Normal functions *over* locally symmetric varieties[1]

Matt Kerr
(Washington University in St. Louis)

Steklov Institute, August 2, 2017

[1] joint work with Ryan Keast

§0. Motivations

A normal function is a section of a bundle of intermediate Jacobians (complex tori) associated to a variation of Hodge structure. They arise from a family of homologically trivial algebraic cycles on the fibers of a smooth proper morphism of varieties, and were first studied by Poincaré and Lefschetz for families of divisors on curves.

A locally symmetric variety (or Shimura variety[2]) is a quotient of a Hermitian symmetric domain by an arithmetic group. A basic example is furnished by

$$\mathcal{A}_g = Sp_{2g}(\mathbb{Z})\backslash \mathfrak{H}_g,$$

the moduli space of principally polarized abelian g-folds.

---

[2] In this talk, what we shall mean by "Shimura variety" is a connected component of an $Sh_K(G,X)$, not the inverse limit of the $Sh_K(G,X)$.

(I) For $g > 2$, the Ceresa cycle

$$C^+ - C^- \in Z_1(J(C))$$

produces an interesting normal function, well-defined over a 2:1 cover of $\mathcal{M}_g \subset \mathcal{A}_g$ (or over $\mathcal{M}_g(\ell)$ for $\ell \geq 3$).

Another example is given by the Fano cycle

$$F^+ - F^- \in Z_2\left(J\left(\begin{smallmatrix}\text{cubic}\\ \text{3-fold}\end{smallmatrix}\right)\right),$$

and lives over a cover of the intermediate Jacobian locus in $\mathcal{A}_5$.

Can we find more such examples?

(II) According to the Oort Conjecture, $\overline{\mathcal{M}_g}$ should contain no Shimura varieties of positive dimension for $g \gg 0$.

This *suggests* that the list of locally symmetric varieties over (a finite cover of) which one has normal functions might be *finite*.

Is this true?

(III) The Green-Voisin theorem states that for a very general smooth hypersurface $X \subset \mathbb{P}^{2m}$ ($m \geq 2$) of degree $d \geq 2 + \frac{4}{m-1}$, the image of the Abel-Jacobi map

$$AJ : CH^m(X) \to J^m(X)$$

is torsion.

We would like analogous examples for abelian varieties of PEL type, and other families of varieties parametrized by locally symmetric varieties.

(IV) Let X be a very general principally-polarized complex abelian threefold, $E/\mathbb{C}$ a very general elliptic curve, and ℓ any prime number.

A recent result of Totaro states that:

(i) $|\mathrm{CH}^2(X)/\ell| = \infty$; and

(ii) $|\mathrm{CH}^2(X \times E)[\ell]| = \infty$.

Are there other such families of varieties?

(V) Finally, one has the Friedman-Laza classification of Hermitian variations of Calabi-Yau-type Hodge structure of level three. (By definition, a Hermitian VHS lives over a locally symmetric variety.)

These should have normal functions – again, over a finite pullback. (Since $H^g(J(C))$ has C-Y type, the Ceresa normal function for $g = 3$ falls under this aegis.)

Are they the only ones?

§1. Kostant's theorem

Begin with a complex semisimple Lie algebra $\mathfrak{g}$ of rank n, acting on itself via $\mathrm{ad}(X) = [X, \cdot\,]$, with subalgebras

$$\mathfrak{g} \supset \underset{\substack{\text{Borel} \\ \text{maximal} \\ \text{solvable}}}{\mathfrak{b}} \supset \underset{\substack{\text{Cartan} \\ \text{maximal} \\ \text{toral}}}{\mathfrak{t}}.$$

In terms of the 1-dimensional $\mathrm{ad}(\mathfrak{t})$-eigenspaces indexed by the roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}^*$, these are

$$\mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right) \supset \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \right) \supset \mathfrak{t},$$

where $\Delta = \Delta^+ \amalg \Delta^-$ $(\Delta^- = -\Delta^+)$. Write $\mathcal{R}$ for the (root) lattice generated by Δ.

