Normal functions

over

locally symmetric varieties$^1$

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$^1$joint work with Ryan Keast
§0. Motivations

A normal function is a section of a bundle of intermediate Jacobians (complex tori) associated to a variation of Hodge structure. They arise from a family of homologically trivial algebraic cycles on the fibers of a smooth proper morphism of varieties, and were first studied by Poincaré and Lefschetz for families of divisors on curves.

A locally symmetric variety (or Shimura variety\(^2\)) is a quotient of a Hermitian symmetric domain by an arithmetic group. A basic example is furnished by

\[ \mathcal{A}_g = \text{Sp}_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g, \]

the moduli space of principally polarized abelian \(g\)-folds.

\(^2\)In this talk, what we shall mean by “Shimura variety” is a connected component of an \(\text{Sh}_K(G, X)\), not the inverse limit of the \(\text{Sh}_K(G, X)\).
(1) For $g > 2$, the Ceresa cycle

\[ C^+ - C^- \in Z_1(J(C)) \]

produces an interesting normal function, well-defined over a 2:1 cover of $\mathcal{M}_g \subset \mathcal{A}_g$ (or over $\mathcal{M}_g(\ell)$ for $\ell \geq 3$).

Another example is given by the Fano cycle

\[ F^+ - F^- \in Z_2 \left( J \left( \begin{smallmatrix} \text{cubic} \\ 3\text{-fold} \end{smallmatrix} \right) \right), \]

and lives over a cover of the intermediate Jacobian locus in $\mathcal{A}_5$.

Can we find more such examples?
(II) According to the Oort Conjecture, $\overline{M}_g$ should contain no Shimura varieties of positive dimension for $g \gg 0$.

This suggests that the list of locally symmetric varieties over (a finite cover of) which one has normal functions might be finite.

Is this true?
(III) The Green-Voisin theorem states that for a very general smooth hypersurface $X \subset \mathbb{P}^{2m}$ ($m \geq 2$) of degree $d \geq 2 + \frac{4}{m-1}$, the image of the Abel-Jacobi map

$$AJ : CH^m(X) \to J^m(X)$$

is torsion.

We would like analogous examples for abelian varieties of PEL type, and other families of varieties parametrized by locally symmetric varieties.
(IV) Let $X$ be a very general principally-polarized complex abelian threefold, $E/\mathbb{C}$ a very general elliptic curve, and $\ell$ any prime number.

A recent result of Totaro states that:

(i) $|\text{CH}^2(X)/\ell| = \infty$ ; and

(ii) $|\text{CH}^2(X \times E)[\ell]| = \infty$.

Are there other such families of varieties?
Finally, one has the Friedman-Laza classification of Hermitian variations of Calabi-Yau-type Hodge structure of level three. (By definition, a Hermitian VHS lives over a locally symmetric variety.)

These should have normal functions – again, over a finite pullback. (Since $H^g(J(C))$ has C-Y type, the Ceresa normal function for $g = 3$ falls under this aegis.)

Are they the only ones?
§1. Kostant’s theorem

Begin with a complex semisimple Lie algebra \( \mathfrak{g} \) of rank \( n \), acting on itself via \( \text{ad}(X) = [X, \cdot] \), with subalgebras

\[
\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}.
\]

Borel Cartan
maximal
solvable
maximal
toral

In terms of the 1-dimensional \( \text{ad}(\mathfrak{t}) \)-eigenspaces indexed by the roots \( \Delta = \Delta(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}^* \), these are

\[
\mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right) \supset \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \right) \supset \mathfrak{t},
\]

where \( \Delta = \Delta^+ \uplus \Delta^- \) (\( \Delta^- = -\Delta^+ \)). Write \( \mathcal{R} \) for the (root) lattice generated by \( \Delta \).
The simple roots

\[ \Sigma = \{\sigma_1, \ldots, \sigma_n\} \subset \Delta^+ = \mathbb{Z}_{\geq 0} \langle \Sigma \rangle \cap \Delta \]

give a basis for \( \mathcal{R} \), with the simple grading elements \( \{S^1, \ldots, S^n\} \subset t \) as dual basis. The reflections \( \omega_i \) in \( \sigma_i \) generate the Weyl group \( W = W(g, t) \).

The fundamental weights \( \Omega = \{\omega_1, \ldots, \omega_n\} \subset t^* \) generate the weight lattice \( \Lambda \cong X^*(T) \supseteq \mathcal{R} \), and span the dominant Weyl chamber \( C = \mathbb{R}_{\geq 0} \langle \omega_1, \ldots, \omega_n \rangle \).

