SHIMURA VARIETIES: A HODGE-THEORETIC PERSPECTIVE

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Abstract. These largely expository notes are based on a mini-course given by the author at the 2010 ICTP Summer School on Hodge Theory.

Introduction

In algebraic geometry there is a plethora of objects which *turn out* by big theorems to be algebraic, but which are *defined* analytically:

- projective varieties, and functions and forms on them (by Chow's theorem or GAGA [Serre1956]);
- Hodge-loci, and zero-loci of normal functions (work of Cattani-Deligne-Kaplan [CDK1995], and Brosnan-Pearlstein [BP2009]);
- complex tori with a polarization (using theta functions, or the embedding theorem [Kodaira1954]);
- Hodge classes (if a certain \$1,000,000 problem could be solved);

and of concern to us presently:

- modular (locally symmetric) varieties,

which can be thought of as the $\Gamma \backslash D$'s for the period maps of certain special VHS's. The fact that they are algebraic is the Baily-Borel theorem [BB1966].

What one does *not* know in the Hodge/zero locus setting above is the field of definition — a question related to the existence of Bloch-Beilinson filtrations, which are discussed in M. Green's course in this volume. For certain cleverly constructed *unions* of modular varieties, called **Shimura varieties**, one actually knows the minimal (i.e. reflex) field of definition, and also quite a bit about the interplay between "upstairs" and "downstairs" (in $\check{D}$ resp. $\Gamma \backslash D$) fields of definition of subvarieties. My interest in the subject stems from investigating Mumford-Tate domains of Hodge structures, where for example the reflex fields can still be defined even though the $\Gamma \backslash D$'s are not algebraic varieties in general (cf. §5, and [GGK2010]). Accordingly, I have tried to pack as many Hodge-theoretic punchlines into the exposition below as possible.

Of course, Shimura varieties are of central importance from another point of view, that of the Langlands program. For instance, they provide a major test case for the conjecture, generalizing Shimura-Taniyama, that all motivic L-functions (arising from Galois representations on étale cohomology of varieties over number fields) are automorphic, i.e. arise from automorphic forms (or more precisely from Hecke eigenforms of adelic algebraic groups). The modern theory is largely due to Deligne, Langlands and Shimura (with crucial details by Shih, Milne, and Borovoi), though many others are implicated in the huge amount of underlying mathematics: e.g.

- complex multiplication for abelian varieties (Shimura, Taniyama, Weil);
- algebraic groups (Borel, Chevalley, Harish-Chandra);
- class field theory (Artin, Chevalley, Weil; Hensel for p-adics);
- modular varieties (Hilbert, Hecke, Siegel) and their compactifications (Baily, Borel, Satake, Serre, Mumford).

It seems that much of the impetus, historically, for the study of locally symmetric varieties can be credited to Hilbert's 12th problem generalizing Kronecker's Jugendtraum. Its goal was the construction of abelian extensions (i.e. algebraic extensions with abelian Galois groups) of certain number fields by means of special values of abelian functions in several variables, and it directly underlay the work of Hilbert and his students on modular varieties and the theory of CM.

What follows is based on the course I gave in Trieste, and (though otherwise self-contained) makes free use of the notes in this volume by E. Cattani (variations of Hodge structure), J. Carlson (period maps and period domains), and P. Griffiths (Mumford-Tate groups). It was a pleasure to speak at such a large and successful summer school, and I heartily thank the organizers for the invitation to lecture there. A brief outline follows:

(1) Hermitian symmetric domains — D
(2) Locally symmetric varieties — $\Gamma \backslash D$
(3) The theory of complex multiplication
(4) Shimura varieties — $\amalg_i (\Gamma_i \backslash D)$
(5) The field of definition

1. Hermitian symmetric domains

A. Algebraic groups and their properties.

Definition 1.1. An **algebraic group** G over a field k (of characteristic zero) is a smooth algebraic variety G together with morphisms

$$\begin{array}{rcl} \bullet : \; G \times G \to G & & \text{(multiplication)} \\ (\cdot)^{-1} : \; G \to G & & \text{(inversion)} \end{array}$$

defined over k and an element

$$\mathbf{e} \in G(k) \qquad \text{(identity)} \; ,$$

subject to rules which make $G(L)$ into a group for each L/k. Here $G(L)$ denotes the L-rational points of G, i.e. the morphisms $\mathrm{Spec}L \to G$. In particular, $G(\mathbb{R})$ and $G(\mathbb{C})$ have the structure of real resp. complex Lie groups.

Exercise 1.2. Write out these rules as commutative diagrams.

Example 1.3. As an algebraic variety, the multiplicative group

$$(GL_1 \cong)\, \mathbb{G}_m := \{XY = 1\} \subset \mathbb{A}^2,$$

with $\mathbb{G}_m(k) = k^*$.

We review the definitions of some basic properties,[1] starting with

- G **connected** $\iff$ $G_{\bar{k}}$ irreducible,

where the subscript denotes the extension of scalars: for L/k, $G_L := G \times_{\mathrm{Spec}k} \mathrm{Spec}L$. There are two fundamental building blocks for algebraic groups, simple groups and tori:

- G **simple** $\iff$ G nonabelian, with no normal connected subgroups $\neq \{\mathbf{e}\}$, G.

Example 1.4. (a) $k = \mathbb{C}$: SL_n [Cartan type A], SO_n [types B, D], Sp_n [type C], and the exceptional groups of types E_6, E_7, E_8, F_4, G_2.

(b) $k = \mathbb{R}$: have to worry about real forms (of these groups) which can be isomorphic over $\mathbb{C}$ but not over $\mathbb{R}$.

(c) $k = \mathbb{Q}$: $\mathbb{Q}$-simple does not imply $\mathbb{R}$-simple; in other words, all hell breaks loose.

- G **(algebraic) torus** $\iff$ $G_{\bar{k}} \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$

Example 1.5. (a) $k = \mathbb{C}$: the algebraic tori are all of the form $(\mathbb{C}^*)^{\times n}$.

(b) $k = \mathbb{Q}$: given a number field E, the "Weil restriction" or "restriction of scalars" $G = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ is a torus of dimension $[E : \mathbb{Q}]$ with the property that $G(\mathbb{Q}) \cong E^*$, and more generally $G(k) \cong E^* \otimes_{\mathbb{Q}} k$. If $k \supseteq E$ then G splits, i.e.

$$G(k) \cong (k^*)^{[E:\mathbb{Q}]},$$

[1] for the analogous definitions for real and complex Lie groups, see [Rotger2005]

with the factors corresponding to the distinct embeddings of E into k.

(c) $k \subset \mathbb{C}$ arbitrary: inside GL_2, one has tori $\mathbb{U} \subset \mathbb{S} \supset \mathbb{G}_m$ with k-rational points

$$\mathbb{U}(k) = \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \middle| \begin{array}{c} a^2+b^2=1 \\ a,b \in k \end{array} \right\}$$

$$\mathbb{S}(k) = \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \middle| \begin{array}{c} a^2+b^2 \neq 0 \\ a,b \in k \end{array} \right\}$$

$$\mathbb{G}_m(k) = \left\{ \left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha \end{array} \right) \middle| a \in k^* \right\}.$$

In particular, their complex points take the form

$$\begin{array}{ccccc} \mathbb{C}^* & \hookrightarrow & \mathbb{C}^* \times \mathbb{C}^* & \hookleftarrow & \mathbb{C}^* \\ \\ z & \longmapsto & (z, \bar{z}) \; ; \; (\alpha, \alpha) & \longleftarrow\!\shortmid & \alpha \end{array}$$

by considering the eigenvalues of matrices. Writing S^1 or U_1 for the unit circle in $\mathbb{C}^*$, the real points of these groups are $U_1 \subset \mathbb{C}^* \supset \mathbb{R}^*$, the smaller two of which exhibit distinct real forms of $\mathbb{G}_{m,\mathbb{C}}$. The map

$$\jmath : \mathbb{U} \to \mathbb{G}_m$$

sending

$$\left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \mapsto \left(\begin{array}{cc} a+bi & 0 \\ 0 & a+bi \end{array} \right)$$

is an isomorphism on the complex points but does not respect real points (hence is not defined over $\mathbb{R}$).

Next we put the building blocks together:

- G **semisimple** $\iff$ G an *almost-direct* product of simple subgroups,

i.e. the morphism from the direct product to G is an isogeny (has zero-dimensional kernel). More generally,

- G **reductive** $\iff$ G an almost-direct product of simple groups and tori,

which turns out to be equivalent to the complete reducibility of G's finite-dimensional linear representations.

Example 1.6. One finite-dimensional representation is the **adjoint map**

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & GL(\mathfrak{g}) \\ g & \longmapsto & \{X \mapsto gXg^{-1}\} \end{array},$$

where $\mathfrak{g} = Lie(G) = T_{\mathbf{e}}G$ and we are taking the differential (at $\mathbf{e}$) of $\Psi_g \in Aut(G)$, i.e. conjugation by g.

We conclude the review with a bit of structure theory. For semisimple groups:

- G **adjoint** $\iff$ Ad is injective; and
- G **simply connected** $\iff$ any isogeny $G' \twoheadrightarrow G$ with G' connected is an isomorphism.

Basically, the center $Z := Z(G)$ is zero-dimensional, hence has finitely many points over $\bar{k}$; we can also say that Z is *finite*. Adjointness is equivalent to its triviality; whereas in the simply connected case Z is as large as possible (with the given Lie algebra).

For reductive groups, we have short-exact sequences

$$\begin{tikzcd}
Z' \arrow[drr] \arrow[ddr] & & & \\
& & G^{\mathrm{der}} \arrow[d] \arrow[dr] & \\
& Z \arrow[r] \arrow[dr] & G \arrow[r, "\mathrm{Ad}"] \arrow[d] & G^{\mathrm{ad}} \\
& & T &
\end{tikzcd}$$

with $G^{\mathrm{der}} := [G,G]$ and $G^{\mathrm{ad}} := \mathrm{Ad}(G)$ both semisimple (and the latter adjoint), $Z' := Z \cap G^{\mathrm{der}}$ finite, and T a torus. Clearly, G is semisimple if and only if the "maximal abelian quotient" T is trivial.

Finally, let G be a reductive real algebraic group,

$$\theta : G \to G$$

an involution.

Definition 1.7. θ is **Cartan** $\iff$

$$\{g \in G(\mathbb{C}) \,|\, g = \theta(\bar{g})\} =: G^{(\theta)}(\mathbb{R}) \text{ is compact.}$$

Equivalently, $\theta = \Psi_C$ for $C \in G(\mathbb{R})$ with

- $C^2 \in Z(\mathbb{R})$ and
- $G \hookrightarrow Aut(V,Q)$ for some symmetric bilinear form Q satisfying $Q(\cdot, C\bar{(\cdot)}) > 0$ on $V_{\mathbb{C}}$.

Of course, to a Hodge theorist this last condition suggests polarizations and the Weil operator.

Cartan involutions always exist, and

$$G(\mathbb{R}) \text{ compact} \iff \theta = \mathrm{id}.$$

B. Three characterizations of Hermitian symmetric domains.

I. Hermitian symmetric space of noncompact type. This is the "intrinsic analytic" characterization. The basic object is (X, g), a connected complex manifold with Hermitian metric, or equivalently a Riemannian manifold with integrable almost complex structure such that J acts by isometries. The real Lie group $Is(X, g)$ of holomorphic isometries must

- act **transitively** on X, and
- contain (for each $p \in X$) **symmetries** $s_p : X \to X$ with $s_p^2 = \mathrm{id}_X$ and p as isolated fixed point; moreover,
- the identity connected component $Is(X, g)^+$ must be a semisimple adjoint **noncompact** (real Lie) group.

The noncompactness means that (a) the Cartan involution projects to the identity in *no* factor and (b) X has negative sectional curvatures.

II. Bounded symmetric domain. Next we come to the "extrinsic analytic" approach, with X a connected open subset of $\mathbb{C}^n$ with compact closure such that the (real Lie) group $Hol(X)$ of holomorphic automorphisms

- acts **transitively**, and
- contains **symmetries** s_p as above.

The Bergman metric makes X into a noncompact Hermitian space, and the Satake (or Harish-Chandra) embedding does the converse job. For further discussion of the equivalence between I and II, see [Milne2005, sec. 1].

III. Circle conjugacy class. Finally we have the "algebraic" version, which will be crucial for any field of definition questions. Moreover, up to a square root, this is the definition that Mumford-Tate domains generalize as we shall see later.

Let G be a real adjoint (semisimple) algebraic group. We take X to be the orbit, under conjugation by $G(\mathbb{R})^+$, of a homomorphism

$$\phi : \mathbb{U} \to G$$

of algebraic groups defined over $\mathbb{R}$ subject to the constraints:

- only z, 1, and z^{-1} appear as eigenvalues in the representation $\mathrm{Ad} \circ \phi$ on $Lie(G)_{\mathbb{C}}$;

- $\theta := \Psi_{\phi(-1)}$ is Cartan; and
- $\phi(-1)$ doesn't project to the identity in any simple factor of G.

The points of X are of the form $g\phi g^{-1}$ for $g \in G(\mathbb{R})^+$, and we shall think of it as "a connected component of the conjugacy class of a circle in $G(\mathbb{R})$". Obviously the definition is independent of the choice of ϕ in a fixed conjugacy class.

Under the equivalence of the three characterizations,

$$Is(X, g)^+ = Hol(X)^+ = G(\mathbb{R})^+.$$

If, in any of these groups, K_p denotes the stabilizer of a point $p \in X$, then

$$G(\mathbb{R})^+ / K_p \xrightarrow{\cong} X.$$

We now sketch the proof of the equivalence of the algebraic and (intrinsic) analytic versions.

From (III) to (I). Let p denote a point of X given by a circle homomorphism ϕ. Since the centralizer

$$K := Z_{G(\mathbb{R})^+}(\phi)$$

belongs to $G^{(\theta)}(\mathbb{R})$,

(a) $\mathfrak{k}_{\mathbb{C}} := Lie(K_{\mathbb{C}})$ is the 1-eigenspace of $\mathrm{Ad}\phi(z)$ in $\mathfrak{g}_{\mathbb{C}}$, and

(b) K is compact (in fact, maximally so).

By (a), we have a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^- \oplus \mathfrak{p}^+$$

into $\mathrm{Ad}\phi(z)$-eigenspaces with eigenvalues 1, z, z^{-1}. Identifying $\mathfrak{p}^- \cong \mathfrak{g}_{\mathbb{R}}/\mathfrak{k}$ puts a complex structure on $T_\phi X$ for which $d(\Psi_{\phi(z)})$ is multiplication by z. Using $G(\mathbb{R})^+$ to translate $J := d(\Psi_{\phi(i)})$ to all of TX yields an almost complex structure.

One way to see this is integrable, making X into a (connected) complex manifold, is as follows: define the **compact dual** $\check{X}$ to be the $\mathrm{Ad}G(\mathbb{C})$-translates of the flag

$$\begin{cases} F^1 &= \mathfrak{p}^+ \\ F^0 &= \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \end{cases}$$

on $\mathfrak{g}_{\mathbb{C}}$. We have

$$\check{X} \cong G(\mathbb{C}) / P \cong G^{(\theta)}(\mathbb{R}) / K$$

for P a parabolic subgroup with $Lie(P) = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+$, which exhibits $\check{X}$ as a $\mathbb{C}$-manifold and as compact. (It is in fact projective.) The obvious map from decompositions to flags yields an injection $X \hookrightarrow \check{X}$ which is an isomorphism on tangent spaces, exhibiting X as an (analytic) open subset of $\check{X}$.

