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Hodge Theory and Iterated Integrals.

§ Introduction

Goal: construct a MHS on the fundamental group $\pi_1(X, x)$ of a compact Kähler manifold X. More generally, one would like to (and can) construct a MHS on the fundamental group of any complex algebraic variety.

What does this mean?

- In general, $\pi_1(X, x)$ is a non-abelian infinite group
- mixed Hodge structures ~~on~~ live on vector spaces.
- Such groups are not vector spaces.

So how do we turn a discrete group into a vector space?

Γ = discrete group

coordinate ring

$$\Gamma \overset{?}{\rightsquigarrow} \underset{\text{(affine or linear)}}{\text{(pro) algebraic gp } G} \rightsquigarrow \left\{ \begin{array}{l} \mathcal{O}(G) \\ \mathfrak{g} \end{array} \right.$$

Lie algebra.

VECTOR SPACES

<u>Unipotent Completion</u> (aka Malcev completion)

F = field of char 0.

Recall that an algebraic group over F is <u>unipotent</u> if it is isomorphic to an algebraic subgroup of

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \subseteq GL_N(F)$$

<u>Note</u>:

Have polynomial bijections

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \underset{\exp}{\overset{\log}{\rightleftarrows}} \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

on affine space $\mathbb{A}_F^{(?)}$. isos.

★ Unipotent groups are isomorphic (as varieties) to their Lie algebras, which are nilpotent.

Suppose Γ is a discrete group

<u>Defn</u>: The unipotent completion of Γ is

$$\Gamma^{un} = \varprojlim_{\rho} U_\rho$$

where $\rho : \Gamma \to U_\rho$ $\begin{cases} \text{is Zariski dense} \\ U_\rho \text{ unipotent} \end{cases}$

Lie algebra:

$$\mathrm{Lie}(\Gamma^{un}) = \varprojlim_{\rho} \mathrm{Lie}(U_\rho) \quad \leftarrow \text{nilpotent}$$

$$= \text{pro nilpotent Lie alg.}$$

Coordinate ring:

$$\mathcal{O}(\Gamma^{un}) = \varinjlim_{\rho} \mathcal{O}(U_\rho). \qquad \text{Hopf algebra.}$$

<u>Example</u>: This looks unmanageable — difficult to work with.

$$\Gamma = F_n = \langle x_1, \dots, x_n \rangle$$

$$A = \mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle = \mathbb{Q}\langle X_1, \dots, X_n\rangle^{\wedge} \quad \leftarrow \text{completion.}$$

power series in non commuting indeterminates X_j. (↑ $\mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle$)

free assoc algebra gen by X_j (↑ $\mathbb{Q}\langle X_1, \dots, X_n\rangle$)

$$\theta : F_n \longrightarrow \mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle = A$$

$$x_j \longmapsto e^{X_j}$$

~~Indeed~~

$$A \supseteq \mathbb{L}(X_1, \dots, X_n)^{\wedge} \quad \leftarrow \text{completion.}$$

free Lie algebra (↑ $\mathbb{L}(X_1, \dots, X_n)$)

<u>Easy fact</u>: $\mathrm{Lie}\, F_n^{un} = \mathbb{L}(X_1, \dots, X_n)^{\wedge}$

$$F_n^{un} = \exp \mathbb{L}(X_1, \dots, X_n)^{\wedge} \subseteq A^{\times}.$$

Example: (a general finitely presented group)

$$\Gamma = \langle x_1, \dots, x_n : r_1, \dots, r_m \rangle \quad \leftarrow \text{words in } x_1, \dots, x_n$$

$$\mathrm{Lie}(\Gamma^{un}) = \mathbb{L}(X_1, \dots, X_n)^\wedge / \left(\log(\theta(r_1)), \dots, \log\theta(r_m)\right)$$

§ Chen's Iterated Integrals

Now that we have our vector space, we need to relate it to differential forms so that we can define a Hodge filtration on it.

Q: How can we use differential forms to detect non-abelian elements of the fundamental group not visible in homology?