To relate them, note that the Killing form \( B(X, Y) := Tr(\text{ad}X \circ \text{ad}Y) \) on \( g \) restricts to \( \langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbb{Z} \); then

\[ \langle \omega_i, \sigma_j \rangle = \frac{1}{2} \langle \sigma_j, \sigma_j \rangle \delta_{ij}. \]

We shall write \( \{e_i\} \) for an orthonormal basis of \( t^*_\mathbb{R} \cong \mathbb{R}^n \).
Example

\((g = \mathfrak{sp}_4)\)

\[
\begin{align*}
\sigma_1 &= e_1 - e_2, \quad \sigma_2 = 2e_2 \\
\omega_1 &= e_1, \quad \omega_2 = e_1 + e_2
\end{align*}
\]
According to the **Theorem of the Highest Weight**, we have a bijective correspondence

<table>
<thead>
<tr>
<th>finite-dim irreps of $\mathfrak{g}$</th>
<th>Borel-Weil</th>
<th>dominant integral weights $\lambda \in \mathbb{C} \cap \Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^\lambda$</td>
<td>$\leftrightarrow$</td>
<td>$\lambda = \sum m_i \omega_i \ (m_i \geq 0)$</td>
</tr>
</tbody>
</table>

Given $w \in W$, $\lambda \in \Lambda$, set

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

where

$$\rho := \frac{1}{2} \sum_{\delta \in \Delta^+} \delta = \sum \omega_i.$$
Example

\((g = \mathfrak{sp}_4, \lambda = \omega_1 + \omega_2)\)

Weight diagram for \(V^\lambda\), the irrep with highest weight \(\lambda\):

Note that \(V^\lambda \subset V^{\omega_1} \otimes V^{\omega_2} = \text{st} \otimes (\wedge^2 \text{st})\), where "st" denotes the standard representation.
Fix $E \in \mathfrak{t}$ such that $\frac{1}{2}E(\sigma_i) \in \mathbb{Z}_{\leq 0}$ (\forall i), and write

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}^{j,-j}$$

for the decomposition into $\text{ad}(E)$-eigenspaces (with eigenvalue $2j$ on $\mathfrak{g}^{j,-j}$). For the corresponding decompositions of representations $V$ of $\mathfrak{g}$, see below.

Writing $\mathfrak{n} = \bigoplus_{j < 0} \mathfrak{g}^{j,-j}$ and $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}^{j,-j}$, we have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ and $\Delta(\mathfrak{n}) \subset \Delta^+.$

**Example**

$(\mathfrak{g} = \mathfrak{sp}_4, E = -2\mathfrak{S}^2)$
Set $g^{0,0} = g^0$, $\Delta_0 = \Delta(g^0, t)$, and $\Delta_0^+ = \Delta_0 \cap \Delta^+$. Put $V_0^\xi$ for the irreps of $g^0$, and $W_0 = W(g^0, t)$. The set

$$W^0 := \left\{ w \in W \mid w(\Delta^+) \supseteq \Delta_0^+ \right\}$$

gives the minimal-length representatives, of length

$$|w| := |w(\Delta^+) \cap \Delta^-|,$$

of the right cosets $W_0 \backslash W$. Write $W^0(j) \subset W^0$ for the elements of length $j$.

Finally, recall that **Lie algebra cohomology** $H^k(n, V^\lambda)$ is the $k^{th}$ cohomology of the complex

$$V^\lambda \rightarrow n^\vee \otimes V^\lambda \rightarrow \wedge^2 n^\vee \otimes V^\lambda \rightarrow \cdots,$$

from which it inherits an action of $g^0$. 
Theorem (Kostant, 1961)

\[ H^k(n, V^\lambda) \cong \bigoplus_{w \in W^0(k)} V^w_0 \cdot \lambda. \]

Example

\((g = \mathfrak{sp}_4, E = -2S^2, \lambda = \omega_1 + \omega_2, k = 1)\)

To apply Kostant, note that \(g^0 = \mathfrak{gl}_2\) and \(W^0(1) = \{w_2\}\), with \(w_2\) sending \(\omega_1 \mapsto \omega_1\) and \(\omega_2 \mapsto 2\omega_1 - \omega_2\).

We find \(w_2 \cdot \lambda = w_2(\lambda + \rho) - \rho = 5\omega_1 - 3\omega_2\), so \(H^1(n, V^\lambda)\) is the irred. \(\mathfrak{gl}_2\)-module \(V^5\omega_1 - 3\omega_2_0\) with weights circled in blue.
§2. Homogeneous variations of Hodge structure

Let $\mathfrak{g}_\mathbb{R}$ be a (noncompact) real form of $\mathfrak{g}$, containing a compact Cartan subalgebra $\mathfrak{t}_\mathbb{R}$. We have the decomposition

$$\Delta = \Delta_c \amalg \Delta_n$$

into compact and noncompact roots, and will assume that the grading element $E$ satisfies

$$\frac{1}{2}E(\Delta_c) \subset 2\mathbb{Z} , \quad \frac{1}{2}E(\Delta_n) \subset 2\mathbb{Z} + 1.$$