Now by (b), there exists a K-invariant symmetric and positive-definite bilinear form on $T_\phi X$. Translating this around yields a $G(\mathbb{R})^+$-invariant Riemannian metric g on X. Since $J \in K$, g commutes with (translates of) J and is thereby Hermitian. The symmetry at ϕ is given by $s_\phi := \Psi_{\phi(-1)}$ (acting on X). Since $\Psi_{\phi(-1)}$ (as an element of $Aut(G)$) is Cartan and doesn't project to $\mathbf{e}$ in any factor, G is noncompact. $\square$

From (I) to (III). Since $Is(X,g)^+$ is adjoint and semisimple, by [Borel1991, Thm. 7.9] it is $G(\mathbb{R})^+$ for some algebraic group $G \subset GL(Lie(Is(X,g)^+))$. (Note that this can *only* make sense if $Is(X,g)^+$ is adjoint hence embeds in $GL(Lie(Is(X,g)^+))$. Further, the "plus" on $G(\mathbb{R})^+$ is necessary: if $Is(X,g)^+ = SO(p,q)^+$, this is *not* $G(\mathbb{R})$ for algebraic G.)

Any $p \in X$ is an isolated fixed point of the associated symmetry $s_p \in Aut(X)$ with $s_p^2 = \mathrm{id}_X$, and so ds_p is multiplication by (-1) on T_pX. In fact, by a delicate argument involving sectional curvature (cf. [Milne2005, sec.1]), for any $z = a + bi$ with $|z| = 1$, there exists an unique isometry $u_p(z)$ of (X,g) such that on T_pX, $du_p(z)$ is multiplication by z (i.e. $a + bJ$). Since du_p yields a homomorphism $U_1 \to GL(T_pX)$, the uniqueness means that

$$u_p : U_1 \to Is(X,g)^+$$

is also a homomorphism. It algebraizes to the homomorphism

$$\phi_p : \mathbb{U} \to G$$

of real algebraic groups.

To view X as a conjugacy class, recall that $G(\mathbb{R})^+$ acts transitively. For $g \in G(\mathbb{R})^+$ sending $p \mapsto q$, the uniqueness of u_q means that

$$\phi_q(z) = g \circ \phi_p(z) \circ g^{-1} = (\Psi_g \circ \phi_p)(z).$$

We must show that ϕ_p satisfies the three constraints. In the decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus T_p^{1,0}X \oplus T_p^{0,1}X,$$

$d(\Psi_{\phi_p(z)})$ has eigenvalue z on $T^{1,0}$ hence $\bar{z} = z^{-1}$ on $T^{0,1}$, since ϕ_p is real and $z \in U_1$. Using the uniqueness once more, $\Psi_k \circ \phi_p = \phi_p$ for any $k \in K_p$, and so $\Psi_{\phi_p(z)}$ acts by the identity on $\mathfrak{k} = Lie(K_p)$. Therefore $1,\ z,\ z^{-1}$ are the eigenvalues of $\mathrm{Ad}\phi_p$. Finally, from the fact that X has negative sectional curvatures we deduce that Ψ_{s_p} is Cartan, which together with noncompactness of X implies that s_p projects to $\mathbf{e}$ in no factor of G. $\square$

C. Cartan's classification of irreducible Hermitian symmetric domains. Let X be an irreducible HSD, G the corresponding simple $\mathbb{R}$-algebraic group, and $T \subset G_{\mathbb{C}}$ a maximal algebraic torus. The restriction to T

$$\begin{tikzcd} T \arrow[r, hook] \arrow[rr, bend left] & G_{\mathbb{C}} \arrow[r, "\mathrm{Ad}"'] & GL(\mathfrak{g}_{\mathbb{C}}) \end{tikzcd}$$

of the adjoint representation breaks into 1-dimensional eigenspaces on which T acts through characters:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha} \right),$$

where

$$R^+ \amalg R^- = R \subset Hom(T, \mathbb{G}_m) \cong \mathbb{Z}^n$$

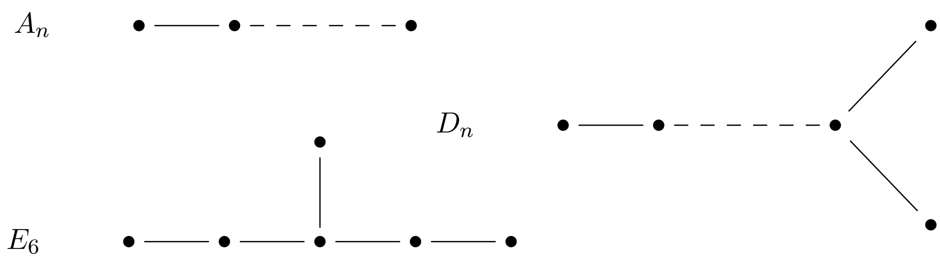
are the roots (with $R^- = -R^+$). With this choice, one has uniquely the

- **simple roots**: $\{\alpha_1, \ldots, \alpha_n\}$ such that each $\alpha \in R^+$ is of the form $\sum m_i \alpha_i$ where $m_i \geq 0$; and the
- **highest root**: $\hat{\alpha} = \sum \hat{m}_i \alpha_i \in R^+$ such that $\hat{m}_i \geq m_i$ for any other $\alpha \in R^+$.

The α_i give the nodes on the Dynkin diagram of G, in which α_i and α_j are connected if they pair nontrivially under a standard inner product, the **Killing form**

$$B(X, Y) := Tr(\mathrm{ad}X \mathrm{ad}Y).$$

Example 1.8.



Over $\mathbb{C}$, our circle map ϕ defines a cocharacter

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, hook] \arrow[rr, bend left, "\mu"] & \mathbb{U} \arrow[r, "\phi_{\mathbb{C}}"'] & G_{\mathbb{C}} \end{tikzcd}.$$

This has a *unique* conjugate factoring through T in such a way that, under the pairing of characters and cocharacters given by

$$\mathbb{G}_m \underset{\mu}{\rightarrow} T \underset{\alpha}{\rightarrow} \mathbb{G}_m$$

$$z \longmapsto z^{\langle \mu, \alpha \rangle},$$

one has $\langle \mu, \alpha \rangle \geq 0$ for all $\alpha \in R^+$. Since μ must act through the eigenvalues z, 1, z^{-1}, we know

$$\begin{array}{rcll} \langle \mu, \alpha \rangle & = & 0 \text{ or } 1 & \text{for all } \alpha \in R^+ \\ \text{and} & \neq & 0 & \text{for some } \alpha \in R^+ \end{array}.$$

By considering $\langle \mu, \hat{\alpha} \rangle$, we deduce from this that $\langle \mu, \alpha_i \rangle = 1$ for a unique i, and that the corresponding α_i is **special**: i.e., $\hat{m}_i = 1$. So we have a 1-to-1 correspondence

$$\boxed{\begin{array}{c} \text{irreducible} \\ \text{Hermitian symmetric domains} \end{array}} \longleftrightarrow \boxed{\begin{array}{c} \text{special nodes on} \\ \text{connected Dynkin diagrams} \end{array}},$$

and hence a list of the number of distinct isomorphism classes of irreducible HSD's corresponding to each simple complex Lie algebra:

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
n	1	1	3	2	1	0	0	0

Example 1.9.
(a) A_n: $X \cong SU(p,q)/S(U_p \times U_q)$ with $p+q = n+1$ (n possibilities).
(b) B_n: $X \cong SO(n,2)^+/SO(n) \times SO(2)$.
(c) C_n: $X \cong Sp_{2n}(\mathbb{R})/U(n) \cong \{Z \in M_n(\mathbb{C}) \,|\, Z = {}^tZ,\, Im(Z) > 0\}$.
By J. Carlson's lectures [Carlson], in (b) X is a period domain for H^2 of $K3$ surfaces; in (c), X is the Siegel upper half-space $\mathfrak{H}^n$, and parametrizes weight/level 1 Hodge structures.

D. Hodge-theoretic interpretation. Let V be a $\mathbb{Q}$-vector space.

Definition 1.10. A Hodge structure on V is a homomorphism

$$\tilde{\varphi} : \mathbb{S} \to GL(V)$$

defined over $\mathbb{R}$, such that the weight homomorphism

$$w_{\tilde{\varphi}} : \mathbb{G}_m \hookrightarrow \mathbb{S} \xrightarrow{\tilde{\varphi}} GL(V)$$

is defined over $\mathbb{Q}$.[2]

Associated to $\tilde{\varphi}$ is

$$\begin{array}{rccc} \mu_{\tilde{\varphi}} : & \mathbb{G}_m & \to & GL(V) \\ & z & \mapsto & \tilde{\varphi}_{\mathbb{C}}(z,1). \end{array}$$

[2]These are precisely the $\mathbb{Q}$-split mixed Hodge structures (no nontrivial extensions). More generally, mixed Hodge structures have weight *filtration* $W_\bullet$ defined over $\mathbb{Q}$ but with the canonical splitting of $W_\bullet$ defined over $\mathbb{C}$, so that the weight *homomorphism* is only defined over $\mathbb{C}$.

Remark 1.11. Recalling that $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$, $V^{p,q} \subset V_{\mathbb{C}}$ is the

$$\begin{cases} z^p w^q\text{-} & \text{eigenspace of } \tilde{\varphi}_{\mathbb{C}}(z,w) \\ z^p\text{-} & \text{eigenspace of } \mu(z) \end{cases},$$

and $w_{\tilde{\varphi}}(r) = \tilde{\varphi}(r,r)$ acts on it by r^{p+q}.

Fix a weight n, Hodge numbers $\{h^{p,q}\}_{p+q=n}$, and polarization $Q : V \times V \to \mathbb{Q}$. Let

- D be the period domain parametrizing Hodge structures of this type, polarized by Q, on V;
- $\mathbf{t} \in \oplus_i V^{\otimes k_i} \otimes \check{V}^{\otimes \ell_i}$ be a finite sum of Hodge tensors;
- $D_{\mathbf{t}}^+ \subset D$ be a connected component of the subset of HS for which these tensors are **Hodge**: $\mathbf{t}_i \in F^{n(k_i - \ell_i)/2}$ for each i; and
- $M_{\mathbf{t}} \subset GL(V)$ be the smallest $\mathbb{Q}$-algebraic subgroup with $M_{\mathbf{t}}(\mathbb{R}) \supset \tilde{\varphi}(\mathbb{S}(\mathbb{R}))$ for all $\tilde{\varphi} \in D_{\mathbf{t}}^+$. (This is reductive.)

Then given any $\tilde{\varphi} \in D_{\mathbf{t}}^+$, the orbit

$$D_{\mathbf{t}}^+ = M_{\mathbf{t}}(\mathbb{R})^+ . \tilde{\varphi} \cong M_{\mathbf{t}}(\mathbb{R})^+ / H_{\tilde{\varphi}}$$

under action by conjugation, is called a **Mumford-Tate domain**. This is a connected component of the **full Mumford-Tate domain** $D_{\mathbf{t}} := M_{\mathbf{t}}(\mathbb{R}).\tilde{\varphi}$, which will become relevant later (cf. §4.B). In either case, $M_{\mathbf{t}}$ is the **Mumford-Tate group** of $D_{\mathbf{t}}^{(+)}$.

Exercise 1.12. Check that $\Psi_{\mu_{\tilde{\varphi}}(-1)}$ is a Cartan involution.

Now, consider the condition that the tautological family $\mathcal{V} \to D_{\mathbf{t}}^+$ be a variation of Hodge structure, i.e. that Griffiths's infinitesimal period relation (IPR) $\mathfrak{I} \subset \Omega^\bullet(D_{\mathbf{t}}^+)$ be trivial. This is equivalent to the statement that the HS induced on $Lie(M_{\mathbf{t}}) \subset End(V)$ "at $\tilde{\varphi}$" (by $\mathrm{Ad} \circ \tilde{\varphi}$) be of type $(-1,1) + (0,0) + (1,-1)$, since terms in the Hodge decomposition of type $(-2,2)$ or worse would violate Griffiths transversality. Another way of stating this is that

$$\text{(1.1)} \qquad \mathrm{Ad} \circ \mu_{\tilde{\varphi}}(z) \text{ has only the eigenvalues } z, 1, z^{-1},$$

and so we have proved part (a) of

Proposition 1.13. *(a) A Mumford-Tate domain with trivial IPR (and $M_{\mathbf{t}}$ adjoint) admits the structure of a Hermitian symmetric domain with G defined over $\mathbb{Q}$, and*

(b) conversely — that is, such Hermitian symmetric domains parametrize VHS.

Remark 1.14. (i) Condition (1.1) implies that $\Psi_{\mu_{\tilde{\varphi}}(-1)}$ gives a symmetry of $D_{\mathbf{t}}^+$ at $\tilde{\varphi}$, but not conversely: e.g., an example of a Hermitian symmetric MT domain with nontrivial IPR is the period domain for HS of weight 6 and type $(1, 0, 1, h, 1, 0, 1)$.

(ii) This doesn't contradict (b), because the same Hermitian symmetric domain can have different MT domain structures.

(iii) Strictly speaking, to get the HSD structure in (a) one must put $\phi := \mu_{\tilde{\varphi}} \circ \jmath$, which unlike $\mu_{\tilde{\varphi}}$ is actually defined over $\mathbb{R}$.

(iv) The isotropy group $H_{\tilde{\varphi}}$ above is a maximal compact subgroup of $M_{\mathbf{t}}(\mathbb{R})^+$ iff $D_{\mathbf{t}}^+$ is Hermitian symmetric.

Proof. To show (b), let X be a HSD with real circle $\mathbb{U} \xrightarrow{\phi} G$. Since a product of MT domains is a MT domain, we may assume G $\mathbb{Q}$-simple. The composition

$$\begin{tikzcd} \mathbb{S} \arrow[r, "(z,w) \mapsto z/w"] \arrow[rrr, bend right, "=:\tilde{\varphi}"'] & \mathbb{U} \arrow[r, "\phi"'] & G \arrow[r] & Aut(\mathfrak{g}, B) \end{tikzcd}$$

is a Hodge structure on $V = \mathfrak{g}$, polarized by $-B$ since $\Psi_{\phi(-1)}$ is Cartan. The $\mathbb{Q}$-closure of a generic $G(\mathbb{R})$-conjugate is G by nontriviality of $\phi(-1)$, and G is of the form $M_{\mathbf{t}}$ by Chevalley's theorem (cf. Griffiths's lectures in this volume). Consequently, $X = G(\mathbb{R})^+.\tilde{\varphi}$ is a MT domain. The IPR vanishes because $\mathrm{Ad} \circ \phi$ has eigenvalues z, 1, z^{-1}. □

The proposition is essentially a theorem of [Deligne1979].

Example 1.15. [due to Mark Green] Applied to one of the HSD's for E_6, this procedure yields a MT domain parametrizing certain HS of type $(h^{2,0}, h^{1,1}, h^{0,2}) = (16, 46, 16)$ — a submanifold $D_{\mathbf{t}}^+$ of the period domain D for such HS. The IPR $\mathfrak{I} \subset \Omega^\bullet(D)$ is nontrivial but *pulls back to zero* on $D_{\mathbf{t}}^+$.

Problem 1.16. Find a family of varieties over $D_{\mathbf{t}}^+$ with this family of HS. More generally, it is conjectured that the tautological VHS over every MT domain with trivial IPR, is *motivic* – i.e., comes from algebraic geometry.