The simple roots

$$\Sigma = \{\sigma_1, \ldots, \sigma_n\} \subset \Delta^+ = \mathbb{Z}_{\geq 0}\langle \Sigma \rangle \cap \Delta$$

give a basis for $\mathcal{R}$, with the simple grading elements $\{\mathrm{S}^1, \ldots, \mathrm{S}^n\} \subset \mathfrak{t}$ as dual basis. The reflections w_i in σ_i generate the Weyl group $W = W(\mathfrak{g}, \mathfrak{t})$.

The fundamental weights $\Omega = \{\omega_1, \ldots, \omega_n\} \subset \mathfrak{t}^*$ generate the weight lattice $\Lambda \cong \mathrm{X}^*(T) \supseteq \mathcal{R}$, and span the dominant Weyl chamber $\mathsf{C} = \mathbb{R}_{\geq 0}\langle \omega_1, \ldots, \omega_n \rangle$.

To relate them, note that the Killing form $B(X, Y) := Tr(\mathrm{ad}X \circ \mathrm{ad}Y)$ on $\mathfrak{g}$ restricts to $\langle\ ,\ \rangle : \Lambda \times \Lambda \to \mathbb{Z}$; then

$$\langle \omega_i, \sigma_j \rangle = \tfrac{1}{2}\langle \sigma_j, \sigma_j \rangle \delta_{ij}.$$

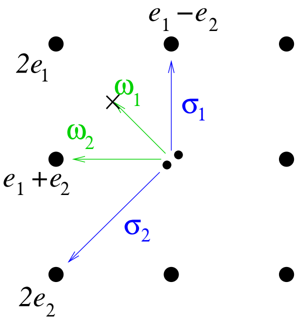
We shall write $\{e_i\}$ for an orthonormal basis of $\mathfrak{t}^*_{\mathbb{R}} \cong \mathbb{R}^n$.

Example

($\mathfrak{g} = \mathfrak{sp}_4$)

$$\sigma_1 = e_1 - e_2,\ \sigma_2 = 2e_2$$

$$\omega_1 = e_1,\ \omega_2 = e_1 + e_2$$



According to the Theorem of the Highest Weight, we have a bijective correspondence

finite-dim irreps of $\mathfrak{g}$	$\overset{\text{Borel-Weil}}{\underset{\text{highest weight}}{\leftrightarrows}}$	dominant integral weights $\lambda \in C \cap \Lambda$
V^λ		$\lambda = \sum m_i \omega_i \ (m_i \geq 0)$.

Given $w \in W$, $\lambda \in \Lambda$, set

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

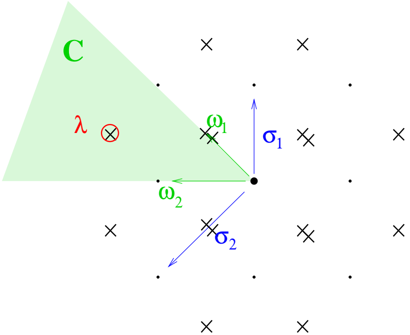
where

$$\rho := \frac{1}{2} \sum_{\delta \in \Delta^+} \delta = \sum \omega_i.$$

Example

$(\mathfrak{g} = \mathfrak{sp}_4,\ \lambda = \omega_1 + \omega_2)$

Weight diagram for V^λ, the irrep with highest weight λ:



"×"= 1–diml weight spaces of V^λ

Note that $V^\lambda \subset V^{\omega_1} \otimes V^{\omega_2} = \mathrm{st} \otimes (\wedge^2 \mathrm{st})$, where "st" denotes the standard representation.