The (finite-dimensional) irreps of $\mathfrak{g}_\mathbb{R}$ take the form $(d\rho_\lambda, \tilde{V}^\lambda)$, with

$$\tilde{V}_C^\lambda = \begin{cases} V^\lambda & \text{"real case"} \\ V^\lambda \oplus V^{\tau(\lambda)} & \begin{cases} \tau(\lambda) \neq \lambda & \text{"complex case"} \\ \tau(\lambda) = \lambda & \text{"quaternionic case"} \end{cases} \end{cases},$$

where $V^{\tau(\lambda)} = \overline{V^\lambda}$ and $\tau = -w_0$ (for $w_0 \in W$ the longest element). The "complex case" occurs only for $A_n$, $D_{odd}$, $E_6$ and in this talk will be partially suppressed.
We also assume that $E(\lambda) \in 2\mathbb{Z} + 1$, so that the decomposition

$$\tilde{\mathcal{V}}_C^\lambda = \bigoplus_{p \in \mathbb{Z}} (\tilde{\mathcal{V}}^\lambda)^{p,-p-1}$$

into $(2p + 1) - d\rho_\lambda(E)$-eigenspaces defines a (real) Hodge structure of weight $(-1)$ and level $-E(\lambda)$ on $\tilde{\mathcal{V}}^\lambda$.

By our assumptions on $E$, this Hodge structure is polarized by the unique (up to scale) $g$-invariant alternating form

$$Q : \tilde{\mathcal{V}}^\lambda \times \tilde{\mathcal{V}}^\lambda \to \mathbb{R};$$

that is, we have $\sqrt{-1}^{2p+1} Q(v, \bar{v}) > 0$ for $v \in (\tilde{\mathcal{V}}^\lambda) \setminus \{0\}$. 
Now take $G$ to be a semisimple $\mathbb{Q}$-algebraic group of Hermitian type, such that $G_\mathbb{R}$ contains a compact Cartan $T_\mathbb{R}$. Choose a co-character

$$\chi_0 : G_{m,\mathbb{C}} \to T_\mathbb{C}$$

so that $E := \chi'_0(1)$ satisfies $E(\Delta_c) = 0$, $E(\Delta_n) = \{\pm 2\}$. That is, the $\text{ad}(E)$ (Hodge) decomposition on $g_\mathbb{C}$ takes the form

$$g_\mathbb{C} = g^{-1,1} \oplus g^{0,0} \oplus g^{1,-1}.$$ 

Then $\Delta_n \cap \Sigma = \{\sigma_I\}$ is a special simple root, i.e.

$$\lambda_{\text{ad}} = \sigma_I + \sum_{j \neq I} m_j \sigma_j,$$

and

$$E(\sigma_j) = -2\delta_{Ij}.$$ 

In this way, the choice of $I$ (from amongst the special nodes on the Dynkin diagram) determines the real form $G_\mathbb{R}$ of $G_\mathbb{C}$. 
The $\rho_\lambda \circ \chi_0$ resp. $\text{Ad} \circ \chi_0$ eigenspaces recover the (compatible) Hodge decompositions on $\tilde{V}_C^\lambda$ resp. $g_C$. To vary them, compose

$$\varphi_0 : \mathbb{G}_m \xrightarrow{\chi_0} T_C \hookrightarrow G_C$$

and take the orbit under conjugation

$$D := G(\mathbb{R}).\varphi_0 \cong G(\mathbb{R})/\text{centralizer of } \varphi_0.$$ 

This is a Hermitian symmetric domain of $\dim_C D = \dim g^{-1,1}$.

Taking $\Gamma \leq G(\mathbb{Z})$ torsion-free of finite index, the quotient

$$X := \Gamma \backslash D$$

is a quasi-projective (locally symmetric) variety by Baily-Borel.

**Example** ($g = sp_4$)

The only choice is $I = 2$, which gives $E = -2S^2$ as above, and $D = S_2$ (of dimension 3). Taking $\Gamma = Sp_4(\mathbb{Z})$ gives $X = A_2$. 
Taking $G_\mathbb{C}$ to be simple, and varying the choice of root system and special node, we get the classification of irreducible Hermitian symmetric domains:\(^3\)

<table>
<thead>
<tr>
<th>$D$</th>
<th>$(\mathcal{R}, \sigma_\mathcal{I})$</th>
<th>$G(\mathbb{R})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I$_{p, n-p+1}$</td>
<td>$(A_{n \geq 2}, \sigma_p)$</td>
<td>$SU(p, n-p + 1)$</td>
</tr>
<tr>
<td>II$_{n \geq 4}$</td>
<td>$(D_n, \sigma_n)$</td>
<td>$Spin^*(2n)$</td>
</tr>
<tr>
<td>III$_{n \geq 1}$</td>
<td>$(C_n, \sigma_n)$</td>
<td>$Sp(2n, \mathbb{R})$</td>
</tr>
<tr>
<td>IV$_{2n-1 \geq 7}$</td>
<td>$(B_n, \sigma_1)$</td>
<td>$Spin(2, 2n-1)$</td>
</tr>
<tr>
<td>IV$_{2n-2 \geq 6}$</td>
<td>$(D_n, \sigma_1)$</td>
<td>$Spin(2, 2n-2)$</td>
</tr>
<tr>
<td>EIII</td>
<td>$(E_6, \sigma_1)$</td>
<td>$E_6(-14)$</td>
</tr>
<tr>
<td>EVII</td>
<td>$(E_7, \sigma_7)$</td>
<td>$E_7(-25)$</td>
</tr>
</tbody>
</table>

The example above is III$_2(\cong \mathfrak{H}_2)$.