The proof of (b) always produces HS of even weight. Sometimes, by replacing the adjoint representation by a "standard" representation, we can parametrize HS of odd weight: For instance, $\mathfrak{H}^4 \cong Sp_8(\mathbb{R})/U(4)$ parametrizes HS of weight/level 1 and rank 8, or equivalently abelian varieties of dimension 4. There are "two" types of MT subdomains in $\mathfrak{H}^4$:

(a) Those corresponding to $End_{\mathrm{HS}}(V)$ ($= End(A)_{\mathbb{Q}}$) containing a non-trivial fixed subalgebra $\mathcal{E}$ isomorphic to a product of matrix algebras over $\mathbb{Q}$-division algebras. These must be of the four types occurring in the Albert classification:
(I) totally real field;
(II) indefinite quaternion algebra over a totally real field;
(III) definite quaternion algebra over a totally real field;
(IV) division algebra over a CM field.
For example, an imaginary quadratic field is a type (IV) division algebra. All four types do occur in $\mathfrak{H}^4$.

(b) Those corresponding to fixed endomorphisms $\mathcal{E}$ plus *higher* Hodge tensors. We regard $End_{\mathrm{HS}}(V)$ as a subspace of $T^{1,1}V$ and the polarization Q as an element of $T^{0,2}V$; "higher" means [in a $T^{k,\ell}V$] of degree $k+\ell>2$.

Here are two such examples:

Example 1.17. (Type (a)) Fix an embedding

$$\mathbb{Q}(i) \overset{\beta}{\hookrightarrow} End(V),$$

and write $Hom(\mathbb{Q}(i),\mathbb{C}) = \{\eta,\bar{\eta}\}$. We consider Hodge structures on V such that

$$V^{1,0} = V^{1,0}_{\eta} \oplus V^{1,0}_{\bar{\eta}}$$

with $\dim V^{1,0}_{\eta} = 1$; in particular, for such HS the image of $\mathbb{Q}(i)$ lies in $End_{\mathrm{HS}}(V)$. The resulting MT domain is easily presented in the three forms from §1.**B**: as the circle conjugacy class $D^+_{\mathbf{t}}$ (taking $\mathbf{t} := \{\beta(i), Q\}$); as the noncompact Hermitian space $SU(1,3)/S(U_1\times U_3)$; and as a complex 3-ball. It is irreducible of Cartan type A_3.

Exercise 1.18. Show that the MT group of a generic HS in $D^+_{\mathbf{t}}$ has real points $M(\mathbb{R})/w_{\tilde{\varphi}}(\mathbb{R}^*) \cong U(2,1)$.

Example 1.19. (Type (b)) [Mumford1969] constructs a quaternion algebra $\mathfrak{Q}$ over a totally real cubic field K, such that $\mathfrak{Q}\otimes_{\mathbb{Q}}\mathbb{R} \cong \mathbb{H}\oplus\mathbb{H}\oplus M_2(\mathbb{R})$, together with an embedding $\mathfrak{Q}^* \hookrightarrow GL_8(\mathbb{Q})$. This yields a $\mathbb{Q}$-simple algebraic group

$$G := Res_{K/\mathbb{Q}} U_{\mathfrak{Q}} \subset Sp_8 \subset GL(V)$$

with $G(\mathbb{R}) \cong SU(2)^{\times 2}\times SL_2(\mathbb{R})$, and the $G(\mathbb{R})$-orbit of

$$\varphi_0 : \mathbb{U} \to G$$
$$a+ib \mapsto \mathrm{id}^{\times 2} \times \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

yields a MT domain. Mumford shows that the generic HS it supports has trivial endomorphisms $\mathcal{E} = \mathbb{Q}$, so G is cut out by higher Hodge tensors.

Remark 1.20. Though we limited ourselves to pure HS above to simplify the discussion, the Mumford-Tate business, and Proposition 1.13(a) in particular, still works in the more general setting of Definition 1.10.

2. Locally symmetric varieties

To construct quotients of Hermitian symmetric domains we'll need the basic

Proposition 2.1. *Let X be a topological space, with $x_0 \in X$; $\mathcal{G}$ be a locally compact group acting on X; and $\Gamma \leq \mathcal{G}$ be a discrete subgroup (i.e. one with no limit points). Assume*
(i) $K := stab(x_0)$ is compact, and
(ii) $gK \mapsto gx_0 : \mathcal{G}/K \to X$ is a homeomorphism.
Then $\Gamma\backslash X$ is Hausdorff.

The proof is a nontrivial topology exercise. Writing $\pi : X \to \Gamma\backslash X$, the key points are:

- π^{-1} of a compact set is compact; and
- the intersection of a discrete and compact set is finite.

Corollary 2.2. *Let $X = G(\mathbb{R})^+/K$ be a Hermitian symmetric domain, and $\Gamma \leq G(\mathbb{R})^+$ discrete and torsion-free. Then $\Gamma\backslash X$ has a unique complex-manifold structure for which π is a local isomorphism.*

Remark 2.3. If Γ isn't torsion-free then we get an orbifold.

Example 2.4. (a) $X = \mathfrak{H}$, $G = SL_2$ acting in the standard way, and

$$\Gamma = \Gamma(N) := \ker\{SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\}$$

with $N \geq 3$. The quotients $\Gamma\backslash X =: Y(N)$ are the **classical modular curves** classifying elliptic curves with *marked N-torsion*, an example of **level structure**.

(b) $X = \mathfrak{H}^n$, $G = Sp_{2n}$, $\Gamma = Sp_{2n}(\mathbb{Z})$. Then $\Gamma\backslash X$ is the **Siegel modular variety** classifying abelian n-folds with a fixed polarization.

(c) $X = \mathfrak{H} \times \cdots \times \mathfrak{H}$ (n times), $G = Res_{F/\mathbb{Q}}SL_2$ with F a totally real field of degree n over $\mathbb{Q}$, and $\Gamma = SL_2(\mathcal{O}_F)$. The quotient $\Gamma\backslash X$ is a **Hilbert modular variety** classifying abelian n-folds with $\mathcal{E} \supset F$: the general member is of Albert type (I). We may view X as a proper MT subdomain of $\mathfrak{H}^n$. To get a more interesting level structure here, one could replace $\mathcal{O}_F$ by a proper ideal.

(d) $X = \{[v] \in \mathbb{P}^n(\mathbb{C}) \mid -|v_0|^2 + \sum_{i=1}^n |v_i|^2 < 0\}$ (the complex n-ball), $G = Res_{K/\mathbb{Q}} SU(n,1)$ with K a quadratic imaginary field ($Hom(K,\mathbb{C}) = \{\theta, \bar{\theta}\}$) and $\Gamma = SU((n,1), \mathcal{O}_K)$. The quotient $\Gamma\backslash X$ is a **Picard modular variety**, classifying abelian $(n+1)$-folds with $\mathcal{E} \supset K$ in such a way that the dimension of the θ- resp. $\bar{\theta}$-eigenspaces in $T_0 A$ are 1 resp. n. See the interesting treatment for $n = 2$ in [Holzapfel1995], especially sec. 4.9.

All the discrete groups Γ arising here have been rather special.

Definition 2.5. (a) Let G be a $\mathbb{Q}$-*algebraic group*. Fix an embedding $G \overset{\iota}{\hookrightarrow} GL_n$. A subgroup $\Gamma \leq G(\mathbb{Q})$ is

arithmetic $\iff$ Γ commensurable[3] with $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$; and

congruence $\iff$ for some N, Γ contains
$$\Gamma(N) := G(\mathbb{Q}) \cap \{g \in GL_n(\mathbb{Z}) \mid g \underset{(N)}{\equiv} \text{id.}\}$$
as a subgroup of finite index.

Congruence subgroups are arithmetic, and both notions are independent of the embedding ι.

(b) Let $\mathcal{G}$ be a connected *real Lie group*. A subgroup $\Gamma \leq \mathcal{G}$ is **arithmetic** if there exist
- a $\mathbb{Q}$-algebraic group G,
- an arithmetic $\Gamma_0 \leq G(\mathbb{Q})$, and
- a homomorphism $G(\mathbb{R})^+ \underset{\rho}{\twoheadrightarrow} \mathcal{G}$ with compact kernel,

such that $\rho(\Gamma_0 \cap G(\mathbb{R})^+) = \Gamma$.

Part (b) is set up so that Γ will always contain a torsion-free subgroup of finite index.

Theorem 2.6. [BB1966] *Let $X = G(\mathbb{R})^+/K$ be a Hermitian symmetric domain, and $\Gamma < G(\mathbb{R})^+$ a torsion-free arithmetic subgroup. Then $X(\Gamma) := \Gamma\backslash X$ is canonically a smooth quasi-projective algebraic variety, called a* ***locally symmetric variety****.*

Remark 2.7. If we don't assume Γ torsion-free, we still get a quasi-projective algebraic variety, but it is an orbifold, hence not smooth and *not* called a locally symmetric variety.

Idea of proof. Construct a minimal, highly singular (Baily-Borel) compactification
$$X(\Gamma)^* := \Gamma \backslash \{X \amalg B\}$$

[3] i.e. the intersection is of finite index in each

where B stands for "rational boundary components". Embed this in $\mathbb{P}^N$, using automorphic forms of sufficiently high weight, as a projective analytic — hence (by Chow/GAGA) projective algebraic — variety. The existence of enough automorphic forms to yield an embedding is a convergence question for certain Poincaré-Eisenstein series. $\square$

Example 2.8. In the modular curve context (with $\Gamma = \Gamma(N)$), $B = \mathbb{P}^1(\mathbb{Q})$ and $X(\Gamma)^* \backslash X(\Gamma)$ is a finite set of points called **cusps**. We write $Y(N)$ resp. $X(N)$ for $X(\Gamma)$ resp. $X(\Gamma)^*$.

Recalling from §1.**D** that X is always a MT domain, we can give a Hodge-theoretic interpretation to the Baily-Borel compactification:

Proposition 2.9. *The boundary components B parametrize the possible $\oplus_i Gr_i^W H_{lim}$ for VHS into $X(\Gamma)$.*

Heuristic idea of proof. Assuming PGL_2 is not a quotient of G, the automorphic forms are Γ-invariant sections of $K_X^{\otimes N}$ for some $N \gg 0$. The canonical bundle K_X, which is pointwise isomorphic to $\bigwedge^d \mathfrak{g}^{(-1,1)}$, measures the change of the Hodge flag in every direction. So the boundary components parametrized by these sections must consist of *naive* limiting Hodge flags in $\partial \bar{X} \subset \check{X}$. In that limit, thinking projectively, the relation between periods that blow up at different rates (arising from different Gr_i^W) is fixed, which means *we cannot see extension data.* On the other hand, since $\exp(zN)$ does not change the $Gr_i^W F^\bullet$, *this information is the same* for the naive limiting Hodge flag and the limiting mixed Hodge structure. $\square$

Remark 2.10. (a) The proposition is due to Carlson, Cattani and Kaplan [CCK1980] for Siegel domains, but there seems to be no reference for the general statement.

(b) There are other compactifications with different Hodge-theoretic interpretations:

- the compactification of Borel-Serre [BS1973], which records Gr_i^W and adjacent extensions, at least in the Siegel case; and
- the smooth toroidal compactifications of [AMRT1975], which capture the entire limit MHS.

The latter is what the monograph of Kato and Usui [KU2009] generalizes (in a sense) to the non-Hermitian symmetric case, where $X(\Gamma)$ is not algebraic.

For the discussion of canonical models to come, we will need the

Theorem 2.11. [Borel1972] *Let $\mathcal{Y}$ be a quasi-projective algebraic variety over $\mathbb{C}$, and $X(\Gamma)$ a locally symmetric variety. Then any analytic map $\mathcal{Y} \to X(\Gamma)$ is algebraic.*

Idea of proof. Extend this to an analytic map $\bar{\mathcal{Y}} \to X(\Gamma)^*$, then use GAGA.

Suppose $X = \mathfrak{H}$ and $\mathcal{Y}$ is a curve; since Γ is torsion-free, $X(\Gamma) \cong \mathbb{C}\backslash\{\geq 2 \text{ points}\}$. Denote by $\mathbb{D}$ a small disk about the origin in $\mathbb{C}$. If a holomorphic $f : (\mathbb{D}\backslash\{0\}) \to X(\Gamma)$ does not extend to a holomorphic map from $\mathbb{D}$ to $\mathbb{P}^1$, then f has an essential singularity at 0. By the "big" Picard theorem, f takes all values of $\mathbb{C}$ except possibly one, a contradiction. Applying this argument to a neighborhood of each point of $\bar{\mathcal{Y}}\backslash\mathcal{Y}$ gives the desired extension.

The general proof uses the existence of a good compactification $\mathcal{Y} \subset \bar{\mathcal{Y}}$ (Hironaka) so that $\mathcal{Y}$ is locally $\mathbb{D}^{\times k} \times (\mathbb{D}^*)^{\times \ell}$. □

3. Complex multiplication

A. CM Abelian varieties. A **CM field** is a totally imaginary field E possessing an involution $\rho \in Gal(E/\mathbb{Q}) =: \mathcal{G}_E$ such that $\phi \circ \rho = \bar{\phi}$ for each $\phi \in Hom(E, \mathbb{C}) =: \mathcal{H}_E$.

Exercise 3.1. Show that then E^ρ is totally real, and $\rho \in Z(\mathcal{G}_E)$.

Denote by E^c a normal closure.

For any decomposition

$$\mathcal{H}_E = \Phi \amalg \bar{\Phi},$$

(E, Φ) is a **CM type**; this is equipped with a **reflex field**

$$\begin{aligned} E' \;&:=\; \mathbb{Q}\left(\left\{ \textstyle\sum_{\phi\in\Phi} \phi(e) \,\middle|\, e \in E \right\}\right) \;\subset\; E^c \\ &=\; \text{fixed field of } \left\{ \sigma \in \mathcal{G}_{E^c} \mid \sigma\tilde{\Phi} = \tilde{\Phi} \right\} \end{aligned}$$

where $\tilde{\Phi} \subset \mathcal{H}_{E^c}$ consists of embeddings restricting on E to those in Φ. Composing Galois elements with a fixed choice of $\phi_1 \in \Phi$ gives an identification

$$\mathcal{H}_{E^c} \xleftarrow{\cong} \mathcal{G}_{E^c}$$

and a notion of inverse on $\mathcal{H}_{E^c}$. Define the **reflex type** by

$$\Phi' := \left\{ \tilde{\phi}^{-1}|_{E'} \,\middle|\, \tilde{\phi} \in \tilde{\Phi} \right\},$$

and **reflex norm** by

$$\begin{aligned} N_{\Phi'} : (E')^* &\to E^* \\ e' &\mapsto \prod_{\phi' \in \Phi'} \phi'(e'). \end{aligned}$$

Example 3.2. (a) All imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ are CM; in this case $N_{\Phi'}$ is the identity or complex conjugation.

(b) All cyclotomic fields $\mathbb{Q}(\zeta_n)$ are CM; and if $E/\mathbb{Q}$ is an abelian extension, then $E' = E^c = E$ and E is contained in some $\mathbb{Q}(\zeta_n)$. For cyclotomic fields we will write

$$\phi_j \ := \ \text{embedding sending } \zeta_n \mapsto e^{2\pi i j/n}.$$

(c) The CM type $(\mathbb{Q}(\zeta_5); \{\phi_1, \phi_2\})$ has reflex $(\mathbb{Q}(\zeta_5); \{\phi_1, \phi_3\})$.

The relationship of this to algebraic geometry is contained in

Proposition 3.3. *(a) For a simple complex abelian g-fold A, the following are equivalent:*

(i) the MT group of $H^1(A)$ is a torus;

(ii) $End(A)_{\mathbb{Q}}$ has (maximal) rank $2g$ over $\mathbb{Q}$;

(iii) $End(A)_{\mathbb{Q}}$ is a CM field; and

(iv) $A \cong \mathbb{C}^g/\Phi(\mathfrak{a}) =: A_{\mathfrak{a}}^{(E,\Phi)}$ for some CM type (E, Φ) and ideal $\mathfrak{a} \subset \mathcal{O}_E$.