$M = C^\infty$ manifold. $E^\bullet M = C^\infty$ deRham complex

Perhaps we can use non-closed 1-forms?

$$PM = \{\gamma : [0,1] \to M\} = \text{path space.}$$

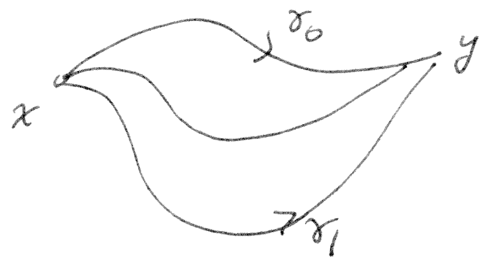
$$P_{x,y}M = \{\gamma : \gamma(0) = x,\ \gamma(1) = y\}$$

~~Def: A function $F : PM \to$~~

Let A = an associative $\mathbb{R}$-algebra

ex: $A = \begin{cases} \mathbb{R} \\ \mathbb{C} \\ M_N(\mathbb{R}) \\ \mathbb{R}\langle\langle X_1, \dots, X_n \rangle\rangle \end{cases}$

<u>Defn</u>: A function $F: PM \to A$ is a <u>homotopy</u> <u>functional</u> if its value on $\gamma \in P_{x,y}M$ depends only on its homotopy class.



$F(\gamma_0) = F(\gamma_1)$

~~If~~ Homotopy functionals induce <u>functions</u>

$\pi_1(M, x) \to A$

↑ not-necess homomorphs!

<u>Basic question</u>: How can we construct interesting homotopy functionals $PM \to A$ using differential forms?

Example: $\omega \in E^1(M)$

$$\int \omega : PM \to \mathbb{R}$$
$$\gamma \mapsto \int_\gamma \omega$$

is a htpy functional $\Leftrightarrow$ $d\omega = 0$.

So we need to do something more general than a standard line integral.

Chen's iterated line integrals:

$\omega_1, \ldots, \omega_r \in E^1(M) \otimes A \cong \left\{ \begin{array}{l} A\text{-valued} \\ 1\text{-forms on } M \end{array} \right.$

Define

$$\int_\gamma \omega_1 \ldots \omega_r = \int \cdots \int_{0 \le t_1 \le \cdots \le t_r \le 1} f_1(t_1) \ldots f_r(t_r)\, dt_1 \ldots dt_r$$

"time ordered simplex"

where $\gamma \in PM$

$$f_j(t)\,dt = \gamma^* \omega_j$$

Ex:
$$\int_0^x \frac{dz}{1-z} \frac{dz}{z} = \int_0^x -\log(1-z)\, \frac{dz}{z}$$
$$= \int_0^x \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots\right) \frac{dz}{z}$$
$$= \left[z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \ldots \right]_0^x$$

$$= \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots$$

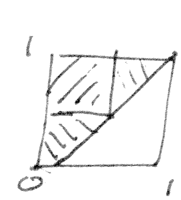
$$= \mathrm{Li}_2(x) \quad \text{dilogarithm.}$$

note $\quad \int_0^1 \frac{dz}{1-z}\,\frac{dz}{z} = \zeta(2).$

PROPERTIES :

coproduct (1) $\quad \int_{\alpha\beta} \omega_1 \ldots \omega_r = \sum_{j=0}^{r} \int_\alpha \omega_1 \ldots \omega_j \int_\beta \omega_{j+1} \ldots \omega_r.$

ex: $\int_{\alpha\beta} \omega_1 \omega_2 = \int_\alpha \omega_1 \omega_2 + \int_\alpha \omega_1 \int_\beta \omega_2 + \int_\beta \omega_1 \omega_2.$



prod (2)

$$\int_\alpha \omega_1 \ldots \omega_r \int_\alpha \omega_{r+1} \ldots \omega_s$$

$$= \sum_{\sigma \in sh(r,s)} \int_\alpha \omega_{\sigma(1)} \ldots \omega_{\sigma(r+s)} \qquad \underline{\text{shuffles}}$$

ex $\quad \int_\alpha \omega_1 \int_\alpha \omega_2 \omega_3 = \int_\alpha \omega_1 \omega_2 \omega_3 + \int_\alpha \omega_2 \omega_1 \omega_3 + \int_\alpha \omega_2 \omega_3 \omega_1$

antipode (3)

$$\int_{\gamma^{-1}} \omega_1 \ldots \omega_r = (-1)^r \int_\gamma \omega_r \ldots \omega_1$$

(4) $\int_\gamma \omega_1 \ldots \omega_r$ does not depend on the parameterization of γ.