\(^3\)In the table, we take for each $G(\mathbb{R})$ the simply-connected form.
Now fix
- a locally symmetric variety $X = \Gamma \backslash D = \Gamma \backslash G(\mathbb{R})/G^0(\mathbb{R})$,
- a point $\{\varphi_0 : \mathbb{G}_m \to G_{\mathbb{C}}\} \in D$,
- a symplectic or orthogonal $\mathbb{Q}$-vector space $(V, Q)$, and
- a $\mathbb{Q}$-linear representation $\rho : G \to Aut(V, Q)$

such that $\rho \circ \varphi_0$ is a Hodge structure on $V$ polarized by $Q$. Then the $\{\rho \circ g \varphi_0 g^{-1}\}_{g \in G(\mathbb{R})}$ give a variation of Hodge structure over $X$ with geometric monodromy (and derived Mumford-Tate) group $G$.\textsuperscript{4} We shall call this an (irreducible) Hermitian ($\mathbb{R}$-)VHS, and the construction yields bijections

\[
\begin{array}{ccc}
\text{irreducible} & \leftrightarrow & \text{finite-dim. irreps of} \\
\text{Hermitian} & & G(\mathbb{R}) \\
\mathbb{R}\text{-VHS}/X & \leftrightarrow & \left\{ \text{dominant integral } \lambda \right\} \\
\end{array}
\]

\textsuperscript{4}or a finite-group quotient thereof
Examples

(1) $V = g$, $Q = -B \leadsto \text{“adjoint VHS” of weight 0 and level 2.}$

(2) $V = \tilde{V}^\lambda$, $E(\lambda)$ odd, $Q$ alternate $\leadsto$ VHS $\tilde{V}^\lambda$ of weight -1:
   - If $\lambda = \tau(\lambda)$, then $\tilde{V}^\lambda$ has level $-E(\lambda)$.
   - $\tilde{V}^\lambda$ is \textit{a priori} an $\mathbb{R}$-VHS, but in cases of interest will be defined over $\mathbb{Q}$ (or we can obtain this by Weil restriction).

(3) Specific examples of (2):
   - $H^1(\text{abelian family})$: $-E(\lambda) = 1$ ($\iff \lambda = \omega_i$ for some $i$)
   - Calabi-Yau VHS: $\tilde{V}^{k\omega_1}$ ($k \geq 1$)
   - running example: $V^{\omega_1+\omega_2} \subset H^1(A) \otimes H^2(A)$ (weight 3)
§3. Infinitesimal normal functions

Let $\mathcal{V}$ be a $\mathbb{Q}$-PVHS\(^5\) of weight $-1$ over a complex manifold $S$, with underlying (flat) local system $\nabla$ and associated intermediate Jacobian bundle $J(\mathcal{V})$. Form the complexes

$$C^\bullet := \mathcal{V} \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{V} \xrightarrow{\nabla} \Omega^2_S \otimes \mathcal{V} \xrightarrow{\nabla} \cdots$$

$$F^p C^\bullet := \mathcal{F}^p \mathcal{V} \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{F}^{p-1} \mathcal{V} \xrightarrow{\nabla} \Omega^2_S \otimes \mathcal{F}^{p-2} \mathcal{V} \xrightarrow{\nabla} \cdots$$

$$\text{Gr}_F^p C^\bullet := \text{Gr}_F^p \mathcal{V} \xrightarrow{\bar{\nabla}} \Omega^1_S \otimes \text{Gr}_F^{p-1} \mathcal{V} \xrightarrow{\bar{\nabla}} \Omega^2_S \otimes \text{Gr}_F^{p-2} \mathcal{V} \xrightarrow{\bar{\nabla}} \cdots$$

of sheaves on $S$, noting that $\bar{\nabla}$ is $\mathcal{O}_S$-linear, and the exact sequence

$$0 \rightarrow F^0 C^\bullet \oplus \mathcal{V} \rightarrow C^\bullet \rightarrow \frac{C^\bullet}{F^0 C^\bullet \oplus \mathcal{V}} \rightarrow 0,$$

noting that the hypercohomology sheaf $\mathcal{H}^0 \left( \frac{C^\bullet}{F^0 C^\bullet \oplus \mathcal{V}} \right) =: \mathcal{J}^Q_{\text{hor}}$ is the sheaf of quasi-horizontal sections of $J(\mathcal{V})$.