(b) Furthermore, any complex torus of the form $A_{\mathfrak{a}}^{(E,\Phi)}$ is algebraic.

A **CM abelian variety** is just a product of simple abelian varieties, each satisfying the conditions in *(a)*. We will suppress the superscript (E, Φ) when the CM type is understood. Note that in *(i)*, the torus may be of dimension less than g, the so-called *degenerate* case.

Example 3.4. $\Phi(\mathfrak{a})$ means the $2g$-lattice

$$\left\{ \left(\begin{array}{c} \phi_1(a) \\ \vdots \\ \phi_g(a) \end{array} \right) \middle| \, a \in \mathfrak{a} \right\}.$$

For Example 3.2 (c) above,

$$A_{\mathcal{O}_{\mathbb{Q}(\zeta_5)}} = \mathbb{C}^2 \Big/ \mathbb{Z} \left\langle \left(\begin{array}{c} 1 \\ 1 \end{array} \right), \left(\begin{array}{c} e^{2\pi i/5} \\ e^{4\pi i/5} \end{array} \right), \left(\begin{array}{c} e^{4\pi i/5} \\ e^{3\pi i/5} \end{array} \right), \left(\begin{array}{c} e^{6\pi i/5} \\ e^{2\pi i/5} \end{array} \right) \right\rangle .$$

The interesting points in Proposition 3.3 are:

- where does the CM field come from?
- why is $A_{\mathfrak{a}}^{(E,\Phi)}$ polarized?

In lieu of a complete proof we address these issues.

Proof of (i) $\Longrightarrow$ (iii). $V = H^1(A)$ is polarized by some Q. Set

$$\mathcal{E} := End_{\text{HS}}(H^1(A)) = \left(Z_{GL(V)}(M) \right)(\mathbb{Q}) \cup \{0\},$$

where the centralizer $Z_{GL(V)}(M)$ is a $\mathbb{Q}$-algebraic group and contains a maximal torus T. Since T commutes with M and is maximal, T contains M (possibly properly). One deduces that

- $T(\mathbb{Q}) \cup \{0\} =: E\ (\underset{\eta}{\hookrightarrow} \mathcal{E})$ is a field,
- V is a 1-dimensional vector space over E, and
- E is actually all of $\mathcal{E}$.

M diagonalizes with respect to a Hodge basis

$$\omega_1, \ldots, \omega_g; \bar{\omega}_1, \ldots, \bar{\omega}_g$$

such that $\sqrt{-1}Q(\omega_i, \bar{\omega}_j) = \delta_{ij}$. The maximal torus in $GL(V)$ this basis defines, centralizes M hence must be T.

Now write $\mathcal{H}_E = \{\phi_1, \ldots, \phi_{2g}\}$, $E = \mathbb{Q}(\xi)$ and

$$m_\xi(\lambda) = \prod_{i=1}^{2g}(\lambda - \phi_i(\xi))$$

for the minimal polynomial of ξ, hence $\eta(\xi)$. Up to reordering, we therefore have

$$[\eta(\xi)]_\omega = diag(\{\phi_i(\xi)\}_{i=1}^{2g})$$

for the matrix of "multiplication by ξ" with respect to the Hodge basis. Since $\eta(\xi) \in GL(V)$ (*a fortiori* $\in GL(V_\mathbb{R})$), and $\phi_j(\xi)$ determines ϕ_j,

$$\omega_{i+g} = \bar{\omega}_i \quad \Longrightarrow \quad \phi_{i+g} = \bar{\phi}_i.$$

Define the **Rosati involution** $\dagger : \mathcal{E} \to \mathcal{E}$ by

$$Q(\varepsilon^\dagger v, w) = Q(v, \varepsilon w) \quad \forall v, w \in V.$$

This produces $\rho := \eta^{-1} \circ \dagger \circ \eta \in \mathcal{G}_E$, and we compute

$$\begin{aligned} \phi_{i+g}(e)Q(\omega_i, \omega_{i+g}) \ &= \ Q(\omega_i, \eta(e)\omega_{i+g}) \ = \ Q(\eta(e)^\dagger \omega_i, \omega_{i+g}) \\ &= \ Q(\eta(\rho(e))\omega_i, \omega_{i+g}) \ = \ \phi_i(\rho(e))Q(\omega_i, \omega_{i+g}), \end{aligned}$$

which yields $\phi_i \circ \rho = \bar{\phi}_i$. □

Proof of (b). We have the following construction of $H^1(A)$: let V be a $2g$-dimensional $\mathbb{Q}$-vector space with identification

$$\beta :\ E \overset{\cong}{\to} V$$

inducing (via multiplication in E)

$$\eta :\ E \hookrightarrow End(V).$$

Moreover, there is a basis $\omega = \{\omega_1, \ldots, \omega_g; \bar{\omega}_1, \ldots, \bar{\omega}_g\}$ of $V_\mathbb{C}$ with respect to which

$$[\eta_\mathbb{C}(e)]_\omega = diag\{\phi_1(e), \ldots, \phi_g(e); \bar{\phi}_1(e), \ldots, \bar{\phi}_g(e)\},$$

and we set $V^{1,0} := \mathbb{C}\langle\omega_1,\ldots,\omega_g\rangle$. This gives $\eta(e) \in End_{\mathrm{HS}}(V)$ and $V \underset{\mathrm{HS}}{\cong} H^1(A)$.

Now, there exists a $\xi \in E$ such that $\sqrt{-1}\phi_i(\xi) > 0$ for $i = 1,\ldots,g$, and we can put

$$Q(\beta(e),\beta(\tilde{e})) := Tr_{E/\mathbb{Q}}(\xi\cdot e\cdot\rho(\tilde{e})) : \; V\times V \;\longrightarrow\; \mathbb{Q}.$$

Over $\mathbb{C}$, this becomes

$$[Q]_\omega = \begin{pmatrix} & 0 & & \phi_1(\xi) & & \\ & & & & \ddots & \\ & & & & & \phi_g(\xi) \\ \bar{\phi}_1(\xi) & & & & & \\ & \ddots & & & 0 & \\ & & \bar{\phi}_g(\xi) & & & \end{pmatrix}.$$

□

Remark 3.5. $N_{\Phi'}$ algebraizes to a homomorphism of algebraic groups

$$\mathcal{N}_{\Phi'} : \mathrm{Res}_{E'/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$$

which gives $N_{\Phi'}$ on the $\mathbb{Q}$-points, and the MT group

$$M_{H^1(A)} \cong \mathrm{image}(\mathcal{N}_{\Phi'}).$$

Let E be a CM field with $[E:\mathbb{Q}] = 2g$. In algebraic number theory we have

$\mathcal{I}(E)$	the monoid of nonzero ideals in $\mathcal{O}_E$,
$\mathcal{J}(E)$	the group of fractional ideals (of the form $e\cdot I$, $e\in E^*$ and $I\in\mathcal{I}(E)$), and
$\mathcal{P}(E)$	the subgroup of principal fractional ideals (of the form $(e) := e\cdot\mathcal{O}_E$, $e\in E^*$).

The (abelian!) **ideal class group**

$$Cl(E) := \frac{\mathcal{J}(E)}{\mathcal{P}(E)},$$

or more precisely the **class number**

$$h_E := |Cl(E)|,$$

expresses (if $\neq 1$) the failure of $\mathcal{O}_E$ to be a principal ideal domain (and to have unique factorization). Each class $\tau \in Cl(E)$ has a representative $I \in \mathcal{I}(E)$ with norm bounded by the **Minkowski bound**, which implies h_E is finite.

Now let

$$\mathrm{Ab}(\mathcal{O}_E,\Phi) := \frac{\left\{ A_{\mathfrak{a}}^{(E,\Phi)} \,\middle|\, \mathfrak{a}\in\mathcal{J}(E)\right\}}{\text{isomorphism}},$$

where the numerator denotes abelian g-folds with $\mathcal{O}_E \subset End(A)$ acting on T_0A through Φ. They are all isogenous, and multiplication by any $e \in E^*$ gives an isomorphism

$$A_{\mathfrak{a}} \underset{\cong}{\longrightarrow} A_{e\mathfrak{a}}.$$

Hence we get a bijection

$$\begin{array}{ccc} Cl(E) & \underset{\cong}{\longrightarrow} & \mathrm{Ab}(\mathcal{O}_E, \Phi) \\ [\mathfrak{a}] & \longmapsto & A_{\mathfrak{a}} \end{array}.$$

Example 3.6. The class number of $\mathbb{Q}(\sqrt{-5})$ is 2. Representatives of $\mathrm{Ab}(\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}, \{\phi\})$ are the elliptic curves $\mathbb{C}/\mathbb{Z}\left\langle 2, 1+\sqrt{-5}\right\rangle$ and $\mathbb{C}/\mathbb{Z}\left\langle 1, \sqrt{-5}\right\rangle$.

The key point now is to notice that for $A \in \mathrm{Ab}(\mathcal{O}_E, \Phi)$ and $\sigma \in Aut(\mathbb{C})$,

$${}^{\sigma}A \in \mathrm{Ab}(\mathcal{O}_E, \sigma\Phi).$$

This is because endomorphisms are given by algebraic cycles, so that the internal ring structure of $\mathcal{E}$ is left unchanged by Galois conjugation; what changes are the eigenvalues of its action on T_0A. From the definition of E' we see that

$$Gal(\mathbb{C}/E') \text{ acts on } \mathrm{Ab}(\mathcal{O}_E, \Phi),$$

which suggests that the individual abelian varieties should be defined over an extension of E' of degree $h_{E'}$. This isn't exactly true if $Aut(A) \neq \{\mathrm{id}\}$, but the argument does establish that any CM abelian variety is defined over $\bar{\mathbb{Q}}$.

B. Class field theory. In fact, it gets much better: not only is there a distinguished field extension H_L/L of degree h_L for any number field L; there is an isomorphism

$$Gal(H_L/L) \overset{\cong}{\longleftarrow} Cl(L)\ . \tag{3.1}$$

To quote Chevalley, "L contains within itself the elements of its own transcendence".

Idea of the construction of (3.1). Let $\tilde{L}/L$ be an extension of degree d which is

- **abelian**: Galois with $Gal(\tilde{L}/L)$ abelian
- **unramified**: for each prime $\mathfrak{p} \in \mathcal{I}(L)$, $\mathfrak{p}.\mathcal{O}_{\tilde{L}} = \prod_{i=1}^{r} \mathfrak{P}_i$ for some $r|d$.

Writing $N(\mathfrak{p}) := |\mathcal{O}_L/\mathfrak{p}|$, the extension $(\mathcal{O}_{\tilde{L}}/\mathfrak{P}_i)/(\mathcal{O}_L/\mathfrak{p})$ of finite fields has degree $N(\mathfrak{p})^{d/r}$. The image of its Galois group in $Gal(\tilde{L}/L)$ by

$$\begin{tikzcd}
Gal(\frac{(\mathcal{O}_{\tilde{L}}/\mathfrak{P}_i)}{(\mathcal{O}_L/\mathfrak{p})}) & \left\{\sigma \in Gal(\tilde{L}/L) \,\middle|\, \sigma\mathfrak{P}_i = \mathfrak{P}_i\right\} \arrow[l, "\cong"'] \arrow[r, hook] & Gal(\tilde{L}/L) \\
\left\{\begin{matrix} \alpha \to \alpha^{N(p)} \\ (\text{mod } \mathfrak{P}_i) \end{matrix}\right\} \arrow[u, squiggly, "\text{generated by}"] \arrow[rr, mapsto] & & =: Frob_{\mathfrak{p}}
\end{tikzcd}$$

is (as the notation suggests) independent of i, yielding a map from

$$\left\{\begin{matrix}\text{prime ideals} \\ \text{of } L\end{matrix}\right\} \longrightarrow Gal(\tilde{L}/L).$$

Taking $\tilde{L}$ to be the Hilbert class field

$$H_L := \text{maximal unramified abelian extension of } L,$$

this leads (eventually) to (3.1). □

More generally, given $I \in \mathcal{I}(L)$, we have the **ray class group mod I**

$$Cl(I) = \frac{\text{fractional ideals prime to } I}{\text{principal fractional ideals with generator } \equiv 1 \pmod{I}}$$

and **ray class field mod I**

$$L_I = \begin{matrix}\text{the maximal abelian extension of } L \text{ in which} \\ \text{all primes } \equiv 1 \text{ mod } I \text{ split completely.}\end{matrix}$$

(Morally, L_I should be the maximal abelian extension in which primes dividing I are allowed to ramify, but this isn't quite correct.) There is an isomorphism

$$Gal(L_I/L) \underset{\cong}{\longleftarrow} Cl(I),$$

and $L_I \supseteq H_L$ with equality when $I = \mathcal{O}_L$.

Example 3.7. For $L = \mathbb{Q}$, $L_{(n)} = \mathbb{Q}(\zeta_n)$.

To deal with the infinite extension

$$L^{ab} := \text{maximal abelian extension of } L \ \ (\subset \bar{\mathbb{Q}}),$$

of L, we have to introduce the adéles.

In studying abelian varieties one considers, for $\ell \in \mathbb{Z}$ prime, the finite groups of ℓ-torsion points $A[\ell]$; multiplication by ℓ gives maps

$$\cdots \to A[\ell^{n+1}] \to A[\ell^n] \to \cdots .$$

If we do the same thing on the unit circle $S^1 \subset \mathbb{C}^*$, we get

$$\begin{array}{ccccccc} \cdots & \to & S^1[\ell^{n+1}] & \stackrel{\cdot\ell}{\to} & S^1[\ell^n] & \to & \cdots \\ & & \| & & \| & & \\ & \to & \mathbb{Z}/\ell^{n+1}\mathbb{Z} & \underset{\substack{\text{natural}\\ \text{map}}}{\to} & \mathbb{Z}/\ell^n\mathbb{Z} & \to & \end{array}$$

and one can define the **ℓ-adic integers** by the inverse limit

$$\mathbb{Z}_\ell := \lim_{\substack{\longleftarrow \\ n}} \mathbb{Z}/\ell^n\mathbb{Z}.$$

An element of this limit is by definition an infinite sequence of elements in the $\mathbb{Z}/\ell^n\mathbb{Z}$ mapping to each other. There is the natural inclusion

$$\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell,$$

and

$$\mathbb{Q}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Elements of $\mathbb{Q}_\ell$ can be written as power series $\sum_{i\geq n} a_i\ell^i$ for some $n \in \mathbb{Z}$ ($n \geq 0$ for elements of $\mathbb{Z}_\ell$). $\mathbb{Q}_\ell$ can also be thought of as the completion of $\mathbb{Q}$ with respect to the metric given by

$$d(x,y) = \frac{1}{\ell^n} \text{ if } x - y = \ell^n\frac{a}{b} \text{ with } a, b \text{ relatively prime to } \ell.$$

The resulting topology on $\mathbb{Z}_\ell$ makes

$$U_n(\alpha) := \{\,\alpha + \lambda\ell^n \mid \lambda \in \mathbb{Z}_\ell\}$$

into "the open disk about $\alpha \in \mathbb{Z}_\ell$ of radius $\frac{1}{\ell^n}$". $\mathbb{Z}_\ell$ itself is compact and totally disconnected.