Fix $\mathtt{E} \in \mathfrak{t}$ such that $\frac{1}{2}\mathtt{E}(\sigma_i) \in \mathbb{Z}_{\leq 0}$ $(\forall i)$, and write

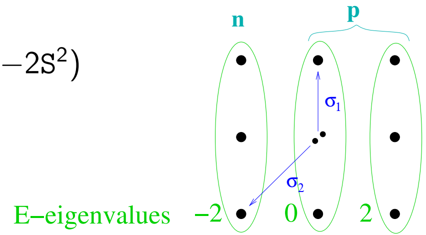
$$\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}^{j,-j}$$

for the decomposition into ad($\mathtt{E}$)-eigenspaces (with eigenvalue $2j$ on $\mathfrak{g}^{j,-j}$). For the corresponding decompositions of representations V of $\mathfrak{g}$, see below.

Writing $\mathfrak{n} = \oplus_{j<0} \mathfrak{g}^{j,-j}$ and $\mathfrak{p} = \oplus_{j \geq 0} \mathfrak{g}^{j,-j}$, we have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ and $\Delta(\mathfrak{n}) \subset \Delta^{+}$.

Example

$(\mathfrak{g} = \mathfrak{sp}_4,\ \mathtt{E} = -2\mathrm{S}^2)$



Set $\mathfrak{g}^{0,0} =: \mathfrak{g}^0$, $\Delta_0 = \Delta(\mathfrak{g}^0, \mathfrak{t})$, and $\Delta_0^+ = \Delta_0 \cap \Delta^+$.
Put V_0^ξ for the irreps of $\mathfrak{g}^0$, and $W_0 = W(\mathfrak{g}^0, \mathfrak{t})$. The set

$$W^0 := \left\{ w \in W \mid w(\Delta^+) \supseteq \Delta_0^+ \right\}$$

gives the minimal-length representatives, of length

$$|w| := |w(\Delta^+) \cap \Delta^-|,$$

of the right cosets $W_0 \backslash W$. Write $W^0(j) \subset W^0$ for the elements of length j.

Finally, recall that Lie algebra cohomology $H^k(\mathfrak{n}, V^\lambda)$ is the k^{th} cohomology of the complex

$$V^\lambda \to \mathfrak{n}^\vee \otimes V^\lambda \to \wedge^2 \mathfrak{n}^\vee \otimes V^\lambda \to \cdots ,$$

from which it inherits an action of $\mathfrak{g}^0$.

Theorem (Kostant, 1961)

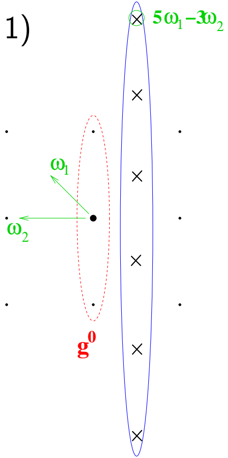
$$H^k(\mathfrak{n}, V^\lambda) \underset{\mathfrak{g}^0\text{-modules}}{\cong} \bigoplus_{w \in W^0(k)} V_0^{w\cdot\lambda}.$$

Example

$(\mathfrak{g} = \mathfrak{sp}_4,\ \mathtt{E} = -2\mathtt{S}^2,\ \lambda = \omega_1 + \omega_2,\ k = 1)$

To apply Kostant, note that $\mathfrak{g}^0 = \mathfrak{gl}_2$ and $W^0(1) = \{w_2\}$, with w_2 sending $\omega_1 \mapsto \omega_1$ and $\omega_2 \mapsto 2\omega_1 - \omega_2$.

We find $w_2 \cdot \lambda = w_2(\lambda + \rho) - \rho = 5\omega_1 - 3\omega_2$, so $H^1(\mathfrak{n}, V^\lambda)$ is the irred. $\mathfrak{gl}_2$-module $V_0^{5\omega_1 - 3\omega_2}$ with weights circled in blue.