\(^5\) Also denotes the sheaf of sections of the corr. vector bundle.
The $J(\mathcal{V})$-valued normal functions over $S$ are defined by

$$\mathbb{H}^0 \left( S, \frac{C^\bullet}{F^0 C^\bullet \oplus V} \right) = \Gamma \left( S, \mathcal{J}_{\text{hor}}^Q \right) =: \text{NF}_S(\mathcal{V}) \supset \text{ANF}_S(\mathcal{V}),$$

where admissibility is a technical condition which is always met for normal functions arising from algebraic cycles. The infinitesimal and topological invariants are defined by

$$\text{ANF}_S(\mathcal{V}) \xrightarrow{\text{conn. hom.}} \mathbb{H}^1( F^0 C^\bullet \oplus V)$$

$$= : (\delta, [\cdot])$$

$$\downarrow \text{edge. hom.}$$

$$\Gamma(S, \mathcal{H}_V^1(F^0 C^\bullet)) \oplus H^1(S, V),$$

where the connecting homomorphism arises from our exact sequence.
Proposition

Assume $H^0(S, \mathbb{V}) = \{0\}$. Then $[\cdot]$ is injective.

**Sketch:** Any $\nu \in \text{ANF}_S(\mathbb{V})$ is equivalent to an extension

\[
(*) \quad 0 \to \mathbb{V} \to \tilde{\mathbb{V}} \to \mathbb{Q}(0)_S \to 0
\]

of AVMHS. If

\[
[
\nu
\] = 0 \in H^1(S, \mathbb{V}) \cong \text{Ext}^1_{\pi_1(S)}(\mathbb{Q}_S, \mathbb{V}),
\]

then $\tilde{\mathbb{V}} \cong \mathbb{V} \oplus \mathbb{Q}$. Applying the assumption, $H^0(S, \tilde{\mathbb{V}}) = \mathbb{Q}$; by the Theorem of the Fixed Part, this underlies a (constant) sub-AVMHS of $\tilde{\mathbb{V}}$. Since it is of rank 1, it can only be of type $(0, 0)$, splitting $(*)$ and rendering $\nu = 0$. □
Corollary

If $\mathcal{V} \to X = \Gamma \backslash G(\mathbb{R})/G^0(\mathbb{R})$ is a Hermitian VHS (with no trivial components) and $\text{rk}_Q G > 1$, then

$$\text{ANF}_U(\mathcal{V}) = \{0\}$$

for any Zariski open $U \subset X$.

**Sketch:** Since $H^0(X, \mathcal{V}) = \{0\}$, this follows from

- extendability: $\text{ANF}_U(\mathcal{V}) = \text{ANF}_X(\mathcal{V})$
- Raghunathan (1967): $\{0\} = H^1(\Gamma, \mathcal{V}) (= H^1(X, \mathcal{V}))$

which implies $[\nu] = 0$. □

So we have to look at étale neighborhoods $\mathcal{T} \to X$, which after all is expected in light of the Ceresa cycle.
Proposition

If $\mathcal{H}_0^\nabla(F^0C^\bullet) = \{0\}$, then $\text{NF}_S(\mathcal{V}) \xrightarrow{\delta} \Gamma(S, \mathcal{H}_\nabla^1(F^0C^\bullet))$.

**Sketch:** By the assumption, it suffices to show that $\text{NF}_S(\mathcal{V})$ injects into $\mathbb{H}^1(F^0C^\bullet)$, which is true if $\mathbb{H}^0(C^\bullet/\mathcal{V})$ vanishes. By the Theorem of the Fixed Part, the assumption also implies $H^0(S, \mathcal{V}) = \{0\}$. But $\mathbb{H}^0(C^\bullet/\mathcal{V}) = H^0(S, \mathcal{V}) \otimes \mathbb{C}/\mathbb{Q}$. □

Let $\mathcal{H}^k(j) := \mathcal{H}_\nabla^k(\text{Gr}_F^iC^\bullet)$. Since

$$\mathcal{E}_1^{p,q} := \begin{cases} \mathcal{H}^{p+q}(p), & p \geq 0 \\ 0, & p < 0 \end{cases} \quad \implies \quad \mathcal{H}_\nabla^*(F^0C^\bullet),$$

we have the

**Corollary**

Assume $\mathcal{H}^0(j)$ and $\mathcal{H}^1(j)$ vanish for $j \geq 0$. Then $\text{ANF}_{\mathcal{T}}(j^*\mathcal{V}) = \{0\}$ for all $\mathcal{T} \rightarrow S$ étale.
Accordingly, we shall say that $\mathcal{V}$ has an INF (infinitesimal normal function) if

$$\mathcal{H}^1(j) \neq 0 \text{ for some } j \geq 0.$$ 

**Exercise:** Any VHS of level 1, or level 3 CY type, has an INF.

Notice that this property makes sense for $\mathbb{R}$- or even $\mathbb{C}$-VHS (i.e. a varying Hodge flag plus $\mathbb{C}$-local system). So consider a Hermitian $\mathbb{C}$-VHS $\mathcal{V}_C^\lambda \to X = \Gamma \backslash D$ of weight $-1$ ($E(\lambda)$ odd). To compute $\mathcal{H}_{\lambda}^{*-0,1}(j)$, fix $\varphi_0 \in D$ and set

$$W^0(k,j) := \left\{ w \in W^0(k) \mid \frac{1}{2} (E(w \cdot \lambda) - 1) = j \right\}$$

$$= \left\{ w \in W \mid w(\Delta^+) \supseteq \Delta_0^+, \ |w| = k, \right.$$

$$\text{and } E(w \cdot \lambda) = 2j + 1 \left. \right\}$$

**Proposition (K-K)**

For any $k$, $\mathcal{H}_{\lambda}^k(j)|_{\varphi_0} \cong \bigoplus_{w \in W^0(k,j)} V_w^{w \cdot \lambda}$. 