Now set

$$\hat{\mathbb{Z}} := \lim_{\substack{\longleftarrow \\ n\,\in\,\mathbb{Z}}} \mathbb{Z}/n\mathbb{Z},$$

which is isomorphic to $\prod_\ell \mathbb{Z}_\ell$ by the Chinese remainder theorem. The **finite adéles** appear naturally as

$$\mathbb{A}_f := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \prod_\ell{}' \mathbb{Q}_\ell,$$

where $\prod'$ means the ∞-tuples with all but finitely many entries in $\mathbb{Z}_\ell$. The "full" **adéles** are constructed by writing

$$\begin{array}{rcccl} \mathbb{A}_{\mathbb{Z}} & := & \mathbb{R} \times \hat{\mathbb{Z}} & & \\ \mathbb{A}_{\mathbb{Q}} & = & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}} & = & \mathbb{R} \times \mathbb{A}_f\,, \end{array}$$

which generalizes for a number field L to

$$\mathbb{A}_L = L \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}} = \mathbb{R}^{[L:\mathbb{Q}]} \times \underbrace{\prod_{\mathfrak{P} \in \mathcal{I}(L)}{}' L_{\mathfrak{P}}}_{\mathbb{A}_{L,f}},$$

where $\prod'$ means all but finitely many entries in $(\mathcal{O}_L)_{\mathfrak{P}}$.

For a $\mathbb{Q}$-algebraic group G, we can define

$$G(\mathbb{A}_f) := \prod{}' G(\mathbb{Q}_\ell)$$

$$G(\mathbb{A}_{\mathbb{Q}}) := G(\mathbb{R}) \times G(\mathbb{A}_f)$$

with generalizations to $\mathbb{A}_L$ and $\mathbb{A}_{L,f}$. Here $\prod'$ simply means that for some (hence every) embedding $G \hookrightarrow GL_N$, all but finitely many entries lie in $G(\mathbb{Z}_\ell)$. The idéles

$$\mathbb{A}^\times_{(f)} = \mathbb{G}_m(\mathbb{A}_{(f)})$$

$$\mathbb{A}^\times_{L(,f)} = (\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m)(\mathbb{A}_{(f)}) = \mathbb{G}_m(\mathbb{A}_{L(,f)})$$

were historically defined first; Weil introduced “adéle” – also (intentionally) a girl’s name – as a contraction of “additive idéle”. The usual norm $N_{\tilde{L}/L}$ and reflex norm $N_{\Phi'}$ extend to maps of idéles, using the formulation of these maps as morphisms of $\mathbb{Q}$-algebraic groups.

Returning to $S^1 \subset \mathbb{C}^*$, let ζ be an N^{th} root of unity and $a = (a_n) \in \hat{\mathbb{Z}}$; then

$$\zeta \mapsto \zeta^a := \zeta^{a_N} \text{ defines an action of } \hat{\mathbb{Z}}$$

$$\text{on the torsion points of } S^1 \text{ (which generate } \mathbb{Q}^{ab}).$$

The cyclotomic character

$$\chi : Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \xrightarrow{\cong} \hat{\mathbb{Z}}^\times \cong \mathbb{Q}^\times \backslash \mathbb{A}^\times_{\mathbb{Q},f}$$

is defined by

$$\sigma(\zeta) =: \zeta^{\chi(\sigma)},$$

and we can think of it as providing a “continuous envelope”[4] for the action of a given σ on any finite order of torsion. The **Artin reciprocity map** is simply its inverse

$$\mathrm{art}_{\mathbb{Q}} := \chi^{-1}$$

[4]I use this term because the automorphisms of $\mathbb{C}$ other than complex conjugation induce highly discontinuous (non-measurable!) maps on the complex points of a variety over $\mathbb{Q}$. But if one specifies a discrete set of points, it is sometimes possible to produce a continuous (even analytic/algebraic) automorphism acting in the same way on those points.

for $\mathbb{Q}$. Assuming now that L is totally imaginary, this generalizes to

$$\begin{tikzcd} L^\times \backslash \mathbb{A}^\times_{L,f} \arrow[r, two heads, "\mathrm{art}_L"] \arrow[d, "(\#)"] & Gal(L^{ab}/L) \arrow[d, two heads] \\ L^\times \backslash \mathbb{A}^\times_{L,f} / N_{\tilde{L}/L}(\mathbb{A}^\times_{\tilde{L},f}) \arrow[r, "\cong"] & Gal(\tilde{L}/L). \end{tikzcd}$$

If $\tilde{L} = L_I$ then the double-coset turns out to be $Cl(I)$, so this recovers the earlier maps for ray class fields. (For $\tilde{L} = H_L$, we can replace $N_{\tilde{L}/L}(\mathbb{A}^\times_{\tilde{L},f})$ by $\widehat{\mathcal{O}_L}$.) The correspondence between

$$\boxed{\begin{array}{c}\text{open subgroups}\\ \text{of } \mathbb{A}^\times_{L,f}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{finite abelian}\\ \text{extensions of } L\end{array}}$$

is the essence of class field theory. Also note that (#) gives compatible maps to all the class groups of L, so that $\mathbb{A}^\times_{L,f}$ acts on them.

C. Main Theorem of CM. Now let's bring the adéles to bear upon abelian varieties. Taking the product of the **Tate modules**

$$T_\ell A := \varprojlim_n A[\ell^n] (= \text{rank-}2g \text{ free } \mathbb{Z}_\ell\text{-module})$$

of an abelian g-fold yields

$$T_f A := \prod_\ell T_\ell A,$$

$$V_f A := T_f A \otimes_{\mathbb{Z}} \mathbb{Q} (= \text{rank-}2g \text{ free } \mathbb{A}_f\text{-module})$$

with (for example)

$$Aut(V_f A) \cong GL_{2g}(\mathbb{A}_f).$$

The "main theorem", due to Shimura and Taniyama, is basically a detailed description of the action of $Gal(\mathbb{C}/E')$ on $\mathrm{Ab}(\mathcal{O}_E, \Phi)$ and the torsion points of the (finitely many isomorphism classes of) abelian varieties it classifies.

Theorem 3.8. *Let $A_{[\mathfrak{a}]} \in Ab(\mathcal{O}_E, \Phi)$ and $\sigma \in Gal(\mathbb{C}/E')$ be given. For any $a \in \mathbb{A}^\times_{E',f}$ with $art_{E'}(a) = \sigma|_{(E')^{ab}}$, we have:*
(a) ${}^\sigma A_{[\mathfrak{a}]} \cong A_{N_{\Phi'}(a).[\mathfrak{a}]}$ (where $N_{\Phi'}(a) \in \mathbb{A}^\times_{E,f}$, and $N_{\Phi'}(a).[\mathfrak{a}]$ depends only on $\sigma|_{H_{E'}}$); and
(b) there exists a unique E-linear isogeny $\alpha : A_{[\mathfrak{a}]} \to {}^\sigma A_{[\mathfrak{a}]}$ such that

$$\alpha\left(N_{\Phi'}(a).x\right) = {}^\sigma x \qquad (\forall x \in V_f A).$$

Sketch of proof for (b).
• ${}^{\sigma}A \in \mathrm{Ab}(\mathcal{O}_E, \Phi) \implies \exists$ (E-linear) isogeny $\alpha : A \to {}^{\sigma}A$;
• $V_f A$ is free of rank 1 over $\mathbb{A}_{E,f}$;
• the composition $V_f(A) \xrightarrow{\sigma} V_f({}^{\sigma}A) \xrightarrow{V_f(\alpha)^{-1}} V_f(A)$ is $\mathbb{A}_{E,f}$-linear, so is just multiplication by some $s \in \mathbb{A}^{\times}_{E,f}$;
• s is independent (up to $E^{\times}$) of the choice of α, defining the horizontal arrow of

$$
\begin{tikzcd}
Gal(\mathbb{C}/E') \arrow[r] \arrow[dr] & E^{\times}\backslash\mathbb{A}^{\times}_{E,f} \\
& Gal((E')^{ab}/E') \arrow[u] \\
& (E')^{\times}\backslash\mathbb{A}^{\times}_{E',f} \arrow[u, "\mathrm{art}_{E'}"'] \arrow[uu, bend right, "(*)"']
\end{tikzcd}
\tag{3.2}
$$

• A is defined over a number field k; the Shimura-Taniyama computation of the prime decompositions of the elements of $E \cong End(A)_{\mathbb{Q}}$ reducing to various Frobenius maps (in residue fields of k) then shows that the vertical map $(*)$ in (3.2) is $N_{\Phi'}$. Hence

$$N_{\Phi'}(a) = s = V_f(\alpha)^{-1} \circ \sigma$$

which gives the formula in (b). □

So what does (b) mean? Like the cyclotomic character, we get a very nice interpretation when we restrict to the action on m-torsion points of A for any fixed $m \in \mathbb{N}$:

Corollary 3.9. *There exists a unique E-linear isogeny $\alpha_m : A \to {}^{\sigma}A$ such that*

$$\alpha_m(x) = {}^{\sigma}x \qquad\qquad (\forall x \in A[m]).$$

That is, α_m provides a "continuous envelope" for the action of automorphisms of $\mathbb{C}$ on special points.

4. Shimura varieties

A. Three key adelic lemmas. Besides the main theorem of CM, there is another (related) connection between the class field theory described in III.B and abelian varieties. The tower of ray class groups associated to the ideals of a CM field E can be expressed as

$$E^{\times}\backslash\mathbb{A}^{\times}_{E,f}/\mathfrak{U}_I \;\cong\; T(\mathbb{Q})\backslash T(\mathbb{A}_f)/\mathfrak{U}_I \tag{4.1}$$

where

$$\mathfrak{U}_I := \left\{ (a_\mathfrak{p})_{\mathfrak{p}\in\mathcal{I}(E)\text{ prime}} \in \mathbb{A}_{E,f}^\times \;\middle|\; \begin{array}{cc} a_\mathfrak{p} \in (\mathcal{O}_E)_\mathfrak{p} & \text{for all } \mathfrak{p}, \\ a_\mathfrak{p} \equiv 1 \bmod \mathfrak{p}^{ord_\mathfrak{p} I} & \text{for the finitely many } \mathfrak{p} \text{ dividing } I \end{array} \right\}$$

is a compact open subgroup of $\mathbb{A}_{E,f}^\times = T(\mathbb{A}_f)$ and

$$T = Res_{E/\mathbb{Q}}\mathbb{G}_m.$$

(4.1) may be seen as parametrizing abelian varieties with CM by a type (E, Φ) and having fixed level structure — which "refines" the set parametrized by $Cl(E)$. Shimura varieties give a way of extending this story to more general abelian varieties with other endomorphism and Hodge-tensor structures, as well as the other families of Hodge structures parametrized by Hermitian symmetric domains.

The first fundamental result we will need is

Lemma 4.1. *For T any $\mathbb{Q}$-algebraic torus, and $K_f \subset T(\mathbb{A}_f)$ any open subgroup, $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_f$ is finite.*

Sketch of Proof. This follows from the definition of compactness if we can show $T(\mathbb{Q})\backslash T(\mathbb{A}_f)$ is compact. For any number field F, the latter is closed in $T(F)\backslash T(\mathbb{A}_{F,f})$, and for some F over which T splits the latter is $(F^\times\backslash\mathbb{A}_{F,f}^\times)^{\dim(T)}$. Finally, by the Minkowski bound

$$F^\times\backslash\mathbb{A}_{F,f}^\times/\widehat{\mathcal{O}_F} = Cl(F) \text{ is finite,}$$

and $\widehat{\mathcal{O}_F}$ is compact (like $\hat{\mathbb{Z}}$), so $F^\times\backslash\mathbb{A}_{F,f}^\times$ is compact. □

For a very different class of $\mathbb{Q}$-algebraic groups, we have the contrasting

Lemma 4.2. *Suppose $G/\mathbb{Q}$ is semisimple and simply connected, of noncompact type*[5]*; then*
(a) [**Strong approximation**] *$G(\mathbb{Q}) \subseteq G(\mathbb{A}_f)$ is dense, and*
(b) For any open $K_f \subseteq G(\mathbb{A}_f)$, $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot K_f$.

Remark 4.3. Lemma 4.2(b) implies that the double coset $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f$ is trivial. Note that double cosets are now essential as we are in the nonabelian setting.

Sketch of (a) $\Longrightarrow$ (b). Given $(\gamma_\ell) \in G(\mathbb{A}_f)$, $U := (\gamma_\ell) \cdot K_f$ is an open subset of $G(\mathbb{A}_f)$, hence by (a) there exists a $g \in U \cap G(\mathbb{Q})$. Clearly $g = (\gamma_\ell) \cdot k$ for some $k \in K_f$, and so $(\gamma_\ell) = g \cdot k^{-1}$. □

[5]i.e. none of its simple almost-direct $\mathbb{Q}$-factors G_i have $G_i(\mathbb{R})$ compact

Nonexample 4.4. $\mathbb{G}_m$, which is of course reductive but not semisimple. If (b) held, then the ray class groups of $\mathbb{Q}$ (which are $\cong (\mathbb{Z}/\ell\mathbb{Z})^\times$) would be trivial. But even more directly, were $\mathbb{Q}^\times$ dense in $\mathbb{A}_f^\times$, there would be $q \in \mathbb{Q}^\times$ close to any $(a_\ell) \in \prod \mathbb{Z}_\ell^\times$. This forces [the image in $\mathbb{A}_f^\times$ of] q to lie in $\prod \mathbb{Z}_\ell^\times$, which means for each ℓ that (in lowest terms) the numerator and denominator of q are prime to ℓ. So $q = \pm 1$, a contradiction.

Finally, for a general $\mathbb{Q}$-algebraic group G, we have

Lemma 4.5. *The congruence subgroups of $G(\mathbb{Q})$ are precisely the $K_f \cap G(\mathbb{Q})$ (intersection in $G(\mathbb{A}_f)$) for compact open $K_f \subseteq G(\mathbb{A}_f)$.*

Sketch of Proof. For $N \in \mathbb{N}$ the

$$K(N) := \left\{ (g_\ell)_{\ell \text{ prime}} \in G(\mathbb{A}_f) \;\middle|\; \begin{array}{cc} g_\ell \in G(\mathbb{Z}_\ell) & \text{for all } \ell, \\ g_\ell \equiv \mathbf{e} \bmod \ell^{ord_\ell N} & \text{for each } \ell | N \end{array} \right\}$$

are compact open in $G(\mathbb{A}_f)$, and

$$\begin{aligned} K(N) \cap G(\mathbb{Q}) &= \left\{ g \in G(\mathbb{Z}) \;\middle|\; \begin{array}{c} g_\ell \equiv \mathbf{e} \bmod \ell^{ord_\ell N} \\ \text{for each } \ell|N \end{array} \right\} \\ &= \left\{ g \in G(\mathbb{Z}) \mid g \equiv \mathbf{e} \bmod N \right\} \quad = \quad \Gamma(N). \end{aligned}$$

In fact, the $K(N)$ are a *basis* of open subsets containing $\mathbf{e}$. So any compact *open* K_f contains some $K(N)$, and

$$(K_f \cap G(\mathbb{Q})) / (K(N) \cap G(\mathbb{Q})) \subseteq K_f / K(N)$$

is a discrete subgroup of a compact set, and therefore finite. □

In some sense, K_f is *itself* the congruence condition.