<u>Exercises:</u>

(1) prove that

$$\int_{\alpha\beta}\omega_1\omega_2 = \int_\alpha \omega_1\omega_2 + \int_\alpha\omega_1\int_\beta\omega_2 + \int_\beta \omega_1\omega_2$$

(2) Show that if $d\omega_j = 0$ then

$$\int\Big(\sum_{j,k} a_{jk}\,\omega_j\omega_k + \xi\Big) : PM \to \mathbb{R}$$

is a homotopy functional

$$\iff \quad \sum a_{jk}\,\omega_j\wedge\omega_k + d\xi = 0.$$

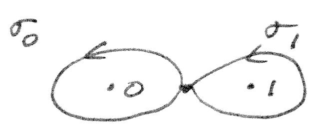
(3)

$$\int_{[\alpha,\beta]}\omega_1\omega_2 = \begin{vmatrix} \int_\alpha\omega_1 & \int_\alpha\omega_2 \\ \int_\beta\omega_1 & \int_\beta\omega_2 \end{vmatrix}$$

where $\quad [\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1} \quad \alpha,\beta\in P_{x,x}M.$

<u>Example:</u> $\quad M = \mathbb{P}^1 - \{0,1,\infty\} = \mathbb{C} - \{0,1\}.$

$\omega_0 = \dfrac{dz}{z} \qquad \omega_1 = \dfrac{dz}{1-z}. \qquad \Big\{ \int\omega_0\omega_1 : PM \to \mathbb{C}$ homotopy functional.



(i) $$\int_{[\sigma_0,\sigma_1]}\omega_0\omega_1 = (-2\pi i)^2 = 4\pi^2$$

Iterated integrals will be viewed as functions

$$PM \to A$$

and not as formal expressions. (There are relations, such as $\int df = 0$ on $P_{xx}M$.)

Will say that an iterated integral is ~~a~~ closed if it is a homotopy functional.

There is one more property of iterated integrals that is relevant:

If $\alpha_1, \ldots, \alpha_s \in P_{x,x}M$ then

1 = const loop.

$$\langle \int \omega_1 \cdots \omega_r, \underbrace{(\alpha_1 - 1)(\alpha_2 - 1) \cdots (\alpha_s - 1)}_{\text{view as a formal linear comb of loops.}} \rangle = 0$$

when $s > r$:

<u>Example:</u>

$$\langle \int \omega, (\alpha - 1)(\beta - 1) \rangle$$

$$= \langle \int \omega, \alpha\beta - \alpha - \beta + 1 \rangle$$

$$= \int_{\alpha\beta} \omega - \int_\alpha \omega - \int_\beta \omega$$

$$= 0.$$

So this generalizes a familiar property of ~~that~~ line integrals

Chen's π_1 de Rham Theorem

Let

$$B_s(M,x) = \left\{ \begin{array}{c} \text{iterated integrals of "length" } \le s \\ \text{restricted to } P_{x,x}M \end{array} \right\}.$$

AND

$B(M,x) = \bigcup_s B_s(M,x)$

$H^0(B(M,x)) = \bigcup_s H^0(B_s(M,x))$

Let $H^0(B_s(M,x)) = \{ I \in B_s(M,x)$ that are homotopy functionals $\}$.

The properties of iterated integrals above imply that $H^0(B(M,x))$ is a Hopf algebra.

π_1 de Rham Theorem (version 1)

$$\mathcal{O}\left(\pi_1(M,x)^{un}\right) \underset{\substack{\uparrow \\ \text{via integration}}}{\cong} H^0(B(M,x)).$$

Hopf algebra isomorphism.

π_1 de Rham Theorem (version 2)

$$H^0(B_s(M,x)) \longrightarrow \operatorname{Hom}_{\mathbb{R}}\left(\mathbb{Z}\pi_1(M,x)/J^{s+1}, \mathbb{R}\right)$$

is an isomorphism for all s.

Cor:
$$H^0(B(M,x)) = \operatorname{Hom}_{cts}(\mathbb{R}\pi_1(M,x), \mathbb{R})$$
$\swarrow$ J-adic top

$$J = \ker\{\mathbb{Z}\pi_1(M,x) \xrightarrow{\varepsilon} \mathbb{Z}\}$$
augmentation ideal.

§ Hodge Theory

Suppose that X is a compact Kähler manifold. Define

$$F^p B_s(X,x)_{\mathbb{C}}$$

to be the span of

Hodge filtration

$$\int \omega_1 \ldots \omega_r \qquad r \le s$$

with $\omega_j \in F^{p_j} E^1_{\mathbb{C}}(X)$ &

$$p_1 + \cdots + p_r \ge p.$$

(i.e. at least p dz's).