§2. Homogeneous variations of Hodge structure

Let $\mathfrak{g}_\mathbb{R}$ be a (noncompact) real form of $\mathfrak{g}$, containing a *compact* Cartan subalgebra $\mathfrak{t}_\mathbb{R}$. We have the decomposition

$$\Delta = \Delta_c \amalg \Delta_n$$

into compact and noncompact roots, and will assume that the grading element $\mathtt{E}$ satisfies

$$\tfrac{1}{2}\mathtt{E}(\Delta_c) \subset 2\mathbb{Z}\,, \quad \tfrac{1}{2}\mathtt{E}(\Delta_n) \subset 2\mathbb{Z}+1.$$

The (finite-dimensional) irreps of $\mathfrak{g}_\mathbb{R}$ take the form $(d\rho_\lambda, \tilde{V}^\lambda)$, with

$$\tilde{V}^\lambda_\mathbb{C} = \begin{cases} V^\lambda & \text{"real case"} \\ V^\lambda \oplus V^{\tau(\lambda)} & \begin{cases} \tau(\lambda) \neq \lambda & \text{"complex case"} \\ \tau(\lambda) = \lambda & \text{"quaternionic case"} \end{cases} \end{cases},$$

where $V^{\tau(\lambda)} = \overline{V^\lambda}$ and $\tau = -w_0$ (for $w_0 \in W$ the longest element). The "complex case" occurs only for A_n, D_{odd}, E_6 and in this talk will be partially suppressed.

We also assume that $\mathtt{E}(\lambda) \in 2\mathbb{Z}+1$, so that the decomposition

$$\tilde{V}^{\lambda}_{\mathbb{C}} = \oplus_{p \in \mathbb{Z}} \left(\tilde{V}^{\lambda} \right)^{p,-p-1}$$

into $(2p+1)$-$d\rho_{\lambda}(\mathtt{E})$-eigenspaces defines a (real) Hodge structure of weight (-1) and level $-\mathtt{E}(\lambda)$ on $\tilde{V}^{\lambda}$.

By our assumptions on $\mathtt{E}$, this Hodge structure is polarized by the unique (up to scale) $\mathfrak{g}$-invariant alternating form

$$Q : \tilde{V}^{\lambda} \times \tilde{V}^{\lambda} \to \mathbb{R};$$

that is, we have $\sqrt{-1}^{2p+1} Q(v, \bar{v}) > 0$ for $v \in \left(\tilde{V}^{\lambda} \right) \backslash \{0\}$.

Now take G to be a semisimple $\mathbb{Q}$-algebraic group of Hermitian type, such that $G_{\mathbb{R}}$ contains a compact Cartan $T_{\mathbb{R}}$. Choose a co-character

$$\chi_0 : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$$

so that $\mathtt{E} := \chi_0'(1)$ satisfies $\mathtt{E}(\Delta_c) = 0$, $\mathtt{E}(\Delta_n) = \{\pm 2\}$. That is, the ad$(E)$ (Hodge) decomposition on $\mathfrak{g}_{\mathbb{C}}$ takes the form

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}.$$

Then $\Delta_n \cap \Sigma = \{\sigma_{\mathtt{I}}\}$ is a special simple root, i.e. $\lambda_{\mathrm{ad}} = \sigma_{\mathtt{I}} + \sum_{j \neq \mathtt{I}} m_j \sigma_j$, and

$$\mathtt{E}(\sigma_j) = -2\delta_{\mathtt{I}j}.$$

In this way, the choice of $\mathtt{I}$ (from amongst the special nodes on the Dynkin diagram) determines the real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$.