B. Shimura data. A $\left\{ \begin{array}{c} \textbf{Shimura datum (SD)} \\ \text{[resp.] } \textbf{connected SD} \end{array} \right.$ is a pair $\left(G, \left\{ \begin{array}{c} \tilde{X} \\ X \end{array} \right. \right)$ consisting of

- $G := \left\{ \begin{array}{c} \text{reductive} \\ \text{semisimple} \end{array} \right.$ algebraic group defined over $\mathbb{Q}$

and

- $\left\{ \begin{array}{c} \tilde{X} \\ X \end{array} \right. := \text{a} \left\{ \begin{array}{c} G(\mathbb{R})\text{-} \\ G^{ad}(\mathbb{R})^+\text{-} \end{array} \right.$ conjugacy class of homomorphisms

$$\tilde{\varphi} : \mathbb{S} \to \left\{ \begin{array}{c} G_{\mathbb{R}} \\ G_{\mathbb{R}}^{ad} \end{array} \right. ,$$

satisfying the axioms:

(**SV1**) only $z/\bar{z}$, 1, $\bar{z}/z$ occur as eigenvalues of

$$Ad \circ \tilde{\varphi} : \mathbb{S} \to GL\left(Lie(G^{ad})_{\mathbb{C}}\right);$$

(**SV2**) $\Psi_{Ad(\tilde{\varphi}(i))} \in Aut(G^{ad}_{\mathbb{R}})$ is Cartan;

(**SV3**) G^{ad} has no $\mathbb{Q}$-factor on which the projection of every $(Ad \circ)\, \tilde{\varphi} \in \left\{ \begin{array}{c} \tilde{X} \\ X \end{array} \right.$ is trivial

(**SV4**) the weight homomorphism $\begin{tikzcd} \mathbb{G}_m \arrow[r, hook] \arrow[rr, bend left, "w_{\tilde{\varphi}}"] & \mathbb{S} \arrow[r, "\tilde{\varphi}"'] & G \end{tikzcd}$ (*a priori* defined over $\mathbb{R}$) is defined over $\mathbb{Q}$;

(**SV5**) $(Z^\circ/w_{\tilde{\varphi}}(\mathbb{G}_m))\,(\mathbb{R})$ is compact;[6]

and

(**SV6**) Z° splits over a CM field.

In the "connected" case, SV4-6 are trivial, while SV1-3 already imply:

- X is a Hermitian symmetric domain (in the precise sense of *III* in §1.**B**, with $\phi = \mu_{\tilde{\varphi}} \circ \jmath$); and
- G is of noncompact type [from SV3], but with ker $(G(\mathbb{R})^+ \twoheadrightarrow Hol(X)^+)$ compact [from SV2],

where the surjective arrow is defined by the action on the conjugacy class of $\tilde{\varphi}$.

In these definitions, axioms SV4-6 are sometimes omitted: for example, canonical models exist for Shimura varieties without them. We include them here from the beginning because they hold in the context of Hodge theory. Indeed, consider a full Mumford-Tate domain $\tilde{X}$ (cf. §1.**D**) for polarized $\mathbb{Q}$-Hodge structures with generic Mumford-Tate group G. Regarded as a pair, $(G, \tilde{X})$ always satisfies

SV2: by the second Hodge-Riemann bilinear relation (cf. Exercise 1.12);

SV3: otherwise the generic MT group would be a proper subgroup of G;

SV4: because the weight filtration is split over $\mathbb{Q}$;

SV5: as G is a Mumford-Tate group, $G/w_{\tilde{\varphi}}(\mathbb{G}_m)$ contains a $G(\mathbb{R})^+$-conjugacy class of anisotropic maximal real tori, and these contain $Z^\circ_{\mathbb{R}}/w_{\tilde{\varphi}}(\mathbb{G}_m)$;

SV6: since G is defined over $\mathbb{Q}$, the conjugacy class contains tori defined over $\mathbb{Q}$; and so $\tilde{X}$ contains a $\tilde{\varphi}$ factoring through some such rational T. This defines a polarized CM Hodge structure, with MT group a $\mathbb{Q}$-torus $T_0 \subseteq T$ split over a CM field (cf. §3). If $T_0 \not\supseteq Z^\circ$, then the projection of $\tilde{\varphi}$ to some $\mathbb{Q}$ -factor of Z° is trivial and then it is trivial for all

[6]SV5 is sometimes weakened: cf. [Milne2005, sec. 5].

its conjugates, contradicting SV3. Hence $T_0 \supseteq Z^\circ$ and Z° splits over the CM field.

Further, SV1 holds *if and only if* the IPR on $\tilde{X}$ is trivial.

What can one say about an arbitrary Shimura datum? First, any SD produces a connected SD by

- replacing G by G^{der} (which has the same G^{ad})
- replacing $\tilde{X}$ by a connected component X (which we may view as a $G^{ad}(\mathbb{R})^+$-conjugacy class of homomorphisms $Ad \circ \tilde{\varphi}$),

and so

$$\tilde{X} \textit{ is a finite union of Hermitian symmetric domains.}$$

Moreover, by SV2 and SV5 one has that $\Psi_{\tilde{\varphi}(i)} \in Aut(G_{\mathbb{R}}/w_{\tilde{\varphi}}(\mathbb{G}_m))$ is Cartan, so that there exist a symmetric bilinear form Q on a $\mathbb{Q}$-vector space V, and an embedding $\rho : G/w_{\tilde{\varphi}}(\mathbb{G}_m) \hookrightarrow Aut(V,Q)$, such that $Q(\cdot, (\rho\circ\tilde{\varphi})(i)\bar{\cdot}) > 0$. As in the "Hodge domains" described in the lectures by Griffiths, for any such faithful representation $\tilde{\rho} : G \hookrightarrow \mathbb{G}O(V,Q)$ with Q polarizing $\tilde{\rho}\circ\tilde{\varphi}$, $\tilde{X}$ is realized as a finite union of MT domains with trivial IPR. Let M denote the MT group of $\tilde{X}$. If $G = M$, then $\tilde{X}$ is exactly a "full MT domain" as described in §1.D. However, in general M is a normal subgroup of G with the same adjoint group; that the center can be smaller is seen by considering degenerate CM Hodge structures. Suffice it to say that (while the correspondence is slightly messy) all Shimura data, hence ultimately all Shimura varieties as defined below, have a Hodge-theoretic interpretation.

Now given a *connected* SD (G, X), we add one more ingredient: let

$$\Gamma \leq G^{ad}(\mathbb{Q})^+$$

be a torsion-free arithmetic subgroup, with inverse image in $G(\mathbb{Q})^+$ a congruence subgroup. Its image $\bar{\Gamma}$ in $Hol(X)^+$ is

(i) [torsion-free and] arithmetic: since $\ker(G(\mathbb{R})^+ \twoheadrightarrow Hol(X)^+)$ compact

(ii) isomorphic to Γ: $\Gamma \cap \ker(G(\mathbb{R})^+ \twoheadrightarrow Hol(X)^+) =$discrete $\cap$ compact = finite, hence torsion (and there is no torsion).

We may write

$$X(\Gamma) := \Gamma\backslash X \underset{(ii)}{=} \bar{\Gamma}\backslash X\,;$$

and $\bar{\Gamma}\backslash X$ is a locally symmetric variety by (i) and Baily-Borel. By Borel's theorem,

$$\Gamma \leq \Gamma' \implies X(\Gamma') \twoheadrightarrow X(\Gamma) \textit{ is algebraic.} \tag{4.2}$$

Definition 4.6. The **connected Shimura variety** associated with (G, X, Γ) is

$$Sh^{\circ}_{\Gamma}(G, X) := X(\Gamma).$$

Remark 4.7. Every $X(\Gamma)$ is covered by an $X(\Gamma')$ with Γ' the image of a congruence subgroup of $G(\mathbb{Q})^+$. If one works with "sufficiently small" congruence subgroups of $G(\mathbb{Q})$, then

- they belong to $G(\mathbb{Q})^+$;
- they have torsion-free image in $G^{ad}(\mathbb{Q})^+$;
- congruence $\implies$ arithmetic.

This will be tacit in what follows.

C. The adélic reformulation. Consider a connected SD (G, X) with G *simply connected*. By a result of Cartan, this means that $G(\mathbb{R})$ is connected, hence acts on X via Ad. Let $K_f \leq G(\mathbb{A}_f)$ be a ("sufficiently small") compact open subgroup and (referring to Lemma 4.5) $\Gamma = G(\mathbb{Q}) \cap K_f$ the corresponding subgroup of $G(\mathbb{Q})$. Replacing the earlier notation we write

$$Sh^{\circ}_{K_f}(G, X)$$

for the associated locally symmetric variety.

Proposition 4.8. *The connected Shimura variety* $Sh^{\circ}_{K_f}(G, X)$ $(\cong \Gamma\backslash X)$ *is homeomorphic to*

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f,$$

where the action defining the double quotient is

$$g.(\tilde{\varphi}, a).k \ := \ (\underbrace{g.\tilde{\varphi}}_{g\tilde{\varphi}g^{-1}}, \, gak\,).$$

Remark 4.9. If $G(\mathbb{R}) = G^{ad}(\mathbb{R})^+$, then the quotient $X \times G(\mathbb{A}_f)/K_f$ can be written $G(\mathbb{A}_{\mathbb{Q}})/K$, where $K = K_{\mathbb{R}} \times K_f$ with $K_{\mathbb{R}} \subset G(\mathbb{R})$ maximal compact.

Sketch of Proof. First note that $[\tilde{\varphi}] \mapsto [(\tilde{\varphi}, 1)]$ gives a well-defined map from $\Gamma\backslash X$ to the double-quotient, since

$$[\tilde{\varphi}] = [\tilde{\varphi}'] \implies \tilde{\varphi}' = \gamma.\tilde{\varphi} \ (\gamma \in \Gamma)$$
$$\implies (\tilde{\varphi}', 1) = \gamma.(\tilde{\varphi}, 1).\gamma^{-1}.$$

Now by assumption G is semisimple, simply connected, and of non-compact type. Lemma 4.2 implies that

$$G(\mathbb{A}_f) = G(\mathbb{Q}).K_f,$$

so that for any $(\tilde{\varphi}, a) \in X \times G(\mathbb{A}_f)$ we have

$$(\tilde{\varphi}, a) = (\tilde{\varphi}, gk) = g.(g^{-1}.\tilde{\varphi}, 1).k.$$

This implies that our map is surjective; for injectivity,

$$[(\tilde{\varphi}, 1)] = [(\tilde{\varphi}', 1)] \implies (\tilde{\varphi}', 1) = g.(\tilde{\varphi}, 1).k^{-1} = (g.\tilde{\varphi}, gk^{-1})$$

$$\implies g = k \in \Gamma,\ [\tilde{\varphi}'] = [\tilde{\varphi}].$$

Finally, since K_f is open, $G(\mathbb{A}_f)/K_f$ is discrete and the map

$$\begin{array}{ccc} X & \to & X \times G(\mathbb{A}_f)/K_f \\ \tilde{\varphi} & \mapsto & (\tilde{\varphi}, [1]) \end{array}$$

is a homeomorphism, which continues to hold upon quotienting both sides by a discrete torsion-free subgroup. □

More generally, given a Shimura datum $(G, \tilde{X})$ and compact open $K_f \leq G(\mathbb{A}_f)$, we simply *define* the **Shimura variety**

$$\boxed{Sh_{K_f}(G, \tilde{X}) \ := \ G(\mathbb{Q})\backslash \tilde{X} \times G(\mathbb{A}_f)/K_f,}$$

which will be a *finite disjoint union of locally symmetric varieties*. The disconnectedness, at first, would seem to arise from two sources:

(a) $\tilde{X}$ not connected (the effect of orbiting by $G(\mathbb{R})$)
(b) the failure of strong approximation for reductive groups (which gives multiple connected components even when $\tilde{X}$ is connected).

The first part of the Theorem below says (a) doesn't contribute: the indexing of the components is "entirely arithmetic". First, some

Notations 4.10.
• X denotes a connected component of $\tilde{X}$;
• $G(\mathbb{R})_+$ is the preimage of $G^{ad}(\mathbb{R})^+$ in $G(\mathbb{R})$, and $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$;
• $\nu : G \to T$ denotes the maximal abelian quotient (from §1.A), and the composition $Z \hookrightarrow G \stackrel{\nu}{\twoheadrightarrow} T$ is an isogeny;
• $T(\mathbb{R})^\dagger := Im(Z(\mathbb{R}) \to T(\mathbb{R}))$;
• $Y := T(\mathbb{Q})^\dagger \backslash T(\mathbb{Q})$.

Theorem 4.11. *(i)* $G(\mathbb{Q})_+\backslash X \times G(\mathbb{A}_f)/K_f \xrightarrow{\cong} G(\mathbb{Q})\backslash \tilde{X} \times G(\mathbb{A}_f)/K_f$.
(ii) The map

$$G(\mathbb{Q})_+\backslash X \times G(\mathbb{A}_f)/K_f \twoheadrightarrow G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K_f =: \mathcal{C}$$

"indexes" the connected components. If G^{der} is simply connected, we have

$$\mathcal{C} \underset{\nu}{\overset{\cong}{\to}} T(\mathbb{Q})^{\dagger}\backslash T(\mathbb{A}_f)/\nu(K_f) \;\cong\; T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/\nu(K_f).$$

Henceforth, "$\mathcal{C}$" denotes a set of representatives in $G(\mathbb{A}_f)$.
(iii) $Sh_{K_f}(G,\tilde{X}) \cong \amalg_{g\in\mathcal{C}}\Gamma_g\backslash X$, a finite *union, where*

$$\Gamma_g := gK_fg^{-1} \cap G(\mathbb{Q})_+.$$

That $|\mathcal{C}| < \infty$ is by Lemma 4.1. The rest is in [Milne2005, sec. 5] with two of the easier points (not requiring the hypothesis on G^{der}) described in the following

Exercise 4.12. (a) The preimage of $[1] \in \mathcal{C}$ is $\Gamma\backslash X \cong Sh^{\circ}_{K_f^{der}}(G^{der}, X)$, where $\Gamma = K_f \cap G(\mathbb{Q})_+$, and $K_f^{der} \leq G^{der}(\mathbb{A}_f)$ is some subgroup containing $K_f \cap G^{der}(\mathbb{A}_f)$.
(b) For $\gamma \in \Gamma_g$, $(\gamma.\tilde{\varphi}, g) \equiv (\tilde{\varphi}, g)$ in $Sh_{K_f}(G,\tilde{X})$.

The key point here is that *(iii)* is an *analytic description* of what will turn out to be $Gal(\mathbb{C}/E)$-conjugates of $\Gamma\backslash X$, in analogy to the formula for ${}^{\sigma}A_{[\mathfrak{a}]}$ in the main theorem of CM. Here E is the reflex field of $(G,\tilde{X})$, which will turn out to be the minimal field of definition of this very clever disjoint union.

Remark 4.13. (a) The definitions of Shimura varieties here are due to Deligne (in the late 1970s), who conjectured that they are fine moduli varieties for motives. The difficulty is in showing that the Hodge structures they parametrize are motivic.
(b) There is a natural construction of vector bundles $\mathcal{V}_{K_f}(\pi)$ on $Sh_{K_f}(G,\tilde{X})$, holomorphic sections of which are (holomorphic) **automorphic forms** of level K_f and type π. (Here, π is a representation of the parabolic subgroup of $G(\mathbb{C})$ stabilizing a point of the compact dual $\check{X}$.) Their higher cohomology groups are called **automorphic cohomology**.
(c) The inverse system[7] $Sh(G,\tilde{X})$ of all $Sh_{K_f}(G,\tilde{X})$ is a reasonably nice scheme on which $G(\mathbb{A}_f)$ operates, and so for example one could study the representation of this on $\varinjlim H^i(Sh_{K_f}(G,\tilde{X}), \mathcal{O}(\mathcal{V}_{K_f}(\pi)))$.