Define

$$F^p H^0(B_s(X,x)) = F^p B_s(X,x) \cap H^0(B(X,x))$$

Define

$$W_m H^0(B(X,x)) = H^0(B_m(X,x)).$$

Thm: For all $s \ge 0$

$$\left.\begin{array}{c}(H^0(B_s(X,x)), W_\bullet, F^\bullet) \\ \cong \\ \mathrm{Hom}(\mathbb{Z}\pi_1(X,x)/J^{s+1}, \mathbb{Z}) \otimes \mathbb{C}\end{array}\right\} \text{ is a } \mathbb{Z}\text{-MHS.}$$

Cor 1

$$\mathcal{O}\left(\pi_1(X,x)^{un}\right)$$

is a Hopf algebra in that cat of Ind-MHS

Morgan. Cor 2: The Lie algebra of $\pi_1(X,x)^{un}$ is a ~~pro~~ Lie algebra in the category of pro-MHS.

Remark: Similar results hold when X is a complex algebraic variety. For a general smooth variety $X = \bar{X} - D$, $\leftarrow$ dnc the weight filtration also takes into account the poles of the ω_j.

ex $\quad \int \frac{dz}{1-z} \, \frac{dz}{z} \in W_4$

§ Applications

① The MHS on $\pi_1(X,x)^{un}$ depends (in general) non-trivially on x.

Thm (Pulte-Hain) If C is a compact RS of genus $g \geq 2$, then the MHS on

$$H^0(B_2(C,x))$$

determines C and x.

POINT: Have the extension:

$$0 \to H^1(C) \to H^0(B_2) \to H^1(C)^{\otimes 2} \xrightarrow{cup} \mathbb{Z}(-1) \to 0$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \; H^2(C).$$

$$\sum a_{ij} \int \omega_i \omega_j + \xi \mapsto \sum a_{ij} [\omega_i] \otimes [\omega_j]$$

$$[\omega] \longmapsto \int \omega.$$

Get elt e_x of

$$Ext^1(K, H^1) \qquad K = \ker\left(H^1(C)^{\otimes 2} \to H^2\right)$$

$$\int$$

This contains $Jac\, C = Ext(\mathbb{Z}, H_1)$

$$e_x - e_y \longleftrightarrow [x] - [y].$$

The extension class e_x also determines

$$[C_x - C_x^-] \in J\left(PH^3(Jac\, C)\right)$$

Ceresa cycle. Intermediate Jacobian.

To tie into Patrick's Lecture:

② unipotent variation of MHS:

$$\left.\begin{array}{ccc} V_x & \subseteq & \mathbb{V} \\ | & & | \\ x & \in & X \end{array}\right\} \text{ admissible VMHS, } X \text{ smooth.}$$

where $\rho_x : \pi_1(X,x) \to \operatorname{Aut}(V_x)$

is unipotent. (ie global monodromy unip!)

$\Updownarrow$

equivalently, each $Gr^W_m \mathbb{V}$ is a constant VHS over X.

The universal mapping property of unipotent completion implies that ρ_x factors

$$\begin{tikzcd} \pi_1(X,x) \arrow[r, "\rho_x"] \arrow[d] & \operatorname{Aut}(V_x) \\ \pi_1(X,x)^{un} \arrow[ur] & \end{tikzcd}$$

to have

$$\theta_x : \underset{\substack{\shortparallel \\ \operatorname{Lie} \pi_1(X,x)^{un}.}}{\mathfrak{g}(X,x)} \longrightarrow \operatorname{End}(V_x).$$

Thm (Hain-Zucker).

(1) θ_x is a morphism of MHS.

(2) $\left\{ \begin{matrix} \mathbb{V} \\ \downarrow \\ X \end{matrix} \right\} \rightsquigarrow \{ \theta_x : \mathfrak{g}_x \to \operatorname{End} V_x \}$ (fiber @ x; Hodge rep)

is an equivalence of categories from unipt VMHS/X to Hodge representations of $\mathfrak{g}_x$.

Cor (Carlson-Hain)

$$\operatorname{Ext}^m_{UMHS(X)}(\mathbb{Q}_X, \mathbb{V}) \cong H^m_{\mathcal{D}}(\mathfrak{g}(X,x), V_x)$$

$$\parallel$$

$$H^m(\operatorname{cone}(W_0 F^0 C^\cdot(\mathfrak{g}_{\mathbb{C}}, V) \oplus W_0 C^\cdot(\mathfrak{g}_{\mathbb{Q}}, V) \to W_0 C^\cdot(\mathfrak{g}_{\mathbb{C}}, V)))$$

~~Ex.~~

Cor: Have s.e.s.