The $\rho_\lambda \circ \chi_0$ resp. $\mathrm{Ad} \circ \chi_0$ eigenspaces recover the (compatible) Hodge decompositions on $\tilde{V}^\lambda_{\mathbb{C}}$ resp. $\mathfrak{g}_{\mathbb{C}}$.To vary them, compose

$$\varphi_0 : \mathbb{G}_m \xrightarrow{\chi_0} T_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}$$

and take the orbit under conjugation

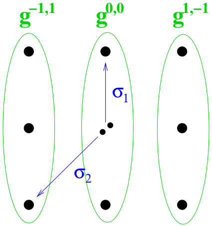
$$D := G(\mathbb{R}).\varphi_0 \cong G(\mathbb{R}) / \overbrace{G^0(\mathbb{R})}^{\substack{\text{centralizer} \\ \text{of } \varphi_0}} .$$

This is a Hermitian symmetric domain of $\dim_{\mathbb{C}} D = \dim \mathfrak{g}^{-1,1}$. Taking $\Gamma \leq G(\mathbb{Z})$ torsion-free of finite index, the quotient

$$\mathrm{X} := \Gamma \setminus D$$

is a quasi-projective (locally symmetric) variety by Baily-Borel.

Example ($\mathfrak{g} = sp_4$)



The only choice is $\mathtt{I} = 2$, which gives $\mathtt{E} = -2\mathtt{S}^2$ as above, and $D = \mathfrak{H}_2$ (of dimension 3). Taking $\Gamma = Sp_4(\mathbb{Z})$ gives $\mathrm{X} = \mathcal{A}_2$.

Taking $G_{\mathbb{C}}$ to be simple, and varying the choice of root system and special node, we get the classification of irreducible Hermitian symmetric domains:[3]

D	$(\mathcal{R}, \sigma_{\mathrm{I}})$	$G(\mathbb{R})$
$\mathrm{I}_{p,n-p+1}$	$(A_{n\geq 2}, \sigma_p)$	$SU(p, n\text{-}p+1)$
$\mathrm{II}_{n\geq 4}$	(D_n, σ_n)	$Spin^*(2n)$
$\mathrm{III}_{n\geq 1}$	(C_n, σ_n)	$Sp(2n, \mathbb{R})$
$\mathrm{IV}_{2n-1\geq 7}$	(B_n, σ_1)	$Spin(2, 2n\text{-}1)$
$\mathrm{IV}_{2n-2\geq 6}$	(D_n, σ_1)	$Spin(2, 2n\text{-}2)$
EIII	(E_6, σ_1)	$E_{6(-14)}$
EVII	(E_7, σ_7)	$E_{7(-25)}$

The example above is $\mathrm{III}_2 (\cong \mathfrak{H}_2)$.

[3] In the table, we take for each $G(\mathbb{R})$ the simply-connected form.

Now fix

- a locally symmetric variety $\mathrm{X} = \Gamma\backslash D = \Gamma\backslash G(\mathbb{R})/G^0(\mathbb{R})$,
- a point $\{\varphi_0 : \mathbb{G}_m \to G_{\mathbb{C}}\} \in D$,
- a symplectic or orthogonal $\mathbb{Q}$-vector space (V, Q), and
- a $\mathbb{Q}$-linear representation $\rho : G \to Aut(V, Q)$

such that $\rho \circ \varphi_0$ is a Hodge structure on V polarized by Q. Then the $\{\rho \circ g\varphi_0 g^{-1}\}_{g \in G(\mathbb{R})}$ give a variation of Hodge structure over X with geometric monodromy (and derived Mumford-Tate) group G.[4] We shall call this an (irreducible) Hermitian ($\mathbb{R}$-)VHS, and the construction yields bijections

$$\boxed{\begin{array}{c}\text{irreducible}\\ \text{Hermitian}\\ \mathbb{R}\text{-VHS}/\mathrm{X}\end{array}} \leftrightarrows \boxed{\begin{array}{c}\text{finite-dim.}\\ \text{irreps of}\\ G(\mathbb{R})\end{array}} \rightleftarrows \boxed{\dfrac{\left\{\begin{array}{c}\text{dominant}\\ \text{integral } \lambda\end{array}\right\}}{\langle\tau\rangle}}$$

[4] or a finite-group quotient thereof

Examples

(1) $V = \mathfrak{g}$, $Q = -B \rightsquigarrow$ "adjoint VHS" of weight 0 and level 2.