Sketch: **Step 1** Commutativity of

\[ V^\lambda \rightarrow n^\vee \otimes V^\lambda \rightarrow \wedge^2 n^\vee \otimes V^\lambda \rightarrow \ldots \]

\[
\left( \bigoplus_j \text{Gr}_F^j \nu \right) |_{\varphi_0} \rightarrow \left( \Omega_1^D \otimes \left( \bigoplus_j \text{Gr}_F^j \nu \right) \right) |_{\varphi_0} \rightarrow \left( \Omega_2^D \otimes \left( \bigoplus_j \text{Gr}_F^j \nu \right) \right) |_{\varphi_0} \rightarrow \ldots
\]

implies \( \bigoplus_j H^k(j) \cong H^k(n, V^\lambda) \).

**Step 2** (e.g. \( k=1 \)) Given \( X^* \in n^\vee, \nu \in (V^\lambda)^{j-1,-j} \), the \( E \)-eigenvalues of \( X^*, \nu, X^* \otimes \nu \) are 2, 2\( j - 1 \), 2\( j + 1 \) respectively. So

\[
\text{im}\{ H^1(j)|_{\varphi_0} \hookrightarrow H^1(n, V^\lambda) \} = \bigoplus_{\xi} H^1(n, V^\lambda)_{\xi}
\]

which by Kostant

\[
= \bigoplus_{\xi: E(\xi) = 2j+1} \left( \bigoplus_{w \in W^0(1)} V_0^{w \cdot \lambda} \right)_{\xi}.
\]

Now use the fact that \( E \) is constant on each \( V^\mu_0 \). \( \square \)
We turn to the consequences of the Proposition.

First, since $E(\lambda) < 0$, we have $\frac{1}{2}(E(id \cdot \lambda) - 1) < 0$ (and of course $W(0) = \{id\}$); so $H_\lambda^0(j) = \{0\} \ (\forall j \geq 0)$.

Next, recalling that our choice of $X$ implies a choice of $\sigma_I$, it turns out that $W(1) = \{w_I\}$. This leads to the

**Corollary (K-K)**

Assume that $\lambda = \tau(\lambda)$. Then $\tilde{V}^\lambda$ has an INF $\iff \mu(\lambda) := \frac{1}{2}(E(w_I \cdot \lambda) - 1) \geq 0$.

**Example**

$(g = sp_4, I = 2, \lambda = \omega_1 + \omega_2)$ From previous Examples, we have $w_2 \cdot \lambda = 5\omega_1 - 3\omega_2$, $E(\omega_1) = -1$, $E(\omega_2) = -2$

$\implies \frac{1}{2}(E(w_2 \cdot \lambda) - 1) = \frac{1}{2}(-5 + 6 - 1) = 0$

and $\tilde{V}^\lambda$ has an INF. In fact, $\mu(\lambda) = 0 \implies H^1(X, \tilde{V}^\lambda)$ is pure of type $(0, 0)$. 
Theorem (K-K)

For $D$ of tube type (and level($\tilde{V}^\lambda$) $> 1$), we have a complete classification, where $a \in \mathbb{Z}_+$ is arbitrary:

<table>
<thead>
<tr>
<th>$D$</th>
<th>INF pairs $(D, \lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_p, p (p \geq 2)$</td>
<td>$(I_2, 2, { \omega_3 \omega_1 } + a\omega_2), (I_3, 3, \omega_3)^*$</td>
</tr>
<tr>
<td>$II_{2m \geq 4}$</td>
<td>$(II_4, \omega_1 + a{ \omega_3 \omega_4 }), (II_6, \omega_6)^*$</td>
</tr>
<tr>
<td>$III_{n \geq 1}$</td>
<td>$(III_1, (2a + 1)\omega_1)^<em>, (III_2, \omega_1 + a\omega_2), (III_3, \omega_3)^</em>$</td>
</tr>
<tr>
<td>$IV_{2n−1 \geq 5}$</td>
<td>$(IV_{2n−1}, a\omega_1 + \omega_n)$</td>
</tr>
<tr>
<td>$IV_{2n−2 \geq 6}$</td>
<td>$(IV_{2n−2}, a\omega_1 + { \omega_n−1 \omega_n })$</td>
</tr>
<tr>
<td>EVII</td>
<td>$(EVII, \omega_7)^*$</td>
</tr>
</tbody>
</table>