D. Examples.

I. 0*-dimensional Shimura varieties.* Let T be a $\mathbb{Q}$-algebraic torus satisfying SV5-6, $\tilde{\varphi} : \mathbb{S} \to T$ a homomorphism of $\mathbb{R}$-algebraic groups satisfying SV4, and $K_f \leq T(\mathbb{A}_f)$ be compact open. Then

$$Sh_{K_f}(T,\{\tilde{\varphi}\}) \;=\; T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_f$$

[7]suppressed in these notes since not needed for canonical models; it uses (4.2).

is finite by Lemma 4.1. These varieties arise as $\mathcal{C}$ in the Theorem, and also from CM Hodge structures, which by definition have a torus as MT group.

II. Siegel modular variety. Begin with a **$\mathbb{Q}$-symplectic space** (V, ψ) — i.e., a vector space $V/\mathbb{Q}$ together with a nondegenerate alternating form $\psi : V \times V \to \mathbb{Q}$ — and set $G =$

$$\mathbb{G}Sp(V,\psi) := \left\{ g \in GL(V) \;\middle|\; \begin{array}{c} \psi(gu, gv) = \chi(g)\psi(u,v)\ (\forall u, v \in V) \\ \text{for some } \chi(g) \in \mathbb{Q}^* \end{array} \right\}.$$

Exercise 4.14. Check that $\chi : G \to \mathbb{G}_m$ defines a character of G.

Now consider the spaces

$$X^{\pm} := \left\{ J \in Sp(V,\psi)(\mathbb{R}) \;\middle|\; \begin{array}{c} J^2 = -\mathbf{e} \text{ and} \\ \psi(u, Jv) \text{ is } \pm\text{-definite} \end{array} \right\}$$

of positive- and negative-definite symplectic complex structures on $V_{\mathbb{R}}$, and regard

$$\tilde{X} := X^+ \amalg X^-$$

as a set of homomorphisms via $\tilde{\varphi}(a + bi) := a + bJ$ (for $a + bi \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$). Then $G(\mathbb{R})$ acts transitively on $\tilde{X}$, and the datum $(G, \tilde{X})$ satisfies SV1-6. For any compact open K_f the attached Shimura variety is a **Siegel modular variety**.

Now consider the set

$$\mathcal{M}_{K_f} := \frac{\left\{ (A, Q, \eta) \;\middle|\; \begin{array}{l} A \text{ an abelian variety}/\mathbb{C}, \\ \pm Q \text{ a polarization of } H_1(A,\mathbb{Q}), \\ \eta : V_{\mathbb{A}_f} \to V_f(A) (\cong H_1(A, \mathbb{A}_f)) \text{ an} \\ \quad \text{isomorphism sending } \psi \mapsto a \cdot Q\ (a \in \mathbb{A}_f) \end{array} \right\}}{\cong},$$

where an isomorphism of triples

$$(A, Q, \eta) \underset{\cong}{\longrightarrow} (A', Q', \eta')$$

is an isogeny $f : A \to A'$ sending $Q' \mapsto q \cdot Q$ ($q \in \mathbb{Q}^*$) such that for some $k \in K_f$ the diagram

$$\begin{array}{ccc} V_{\mathbb{A}_f} & \xrightarrow{\eta} & V_f(A) \\ \downarrow{\scriptstyle \cdot k} & & \downarrow{\scriptstyle f} \\ V_{\mathbb{A}_f} & \xrightarrow{\eta'} & V_f(A') \end{array}$$

commutes. $\mathcal{M}_{K_f}$ is a moduli space for polarized abelian varieties with K_f-level structure. Write $\tilde{\varphi}_A$ for the Hodge structure on $H_1(A)$, and choose an isomorphism $\alpha : H_1(A, \mathbb{Q}) \to V$ sending ψ to Q (up to $\mathbb{Q}^*$).

Proposition 4.15. *The (well-defined) map*

$$\mathcal{M}_{K_f} \longrightarrow Sh_{K_f}(G, \tilde{X})$$

induced by

$$(A, Q, \eta) \longmapsto (\alpha \circ \tilde{\varphi}_A \circ \alpha^{-1}, \alpha \circ \eta)$$

is a bijection.

III. Shimura varieties of PEL type. This time we take (V, ψ) to be a **symplectic** $(B, *)$**-module**, i.e.

- (V, ψ) is a $\mathbb{Q}$-symplectic space;
- $(B, *)$ is a simple $\mathbb{Q}$-algebra with positive involution $*$ (that is, $tr_{(B\otimes_{\mathbb{Q}}\mathbb{R})/\mathbb{R}}(b^*b) > 0$); and
- V is a B-module and $\psi(bu, v) = \psi(u, b^*v)$.

We put $G := Aut_B(V) \cap \mathbb{G}Sp(V, \psi)$, which is of generalized SL, Sp, or SO type (related to the Albert classification) according to the structure of $(B_{\bar{\mathbb{Q}}}, *)$. (Basically, G is cut out of $\mathbb{G}Sp$ by fixing tensors in $T^{1,1}V$.) The (canonical) associated conjugacy class $\tilde{X}$ completes this to a Shimura datum, and the associated Shimura varieties parametrize **P**olarized abelian varieties with **E**ndomorphism and **L**evel structure (essentially a union of quotients of MT domains cut out by a subalgebra $\mathcal{E} \subseteq End(V)$). They include the Hilbert and Picard modular varieties.

IV. Shimura varieties of Hodge type. This is a straightforward generalization of Example *III*, with G cut out of $\mathbb{G}Sp(V, \psi)$ by fixing tensors of *all* degrees.

Remark 4.16. In both Examples *III* and *IV*, X is a subdomain of a Siegel domain, so "of Hodge type" excludes the type D and E Hermitian symmetric domains which still do yield Shimura varieties parametrizing equivalence classes of Hodge structures. So the last example is more general still:

V. Mumford-Tate groups/domains with vanishing IPR. This was already partially dealt with in §4.**B**. In the notation from §1.**D**, let $G := M_{\mathbf{t}}$ and

$$\tilde{X} = M_{\mathbf{t}}(\mathbb{R}).\tilde{\varphi} =: D_{\mathbf{t}}$$

for some $\tilde{\varphi}$ with MT group $M_{\mathbf{t}}$. (Note that X will be $D_{\mathbf{t}}^+$.) Under the assumption that $D_{\mathbf{t}}$ has trivial infinitesimal period relation, each choice of level structure

$$K_f \leq M_{\mathbf{t}}(\mathbb{A}_f)$$

will produce Shimura varieties. These take the form

$$\amalg_{g\in\mathcal{C}}\Gamma_g\backslash M_{\mathbf{t}}^{ad}(\mathbb{R})^+/K_{\mathbb{R}} \ \ (K_{\mathbb{R}} \text{ maximal compact})$$

with components parametrizing Γ_g-equivalence classes of higher weight Hodge structures.

In addition to the examples in §1.**D**, one prototypical example is the MT group and domain for HS of weight 3 and type $(1,n,n,1)$ with endomorphisms by an imaginary quadratic field, in such a way that the two eigenspaces are $V^{3,0}\oplus V^{2,1}$ and $V^{1,2}\oplus V^{3,0}$. (In particular, note that $M_{\mathbf{t}}(\mathbb{R})\cong \mathbb{G}U(1,n)$.) This has vanishing IPR and yields a Shimura variety.

Remark 4.17. (i) Even in the not-necessarily-Hermitian-symmetric case, where $D_{\mathbf{t}}=M(\mathbb{R})/H_{\tilde{\varphi}}$ is a general MT domain, there is still a canonical homeomorphism

(4.3)

$$\begin{aligned} M(\mathbb{Q})\backslash M(\mathbb{A})/H_{\tilde{\varphi}}\times K_f \quad &\cong \quad M(\mathbb{Q})\backslash D_{\mathbf{t}}\times M(\mathbb{A}_f)/K_f \\ &\overset{\text{homeo}}{\cong} \quad \amalg_{g\in\mathcal{C}}\Gamma_g\backslash D_{\mathbf{t}}^+ =: GS_{K_f}(M,D_{\mathbf{t}}). \end{aligned}$$

The right-hand object is a **Griffiths-Schmid variety**, which is in general only complex analytic. The construction of vector bundles $\mathcal{V}_{K_f}(\pi)$ mentioned in Remark 4.13(b) extends to this setting. $GS_{K_f}(M,D_{\mathbf{t}})$ is algebraic if and only if $D_{\mathbf{t}}^+$ fibers holomorphically or antiholomorphically over a Hermitian symmetric domain [GRT2013]. It is considered to be a Shimura variety under the slightly more stringent condition that the IPR be trivial.

(ii) Continuing in this more general setting, let $\alpha\in M(\mathbb{Q})$ and write $K_f\alpha K_f = \overset{\text{finite}}{\amalg_i} K_f a_i$ for some $a_i\in M(\mathbb{A}_f)$. This gives rise to an analytic correspondence

$$\sum_i \left\{\left([m_\infty,m_f],[m_\infty,m_f a_i^{-1}]\right)\right\}_{[m_\infty,m_f]\in M(\mathbb{A})}$$

in the self-product of the LHS of (4.3). The endomorphism induced in automorphic cohomology groups

$$\mathfrak{A}_i(\pi) := H^i\left(GS_{K_f}(M,D_{\mathbf{t}}),\mathcal{O}(\mathcal{V}_{K_f}(\pi))\right)$$

is called the **Hecke operator** associated to α.

(iii) For simplicity, assume that the representations π are 1-dimensional (so that the $\mathcal{V}_{K_f}(\pi)$ are line bundles) and that the Γ_g are co-compact.[8]

[8]Similar remarks apply in the non-co-compact case, except that one has to restrict to a subgroup of cuspidal classes in $\mathfrak{A}_i(\pi)$, whose behavior for $i>0$ is not yet well-understood.

In the Shimura variety case, the eigenvectors of Hecke operators in $\mathfrak{A}_0(\pi)$ are the arithmetically interesting automorphic forms. In the non-algebraic Griffiths-Schmid case, provided π is "regular", we have $\mathfrak{A}_0(\pi) = \{0\}$; however, the automorphic cohomology $\mathfrak{A}_i(\pi)$ will typically be nonzero for some $i > 0$. It doesn't seem too far-fetched to hope that the Hecke eigenvectors in *this* group might hold some yet-to-be-discovered arithmetic significance. Recent work of Carayol [Carayol2005] (in the *non*-co-compact case) seems particularly promising in this regard.

5. Fields of definition

Consider a period domain D for Hodge structures (on a fixed $\mathbb{Q}$-vector space V, polarized by a fixed bilinear form Q) with fixed Hodge numbers. The compact dual of a Mumford-Tate domain $D_M = M(\mathbb{R}).\tilde{\varphi} \subseteq D$ is the $M(\mathbb{C})$-orbit of the attached filtration $F^{\bullet}_{\tilde{\varphi}}$,

$$\check{D}_M = M(\mathbb{C}).F^{\bullet}_{\tilde{\varphi}}.$$

It is a connected component of the "MT Noether-Lefschetz locus" cut out of $\check{D}$ by the criterion of (a Hodge flag in $\check{D}$) having MT group contained in M,

$$\check{D}_M \subseteq \check{NL}_M \subseteq \check{D}.$$

Now, $\check{NL}_M$ is cut out by $\mathbb{Q}$-tensors hence defined over $\mathbb{Q}$, but its components[9] are permuted by the action of $Aut(\mathbb{C})$. The fixed field of the subgroup of $Aut(\mathbb{C})$ preserving $\check{D}_M$, is considered its field of definition; this is defined regardless of the vanishing of the IPR (or D_M being Hermitian symmetric). What is interesting in the Shimura variety case, is that this field has meaning "downstairs", for $\Gamma\backslash D_M$ — even though the upstairs-downstairs correspondence is highly transcendental.

A. Reflex field of a Shimura datum. Let $(G, \tilde{X})$ be a Shimura datum; we start by repeating the definition just alluded to in this context. Recall that any $\tilde{\varphi} \in \tilde{X}$ determines a complex cocharacter of G by $z \mapsto \tilde{\varphi}_{\mathbb{C}}(z,1) =: \mu_{\tilde{\varphi}}(z)$. (Complex cocharacters are themselves more general and essentially correspond to points of the compact dual.) Write, for any subfield $k \subset \mathbb{C}$,

$$\mathfrak{C}(k) := G(k)\backslash Hom_k(\mathbb{G}_m, G_k)$$

for the *set* of $G(k)$-conjugacy classes of k-cocharacters. The Galois group $Gal(k/\mathbb{Q})$ acts on $\mathfrak{C}(k)$, since $\mathbb{G}_m$ and G are $\mathbb{Q}$-algebraic groups.

[9] which are $M(\mathbb{C})$-orbits, as M is (absolutely) connected; note that this does not mean $M(\mathbb{R})$ is connected.

The element

$$c(\tilde{X}) := [\mu_{\tilde{\varphi}}] \in \mathfrak{C}(\mathbb{C})$$

is independent of the choice of $\tilde{\varphi} \in \tilde{X}$.

Definition 5.1. $E(G, \tilde{X})$ is the fixed field of the subgroup of $Aut(\mathbb{C})$ fixing $c(\tilde{X})$ as an element of $\mathfrak{C}(\mathbb{C})$.

Examples 5.2. (a) Given A an abelian variety of CM type (E, Φ), E' the associated reflex field (cf. §3.A), $\tilde{\varphi}$ the Hodge structure on $H^1(A)$, and

$$T = M_{\tilde{\varphi}} \subseteq Res_{E/\mathbb{Q}}\mathbb{G}_m$$

the associated MT group. Then $\mu_{\tilde{\varphi}}(z)$ multiplies $H^{1,0}(A)$ (the Φ-eigenspaces for E) by z, and $H^{0,1}(A)$ (The $\bar{\Phi}$-eigenspaces for E) by 1. Clearly $Ad(\sigma)\mu_{\tilde{\varphi}}(z)$ (for $\sigma \in Aut(\mathbb{C})$) multiplies the $\sigma\Phi$-eigenspaces by z, while $T(\mathbb{C})$ acts trivially on $Hom_{\mathbb{C}}(\mathbb{G}_m, T_{\mathbb{C}})$. Consequently, σ fixes $c(\{\tilde{\varphi}\})$ if and only if σ fixes Φ, and so

$$E(T, \{\tilde{\varphi}\}) = E'.$$

(b) For an inclusion $(G', \tilde{X}') \hookrightarrow (G, \tilde{X})$, one has $E(G', \tilde{X}') \supseteq E(G, \tilde{X})$. Every $\tilde{X}$ has $\tilde{\varphi}$ factoring through rational tori, which are then CM Hodge structures. Each torus arising in this manner splits over a CM field, and so $E(G, \tilde{X})$ is always contained in a CM field. (In fact, it is always either CM or totally real.)
(c) In the Siegel case, $E(G, \tilde{X}) = \mathbb{Q}$; while for a PEL Shimura datum, $E(G, \tilde{X}) = \mathbb{Q}\left(\{tr\,(b|_{T_0A})\}_{b \in B}\right)$.