(2) $V = \tilde{V}^\lambda$, $\mathtt{E}(\lambda)$ odd, Q alternate $\rightsquigarrow$ VHS $\tilde{\mathcal{V}}^\lambda$ of weight -1:

- If $\lambda = \tau(\lambda)$, then $\tilde{\mathcal{V}}^\lambda$ has level $-\mathtt{E}(\lambda)$.
- $\tilde{\mathcal{V}}^\lambda$ is *a priori* an $\mathbb{R}$-VHS, but in cases of interest will be defined over $\mathbb{Q}$ (or we can obtain this by Weil restriction).

(3) Specific examples of (2):

- $H^1(\substack{\text{abelian}\\\text{family}})$: $-\mathtt{E}(\lambda) = 1$ ($\implies \lambda = \omega_i$ for some i)
- Calabi-Yau VHS: $\tilde{\mathcal{V}}^{k\omega_{\mathtt{I}}}$ ($k \geq 1$)
- running example: $\mathcal{V}^{\omega_1+\omega_2} \subset H^1(A) \otimes H^2(A)$ (weight 3)

§3. Infinitesimal normal functions

Let $\mathcal{V}$ be a $\mathbb{Q}$-PVHS[5] of weight -1 over a complex manifold S, with underlying (flat) local system $\mathbb{V}$ and associated intermediate Jacobian bundle $J(\mathcal{V})$. Form the complexes

$$C^\bullet := \mathcal{V} \overset{\nabla}{\to} \Omega^1_S \otimes \mathcal{V} \overset{\nabla}{\to} \Omega^2_S \otimes \mathcal{V} \overset{\nabla}{\to} \cdots$$

$$F^p C^\bullet := \mathcal{F}^p\mathcal{V} \overset{\nabla}{\to} \Omega^1_S \otimes \mathcal{F}^{p-1}\mathcal{V} \overset{\nabla}{\to} \Omega^2_S \otimes \mathcal{F}^{p-2}\mathcal{V} \overset{\nabla}{\to} \cdots$$

$$Gr^p_F C^\bullet := Gr^p_{\mathcal{F}}\mathcal{V} \overset{\bar{\nabla}}{\to} \Omega^1_S \otimes Gr^{p-1}_{\mathcal{F}}\mathcal{V} \overset{\bar{\nabla}}{\to} \Omega^2_S \otimes Gr^{p-2}_{\mathcal{F}}\mathcal{V} \overset{\bar{\nabla}}{\to} \cdots$$

of sheaves on S, noting that $\bar{\nabla}$ is $\mathcal{O}_S$-linear, and the exact sequence

$$0 \to F^0 C^\bullet \oplus \mathbb{V} \to C^\bullet \to \frac{C^\bullet}{F^0 C^\bullet \oplus \mathbb{V}} \to 0,$$

noting that the hypercohomology sheaf $\mathcal{H}^0\left(\frac{C^\bullet}{F^0 C^\bullet \oplus \mathbb{V}}\right) =: \mathcal{J}^{\mathbb{Q}}_{\text{hor}}$ is the sheaf of quasi-horizontal sections of $J(\mathcal{V})$.

---

[5] $\mathcal{V}$ also denotes the sheaf of sections of the corr. vector bundle

The $J(\mathcal{V})$-valued normal functions over S are defined by

$$\mathbb{H}^0\left(S, \frac{C^\bullet}{F^0C^\bullet \oplus \mathbb{V}}\right) = \Gamma\left(S, \mathcal{J}^{\mathbb{Q}}_{\text{hor}}\right) =: \text{NF}_S(\mathcal{V}) \supset \overbrace{\text{ANF}_S(\mathcal{V})}^{\text{admissible NF}},$$

where admissibility is a technical condition which is always met for normal functions arising from algebraic cycles. The infinitesimal and topological invariants are defined by

$$\begin{array}{ccc} \text{ANF}_S(\mathcal{V}) & \xrightarrow{\text{conn. hom.}} & \mathbb{H}^1(F^0C^\bullet \oplus \mathbb{V}) \\ & \searrow^{=:(\delta,[\cdot])} & \downarrow \text{edge. hom.} \\ & & \Gamma(S, \mathcal{H}^1_\nabla(F^0C^\bullet)) \oplus H^1(S, \mathbb{V}), \end{array}$$

where the connecting homomorphism arises from our exact sequence.