$\Gamma = \operatorname{Hom}_{Hodge}(\mathbb{Q}(0), -)$

$$0 \to \operatorname{Ext}^1_{MHS}(\mathbb{Q}, H^{m-1}(\mathfrak{g}_x, V)) \to \operatorname{Ext}^m_{UVMHS(X)}(\mathbb{Q}_X, \mathbb{V}) \to \Gamma H^m(\mathfrak{g}_x, V_x) \to 0$$

Has MHS

Examples: X = product of curves $g \geq 1$ or (configuration space of any curve)

$X = F_n(C) := C^n -$ fat diag, $n \geq 2$

$H^\cdot(X, \mathbb{V})$ — unipt local system.

$\cong H^\cdot(\mathfrak{g}(X,x), \mathbb{V})$

Then have s.e.s.

$$0 \to Ext^1(\mathbb{Q}, H^{m-1}(X, \mathbb{V}))$$

$$\to Ext^m_{UVMHS}(\mathbb{Q}_X, \mathbb{V})$$

$$\to \Gamma H^m(X, \mathbb{V}) \to 0.$$

$\underline{Ex}$: $X = E^m$ E = elliptic curve

$\mathbb{V} = \mathbb{Q}(m)$.

$$0 \to E^m \otimes_{\mathbb{Z}} \mathbb{Q} \to Ext^{2m}_{UVMHS.}(X, \mathbb{Q}(m)) \to \underset{\substack{\parallel \\ H^{m,m}_{\mathbb{Q}}(X)}}{\mathbb{Q}} \to 0$$

$$Ext^1(\mathbb{Q}, \overbrace{H^{2m-1}(X, \mathbb{Q}(m))}^{wt\ -1})$$

$$\left.\begin{aligned} &= Ext^1(\mathbb{Q}, H_1(E)^{\oplus m}) \\ &= E^m \otimes_{\mathbb{Z}} \mathbb{Q} \end{aligned}\right\}.$$

③ Topology of Varieties

1: If $\mathfrak{g}$ (pro-nilpt) has a MHS (with negative wts), then

$$\mathfrak{g}_{\mathbb{C}} \cong \prod_{m\geq 1} Gr^W_{-m}\,\mathfrak{g} =: \left(Gr^W_\bullet \mathfrak{g}\right)^\wedge$$

X smooth $= \bar{X} - D$ (dnc)

2: $H^1(Gr^W_\bullet \mathfrak{g}) \cong Gr^W_\bullet H^1(X)$ ← wts $\{1,2\}$ $\qquad \mathfrak{g} = \mathfrak{g}_X = \operatorname{Lie} \pi_1(X, x)$

$H^2(Gr^W_\bullet \mathfrak{g}) \subseteq Gr^W_\bullet H^2(X)$ ← wts $\{2,3,4\}$

Morgan 1977

3: So $Gr^W_\bullet \mathfrak{g}$ has a (minimal weighted homog.) presentation with generators of weights $\subseteq \{-1,-2\}$, relations with weights $\subseteq \{-2,-3,-4\}$.

↑ at worst quartic rels!

Ex: $\langle x, y : [x[x[x[x,y]\cdots]\rangle$

is not π_1(smooth q. proj var).

Deligne–Griffiths–Morgan–Sullivan

Special Case: If Γ is fund gp of a compact Kähler manifold X, then

$$\mathfrak{g} = \operatorname{Lie}(\Gamma^{un}) \cong \mathbb{L}(H_1(X))^\wedge \Big/ \operatorname{im}\{H_2(X) \xrightarrow{\text{cup}^*} \Lambda^2 H_1(X)\}$$

($H_1(X)$: wt −1 gens; $\Lambda^2 H_1(X) = \mathbb{L}_2(H_1(X))$)

ex: $\Gamma = \pi_1(C, *)$

$$\mathfrak{g} = \mathbb{L}(a_1, \ldots, b_g)^\wedge \Big/ \left(\sum_1^g a_j \wedge b_j\right)$$

References:

[1] R. HAIN: Geometry of the MHS on π_1, PROC. SYMP. PURE MATH.

[2] R. HAIN: Arizona Winter School Lectures. On the AWS website.