The starred items correspond to VHS (over $X$) of CY type.
The case $III_n$ was analyzed previously by Nori, and $(III_3, \omega_3)$ corresponds to the Ceresa cycle on $A_3$. Note that the type IV domains yield two infinite families of examples.
In the non-tube case, even to obtain the VHS appearing in the cohomology of an abelian family, or VHS of CY type, we have to generalize the $\tilde{\mathcal{V}}^\lambda$ construction via half-twists. Given an irrep $V^\lambda$ of $\mathfrak{g}$ and $E \in \mathfrak{t}$ as before, let $\tilde{E} = (E, 1) \in \mathfrak{g} \oplus \mathbb{C} = \tilde{\mathfrak{g}}$, and define irreps $V^\lambda\left\{\frac{a}{2}\right\}$ of $\tilde{\mathfrak{g}}$ by taking

$$V^\lambda\left\{\frac{a}{2}\right\}^{p,-p-1} := (V^\lambda)^{p+\frac{a}{2},-p-\frac{a}{2}-1}$$

for the $(2p + 1)$-eigenspaces of $\tilde{E}$, and

$$\tilde{\mathcal{V}}^\lambda\left\{\frac{a}{2}\right\} := V^\lambda\left\{\frac{a}{2}\right\} \oplus V^\tau(\lambda)\left\{-\frac{a}{2}\right\}$$

for the irreps of $\tilde{G}(\mathbb{R}) = U(1) \cdot G(\mathbb{R})$. For $I_p,n-p$, we study the VHS $\tilde{\mathcal{V}}^\lambda_{\mathbb{R}}\left\{\frac{a}{2}\right\}$ occurring in $H^*$ of $k$-Weil\textsuperscript{6} abelian $n$-folds $A$, i.e. those with an imaginary quadratic field in $End(A)_{\mathbb{Q}}$, whose eigenspaces $H^1_{\pm} \subset H^1(A, \mathbb{C})$ have Hodge type $(\frac{n-k}{2}, \frac{n+k}{2})$. We also show that, for irreducible HSD of any type, the only “minimal-level” C-Y Hermitian VHS with an INF have level 3. (This includes examples over $I_{1,n}$, $I_{2,n}$, $I_{5}$, and $E_{III}$.)

\textsuperscript{6}Weil abelian varieties are the case $k = 0$ (corr. to tube domain $I_{p,p}$).
§4. Applications to algebraic cycles

Now the purpose of normal functions is to study algebraic cycles. The injectivity of $\delta$ has the following consequence:

**Lemma**

Let $\pi : \mathcal{X} \to S$ be a smooth proper family of varieties over $\mathbb{C}$, $\mathcal{V}$ the quotient of the VHS associated to $R^{2p-1}\pi_*\mathbb{Q}(r)$ by its maximal level-one sub-VHS. If $\mathcal{V}$ has $\mathcal{H}^0(j) = \{0\} = \mathcal{H}^1(j)$ for all $j \geq 0$, then the reduced Abel-Jacobi map

$$\overline{AJ}^p_{X_{s_0}} : \text{Griff}^p(X_{s_0}) \to J^p(X_{s_0})/J_{\text{alg}}$$

is zero for very general $s_0 \in S$.

Conversely, one might pose the

**Conjecture**

If $\mathcal{H}^0(j) = \{0\}$ ($\forall j \geq 0$) and $\mathcal{H}^1(0) \neq \{0\}$, then for some étale neighborhood $\mathcal{T} \to S$, $\text{IH}^1(\mathcal{T}, j^*\mathcal{V}) \neq \{0\}$. 

Together with the classification, the Lemma yields the

**Theorem (Nori; K-K)**

(i) \( \overline{AJ}' = 0 \) (\( \forall r \)) for a very general abelian, Weil-abelian or quaternionic-abelian variety of \( \dim > 3, \ 6 \) resp. 8.

(ii) \( \overline{AJ}' = 0 \) for a very general \( k \)-Weil abelian \( n \)-fold (with \( k \leq n - 6 \)) unless \( r \in \left[ \frac{n-k}{2}, \frac{n+k}{2} + 1 \right] \).

because these cases aren’t on the list. Should we get excited about the cases that are?

**Proposition**

Assuming the Conjecture, each tube-type INF pair (except for \((III_1, a\omega_1)\)) arises from a normal function – and if the HC holds, from a family of cycles.
The last slide suggests the question: what about the Weil 4- and 6-folds ($I_2,2$, $I_3,3$), and quaternionic 8-folds ($II_4$), is special? Just as all abelian 1-, 2-, and 3-folds are (up to isogeny) Jacobians,

- Weil 4- and 6-folds are all 3 : 1 Prym varieties, and
- quaternionic 8-folds are all “quaternionic Pryms”.

(A dimension count shows this can’t be true in higher dims.)