Let T be a $\mathbb{Q}$-algebraic torus, μ a cocharacter defined over a finite extension $K/\mathbb{Q}$. Denote by

$$r(T, \mu) : Res_{K/\mathbb{Q}}\mathbb{G}_m \to T$$

the homomorphism given on rational points by

$$K^* \to T(\mathbb{Q})$$

$$k \mapsto \prod_{\phi \in Hom(K, \bar{\mathbb{Q}})} \phi(\mu(k)).$$

As in Example 5.2(b), every $(G, \tilde{X})$ contains a CM-pair $(T, \{\tilde{\varphi}\})$. The field of definition of $\mu_{\tilde{\varphi}}$ and the reflex field $E(T, \{\tilde{\varphi}\})$ are the same; denote this by $E(\tilde{\varphi})$. The map

$$r(T, \mu_{\tilde{\varphi}}) : Res_{E(\tilde{\varphi})/\mathbb{Q}}\mathbb{G}_m \to T$$

yields on $\mathbb{A}_{\mathbb{Q}}$-points

$$\begin{tikzcd} \mathbb{A}^{\times}_{E(\tilde{\varphi})} \arrow[r, "r(T,\mu_{\tilde{\varphi}})"] \arrow[rr, bend right, "=:r_{\tilde{\varphi}}"'] & T(\mathbb{A}_{\mathbb{Q}}) \arrow[r, "\text{project}"] & T(\mathbb{A}_f). \end{tikzcd}$$

Example 5.3. In Example 5.2(a) above, we have $E(\tilde{\varphi}) = E'$; assume for simplicity that $T = Res_{E/\mathbb{Q}}\mathbb{G}_m$. A nontrivial computation shows that the $r(T, \mu_{\tilde{\varphi}})$ part of this map is the adelicized reflex norm

$$\mathcal{N}_{\Phi'}(\mathbb{A}_{\mathbb{Q}}) : \mathbb{A}^{\times}_{E'} \to \mathbb{A}^{\times}_{E}.$$

B. Canonical models. The Shimura varieties we have been discussing — i.e., $Sh_{K_f}(G, \tilde{X})$ — are finite disjoint unions of locally symmetric varieties, and hence algebraic varieties defined *a priori* over $\mathbb{C}$. More generally, if $\mathcal{Y}$ is any complex algebraic variety, and $k \subset \mathbb{C}$ is a subfield, a **model of $\mathcal{Y}$ over** k is

- a variety $\mathcal{Y}_0$ over k, together with
- an isomorphism $\mathcal{Y}_{0,\mathbb{C}} \underset{\cong}{\overset{\theta}{\to}} \mathcal{Y}$.

For general algebraic varieties, it is *not* true that two models over the same field k are necessarily isomorphic *over that field.* But if we impose a condition on how $Gal(\mathbb{C}/E_1')$ acts on a dense set of points on any model, then the composite isomorphism

$$\begin{tikzcd} \mathcal{Y}_{0,\mathbb{C}} \arrow[r, "\theta", "\cong"'] \arrow[rr, bend left] & \mathcal{Y} \arrow[r, "\tilde{\theta}", "\cong"'] & \widetilde{\mathcal{Y}}_{0,\mathbb{C}} \end{tikzcd}$$

is forced to be $Gal(\mathbb{C}/E_1')$-equivariant, making $\mathcal{Y}_0$ and $\widetilde{\mathcal{Y}_0}$ isomorphic over E_1'. Repeating this criterion for more point sets and number fields E_i', forces $\mathcal{Y}_0$ and $\widetilde{\mathcal{Y}_0}$ to be isomorphic over $\cap_i E_i'$.

To produce the dense sets of points we need the following

Definition 5.4. (a) A point $\tilde{\varphi} \in \tilde{X}$ is a **CM point**, if there exists a (minimal, $\mathbb{Q}$-algebraic) torus $\mathbb{T} \subset G$ such that $\tilde{\varphi}(\mathbb{S}(\mathbb{R})) \subset \mathbb{T}(\mathbb{R})$.

(b) $(\mathbb{T}, \{\tilde{\varphi}\})$ is then a **CM pair** in $(G, \tilde{X})$.

Remark 5.5. Such a $\tilde{\varphi}$ exists since in a $\mathbb{Q}$-algebraic group every conjugacy class of maximal real tori contains one defined over $\mathbb{Q}$. To get density in $\tilde{X}$, look at the orbit $G(\mathbb{Q}).\tilde{\varphi}$. To get density, more importantly, in $Sh_{K_f}(G, \tilde{X})$, look at the set $\{[(\tilde{\varphi}, a)]\}_{a \in G(\mathbb{A}_f)}$.

To produce the condition on Galois action, recall for any CM field E' the Artin reciprocity map

$$\mathrm{art}_{E'} : \mathbb{A}^{\times}_{E'} \twoheadrightarrow Gal\left((E')^{ab}/E'\right).$$

Definition 5.6. A model $\left(M_{K_f}(G,\tilde{X})\,,\,\theta\right)$ of $Sh_{K_f}(G,\tilde{X})$ over $E(G,\tilde{X})$ is **canonical**, if for every

- CM $(\mathbb{T},\tilde{\varphi}) \subset (G,\tilde{X})$
- $a \in G(\mathbb{A}_f)$
- $\sigma \in Gal\left(E(\tilde{\varphi})^{ab}/E(\tilde{\varphi})\right)$
- $s \in \mathrm{art}_{E(\tilde{\varphi})}^{-1}(\sigma) \subset \mathbb{A}_{E(\tilde{\varphi})}^{\times}$,

$\theta^{-1}[(\tilde{\varphi},a)]$ is a point defined over $E(\tilde{\varphi})^{ab}$, and

$$\sigma.\theta^{-1}[(\tilde{\varphi},a)] = \theta^{-1}.[(\tilde{\varphi},r_{\tilde{\varphi}}(s)a)]. \tag{5.1}$$

Remark 5.7. In (5.1), the $r_{\tilde{\varphi}}$ is essentially a reflex norm.

The *uniqueness* of the canonical model is clear from the argument above — if one exists — since we can take the E_i' to be various $E(\tilde{\varphi})$ for CM $\tilde{\varphi}$, whose intersections are known to give $E(G,\tilde{X})$.

The *existence* of canonical models is known for all Shimura varieties by work of Deligne, Shih, Milne and Borovoi. To see how it might come about for Shimura varieties of Hodge type, first note that by

- Baily-Borel,[10]

$\mathcal{Y} := Sh_K(G,\tilde{X})$ is a variety over $\mathbb{C}$. Now we know that

- Sh_K is a moduli space for certain abelian varieties,

say $\mathcal{A} \to \mathcal{Y}$. Let $E = E(G,\tilde{X})$ and $\sigma \in Aut(\mathbb{C}/E)$. Given $P \in \mathcal{Y}(\mathbb{C})$ we have an equivalence class $[A_P]$ of abelian varieties, and we define a map

$$({}^{\sigma}\mathcal{Y})(\mathbb{C}) \to \mathcal{Y}(\mathbb{C})$$

by

$$\sigma(P) \longmapsto [{}^{\sigma}A_P].$$

That ${}^{\sigma}A_P$ is still "in the family $\mathcal{A}$" follows from

- the definition of the reflex field

and

- **Deligne's theorem** (cf. [Deligne1982], or [**?**]) that the Hodge tensors determining $\mathcal{A}$ are absolute.

That these maps produce regular (iso)morphisms

$$f_{\sigma} : {}^{\sigma}\mathcal{Y} \to \mathcal{Y}$$

boils down to

- Borel's theorem (§2).

[10] The bullets in this paragraph distinguish the ingredients that go into the proof of existence

Now $\mathcal{Y}$ has (for free) a model $\mathcal{Y}_0$ over some finitely generated extension L of E, and using

- $|Aut(\mathcal{Y})| < \infty$ (cf. [Milne2005], Thm. 3.21)

we may deduce that for σ' fixing L

$$\begin{tikzcd} {}^{\sigma'}\mathcal{Y} \arrow[rr, "f_{\sigma'}"] & & \mathcal{Y} \\ & \mathcal{Y}_0 \arrow[ul, "{}^{\sigma'}\theta"] \arrow[ur, "\theta"'] & \end{tikzcd}$$

commutes. At this point it makes sense to spread $\mathcal{Y}_0$ out over E — i.e. take all $Gal(\mathbb{C}/E)$-conjugates, viewed as a variety via

$$\begin{tikzcd} \mathcal{Y}_0 \arrow[r] \arrow[rr, bend right] & \mathrm{Spec}L \arrow[r] & \mathrm{Spec}E. \end{tikzcd}$$

The diagram

$$\begin{tikzcd} {}^{\sigma}\mathcal{Y} \arrow[r, "f_\sigma", "\cong"'] & \mathcal{Y} \\ {}^{\sigma}\mathcal{Y}_0 \arrow[u, "{}^{\sigma}\theta", "\cong"'] & \mathcal{Y}_0 \arrow[u, "\cong", "\theta"'] \end{tikzcd}$$

shows that the spread is constant; extending it over a quasi-projective base shows that $\mathcal{Y}$ has a model defined over a *finite* extension of E. (To get all the way down to E requires some serious descent theory.) Finally, that the action of $Aut(\mathbb{C}/E)$ on the resulting model implied by the $\{f_\sigma\}$ satisfies (5.1) (hence yields a *canonical* model), is true by

- the main theorem of CM.

In fact, (5.1) is precisely encoding how Galois conjugation acts on various $\mathrm{Ab}(\mathcal{O}_E, \Phi)$ together with the level structure.

So the three key points are:

(1) the *entire* theory is used in the construction of canonical models;

(2) $Sh_{K_f}(G, \tilde{X})$ is defined over $E(G, \tilde{X})$ independently of K_f; and

(3) the field of definition of a connected component $Sh_{K_f}(G, \tilde{X})^+$ is contained in $E(G, \tilde{X})^{ab}$ and gets larger as K_f shrinks (and the number of connected components increases).

C. Connected components and VHS. Assume G^{der} is simply connected. The action on CM points imposed by (5.1) turns out to force the following action on

$$\pi_0\left(Sh_{K_f}(G, \tilde{X})\right) \cong T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/\nu(K_f),$$

where $G \overset{\nu}{\twoheadrightarrow} T$ is the maximal abelian quotient. For *any* $\tilde{\varphi} \in \tilde{X}$, put

$$r = r(T, \nu \circ \mu_{\tilde{\varphi}}) : \mathbb{A}^{\times}_{E(G,\tilde{X})} \to T(\mathbb{A}_{\mathbb{Q}}).$$

Then for $\sigma \in Gal\left(E(G,\tilde{X})^{ab}/E(G,\tilde{X})\right)$ and $s \in \mathrm{art}^{-1}_{E(G,\tilde{X})}(\sigma)$ one has

$$\sigma.[y,a] = [r(s)_{\infty}.y,\ r(s)_f.a].$$

Assume for simplicity Y is trivial. Let $\mathbb{E}$ denote the field of definition of the component $\mathcal{S} := Sh_{K_f}(G,\tilde{X})^+$ over $[(1,1)]$, which is a finite abelian extension of $E' := E(G,\tilde{X})$. From the above description of the Galois action, we get

$$\mathbb{E} \ = \ \text{fixed field of } \mathrm{art}_{E'}\left(r_f^{-1}\left(T(\mathbb{Q}).\nu(K_f)\right)\right).$$

That is, by virtue of the theory of canonical models we can essentially write down a minimal field of definition of the locally symmetric variety $\mathcal{S}$.

Example 5.8. Take the Shimura datum $(G,\tilde{X}) = (\mathbb{T}, \{\tilde{\varphi}\})$ associated to an abelian variety with CM by E, so that E' is the reflex field (and $\mathbb{E}$ the field of definition of the point it lies over in a relevant Siegel modular variety). Let $K_f = \mathfrak{U}_I$ for $I \in \mathcal{I}(E)$ and consider the diagram

$$\begin{tikzcd}
\mathbb{A}^{\times}_{E',f} \arrow[r, "\mathcal{N}_{\Phi'}(=r_f)"'] \arrow[d, "\mathrm{art}_{E'}"] & \mathbb{A}^{\times}_{E,f} \arrow[r, two heads] \arrow[d, "\mathrm{art}_E"] & E^{\times}\backslash\mathbb{A}^{\times}_{E,f}/\mathfrak{U}_I \arrow[d, "\overline{\mathrm{art}_E}", "\cong"'] \\
Gal\left((E')^{ab}/E'\right) \arrow[r, "\mathfrak{N}_{\Phi'}"'] & Gal\left(E^{ab}/E\right) \arrow[r, two heads] & Gal\left(E_I/E\right),
\end{tikzcd}$$

where $\mathfrak{N}_{\Phi'}$ exists (and is continuous) in such a way that the left-hand square commutes. Writing "FF" for fixed field, we get that

$$\mathbb{E} = FF\left(\mathrm{art}_{E'}\left(\mathcal{N}^{-1}_{\Phi'}(E^{\times}\mathfrak{U}_I)\right)\right) = FF\left(\mathfrak{N}^{-1}_{\Phi'}\left(Gal(E^{ab}/E_I)\right)\right).$$

In case the CM abelian variety is an elliptic curve, $\mathcal{N}_{\Phi'}$ and $\mathfrak{N}_{\Phi'}$ are essentially the identity (and $E' = E$), so

$$\mathbb{E} = FF\left(Gal(E^{ab}/E_I)\right) = E_I.$$

It is a well-known result that, for example, the j-invariant of a CM elliptic curve generates (over the imaginary quadratic field E) its Hilbert class field $E_{(1)}$. We also see that the fields of definition of CM points in the modular curve $X(N)$ are ray class fields modulo N.

We conclude by describing a possible application to variations of Hodge structure. Let $\mathcal{V} \to \mathcal{S}$ be a VHS with reference Hodge structure

V_s over $s \in \mathcal{S}$. The underlying local system $\mathbb{V}$ produces a monodromy representation

$$\rho : \pi_1(\mathcal{S}) \to GL(V_s),$$

and we denote $\rho(\pi_1(\mathcal{S})) =: \Gamma_0$ with **geometric monodromy group**

$$\Pi := \text{identity component of } \mathbb{Q}\text{-Zariski closure of } \Gamma_0.$$

Moreover, $\mathcal{V}$ has a MT group M; and we make the following two crucial assumptions:

- $\Pi = M^{der}$; and
- D_M has vanishing IPR.

In particular, this means that the quotient of D_M by a congruence subgroup is a connected component of a Shimura variety, and that Π is as big as it can be.

For any compact open $K_f \subseteq M(\mathbb{A}_f)$ such that $\Gamma := K_f \cap M(\mathbb{Q}) \supseteq \Gamma_0$, $\mathcal{V}$ gives a period (analytic) mapping

$$\Psi^{an}_{K_f} : \mathcal{S}^{an}_{\mathbb{C}} \to \Gamma\backslash D_M \cong \left(Sh_{K_f}(M, D_M)^+ \otimes_{\mathbb{E}} \mathbb{C}\right)^{an}.$$

This morphism is algebraic by Borel's theorem, and has minimal field of definition (trivially) bounded below by the field of definition $\mathbb{E}$ of $Sh_{K_f}(M, D_M)^+$, which henceforth we shall denote $\mathbb{E}(K_f)$.

The period mapping which gives the most information about $\mathcal{V}$, is the one attached to the *smallest* congruence subgroup $\Gamma \subset M(\mathbb{Q})$ containing Γ_0. Taking then the *largest* K_f with $K_f \cap M(\mathbb{Q})$ equal to this Γ, minimizes the resulting $\mathbb{E}(K_f)$. It is this last field which it seems natural to consider as the **reflex field of a VHS** — an "expected *lower* bound" for the field of definition of a period mapping of $\mathcal{V}$. Furthermore, if $\mathcal{V}$ arises (motivically) from $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$, then assuming Deligne's absolute Hodge conjecture (cf. [Deligne1982, ?]), the $\bar{\mathbb{Q}}$-spread of π produces a period mapping into D_M modulo a *larger* Γ, and our "reflex field of $\mathcal{V}$" may be an *upper bound* for the minimal field of definition of *this* period map.

At any rate, the relations between fields of definition of

- varieties X_s,
- transcendental period points in D_M, and
- equivalence classes of period points in $\Gamma\backslash D_M$,

and hence between spreads of

- families of varieties,
- variations of Hodge structure, and
- period mappings,

is very rich. Our suggested definition may be just one useful tool, for investigating the case where the period map is into a Shimura subdomain of a period domain.

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