was first done by Hassett and Tschinkel, and further developed in [4]. From a more positive perspective, an example as above implies that every curve on a very general hypersurface of degree d in $\mathbf{P}^4$ has degree a multiple of $b/6$. That is a partial result in the direction of Griffiths-Harris's conjecture that when $d \geq 6$, every curve on a very general hypersurface of degree d in $\mathbf{P}^4$ should have degree a multiple of d [2].

Example ($d = 64$, Kollár). Let $X = \mathbf{P}^3$ and $A = 4H$, where H is a hyperplane on X. Then it is clear that the intersection number of A with any curve on X is a multiple of 4, and we have $L^3 = 64$.

Example ($d = 48$, Kollár). Let $X = S \times \mathbf{P}^1$ where S is a smooth quartic surface in $\mathbf{P}^3$ with Picard number 1. (Such a surface exists by the Noether-Lefschetz theorem.) Write H_1 be the pullback of a hyperplane divisor from $\mathbf{P}^3$ to S, and let H_2 be the hyperplane divisor on $\mathbf{P}^1$. We use the same names for the pullback of these divisors to X. Let $A = H_1 + 4H_2$. I claim that every curve C on X has $A \cdot C \equiv 0 \pmod{4}$. It suffices to show that every curve C on S has $H_1 \cdot C \equiv 0 \pmod{4}$. That holds because every curve C on S has a class in $\mathrm{Pic}(S) = \mathbf{Z} \cdot H_1$, and $H_1^2 = 4$. Finally, on the 3-fold X, A has degree

$$\begin{aligned} A^3 &= (H_1 + 4H_2)^3 \\ &= 3H_1^2 \cdot 4H_2 \\ &= 3 \cdot 4 \cdot 4 \cdot 1 \\ &= 48. \end{aligned}$$

So far, 48 is the lowest known degree of a counterexample to the integral Hodge conjecture among hypersurfaces in $\mathbf{P}^4$. It seems quite possible that this method could produce an example of lower degree. In high degrees, the best known result is by Debarre-Hulek-Spandaw: for $k \geq 9$, every curve on a very general complex hypersurface of degree $6k$ has degree a multiple of $k/2$ (hence a multiple of k, if k is odd) [1]. Their proof uses the method here, with (X, L) a very general $(1, 1, k)$-polarized abelian 3-fold.

References

[1] O. Debarre, K. Hulek, and J. Spandaw. Very ample linear systems on abelian varieties. *Math. Ann.* **300** (1994), 181–202.

[2] P. Griffiths and J. Harris. On the Noether-Lefschetz theorem and some remarks on codimension two cycles. *Math. Ann.* **271** (1985), 31–51.

[3] J. Kollár. Trento examples. *Classification of irregular varieties* (Trento, 1990), 134–135. Lecture Notes in Mathematics **1515**, Springer (1992).

[4] B. Totaro. On the integral Hodge and Tate conjectures over number fields. *Forum of Mathematics, Sigma*, to appear. `http://journals.cambridge.org/sigma`