A $k : 1$ Prym variety $A$ is (an irreducible component of) the cokernel of an embedding $J(C) \hookrightarrow J(\tilde{C})$ associated to a $k : 1$ étale morphism $\tilde{C} \rightarrow C$ of (smooth, proper, connected) curves. The Prym-Ceresa 1-cycle $Z_{\tilde{C}/C}$ on $A$ is the push-forward of the Ceresa cycle on $J(\tilde{C})$.

**Proposition**

For 2 : 1 Pryms, the Prym-Ceresa cycle is algebraically equivalent to zero.
As a result, these cycles were overlooked for $k > 2$.

**Proposition (K-K)**

For the $3:1$ Prym 6-folds associated to an étale cover with $g_{\mathcal{C}} = 4$ and $g_{\tilde{\mathcal{C}}} = 10$, $AJ$ of the Prym-Ceresa cycle yields an nontrivial admissible normal function $\nu$, so that $\delta\nu$ recovers the INF in the case $(I_{3,3}, \omega_3)$.

**Sketch:** To see this, we can degenerate to the picture shown, where the subscript “0” means “at the degenerate fiber”. Accordingly, we have

$$A_0 = \frac{J(E) \oplus \bigoplus_{i=1}^3 J(C'_i)}{J(E) \oplus J(C')_\Delta}.$$
The main points in the argument are now:

- For general $A$, $\mathbb{C}\langle \omega, \omega' \rangle = H^{3,0}(A)^\alpha \subset H^3(A)_{\mathbb{C}}^\alpha = (\mathcal{V}\omega_3)^{\oplus 2}$, with each $\mathcal{V}\omega_3$ of type $(1, 9, 9, 1)$ and defined over $\mathbb{Q}$.

- Upon degeneration, writing $\Omega^1(J(C'_i)) = \mathbb{C}\langle \omega_1^i, \omega_2^i, \omega_3^i \rangle$, $\omega$ pulls back to $\wedge^3_{j=1}(\omega_1^j + \zeta_3 \omega_2^j + \bar{\zeta}_3 \omega_3^j) \in \Omega^3(J(\tilde{C}_0))^\alpha$.

- The projection of $C'_1^+ - C'_1^- = \partial \Gamma_1 \in Z_1(J(C'_1))$ to $A_0$ has $\int_{\text{pr}(\Gamma_1)} \omega = \int_{\Gamma_1} \omega_1^1 \wedge \omega_2^1 \wedge \omega_3^1 \not\equiv 0$ (i.e. not a period) generically, by Ceresa’s result for $C'_1$.

- The degeneration of the Prym-Ceresa cycle is

$$\sum \pi_A(C'_i^+ - C'_i^-) = \partial \Gamma.$$  

Since this is $\alpha$-invariant, $\int_{\Gamma} \omega = 3 \int_{\text{pr}(\Gamma_1)} \omega \not\equiv 0$.

- So the image in the limit, hence generically, of the P-C cycle is nonzero under $J(H^3(A)^\vee) \rightarrow J((H^3(A)^\alpha)^\vee)$. □
These 3:1 Pryms dominate a locally symmetric family $\mathcal{A} \to X$ of abelian varieties called Faber-Weil 6-folds.

Using Nori’s trick of pulling $\mathbb{Z}\tilde{\mathcal{C}}/\mathbb{C}$ and $\nu$ back under Hecke correspondences, together with Raghunathan, we obtain (i) of:

**Proposition**

For a very general Faber-Weil 6-fold $A/\mathbb{C}$ and $2 \leq r \leq 5$:

(i) $\text{Griff}^r(A)$ and $\text{im}(\overline{AJ}_A^r)$ are countably $\infty$-dim’l; and

(ii) $|\text{CH}^r(A)/\ell| = \infty$ for all primes $\ell$.

Similar results are expected for each INF one is able to geometrically realize, provided $\text{rk}_Q G > 1$ and $\tilde{V}_\lambda$ is “abelian”.

**Example**

The INF $(\text{III}_2, \omega_1 + \omega_2)$ would correspond to $\text{Griff}^3(A \times A)$, for $A$ a very general abelian surface. I am not aware of a geometric realization.
To predict (or rule out) *higher normal functions* arising from indecomposable *higher cycles* in $K_{\text{alg}}^n$ of our family, one can try to classify INF pairs for $\tilde{\mathcal{V}}^\lambda$ of weight $-1 - n$.

For tube domains, one obtains (with $a \in \mathbb{Z}_+$ arbitrary):

$n = 1$

$$(\text{I}_2, a\omega_2), (\text{II}_4, a\omega_4), (\text{III}_1, 2a\omega_1), (\text{III}_2, a\omega_2), (\text{IV}_{m \geq 5}, a\omega_1)$$

$n \geq 2$

$$(\text{III}_1, (2a + n - 3)\omega_1) \text{ (that’s it!) }$$

For instance, $K_{1}^{\text{ind}}$ of a K3 shows up as $(\text{IV}_{19}, \omega_1)$, but the dearth of other cases is striking!
– Thank You –