Proposition

Assume $H^0(S, \mathbb{V}) = \{0\}$. Then $[\cdot]$ is injective.

Sketch: Any $\nu \in \mathrm{ANF}_S(\mathcal{V})$ is equivalent to an extension

$$(*) \quad 0 \to \mathcal{V} \to \tilde{\mathcal{V}} \to \mathbb{Q}(0)_S \to 0$$

of AVMHS. If

$$[\nu] = 0 \in H^1(S, \mathbb{V}) \cong Ext^1_{\pi_1(S)}(\mathbb{Q}_S, \mathbb{V}),$$

then $\tilde{\mathbb{V}} \cong \mathbb{V} \oplus \mathbb{Q}$. Applying the assumption, $H^0(S, \tilde{\mathbb{V}}) = \mathbb{Q}$; by the Theorem of the Fixed Part, this underlies a (constant) sub-AVMHS of $\tilde{\mathcal{V}}$. Since it is of rank 1, it can only be of type $(0, 0)$, splitting $(*)$ and rendering $\nu = 0$. $\square$

Corollary

If $\mathcal{V} \to \mathrm{X} = \Gamma \setminus G(\mathbb{R})/G^0(\mathbb{R})$ is a Hermitian VHS (with no trivial components) and $\mathrm{rk}_{\mathbb{Q}} G > 1$, then

$$\mathrm{ANF}_U(\mathcal{V}) = \{0\}$$

for any Zariski open $U \subset \mathrm{X}$.

Sketch: Since $H^0(\mathrm{X}, \mathbb{V}) = \{0\}$, this follows from

- extendability: $\mathrm{ANF}_U(\mathcal{V}) = \mathrm{ANF}_{\mathrm{X}}(\mathcal{V})$
- Raghunathan (1967): $\{0\} = H^1(\Gamma, V)\,(= H^1(\mathrm{X}, \mathbb{V}))$

which implies $[\nu] = 0$. $\square$

So we have to look at étale neighborhoods $\mathcal{T} \xrightarrow{\jmath} \mathrm{X}$, which after all is expected in light of the Ceresa cycle.

Proposition

If $\mathcal{H}^0_\nabla(F^0 C^\bullet) = \{0\}$, then $\mathrm{NF}_S(\mathcal{V}) \overset{\delta}{\hookrightarrow} \Gamma(S, \mathcal{H}^1_\nabla(F^0 C^\bullet))$.

Sketch: By the assumption, it suffices to show that $\mathrm{NF}_S(\mathcal{V})$ injects into $\mathbb{H}^1(F^0 C^\bullet)$, which is true if $\mathbb{H}^0(C^\bullet/\mathbb{V})$ vanishes. By the Theorem of the Fixed Part, the assumption also implies $H^0(S, \mathbb{V}) = \{0\}$. But $\mathbb{H}^0(C^\bullet/\mathbb{V}) = H^0(S, \mathbb{V}) \otimes \mathbb{C}/\mathbb{Q}$. $\square$

Let $\mathcal{H}^k(j) := \mathcal{H}^k_\nabla(Gr^j_F C^\bullet)$. Since

$$\mathcal{E}_1^{p,q} := \begin{cases} \mathcal{H}^{p+q}(p), & p \geq 0 \\ 0 & p < 0 \end{cases} \implies \mathcal{H}^*_\nabla(F^0 C^\bullet),$$

we have the

Corollary

Assume $\mathcal{H}^0(j)$ and $\mathcal{H}^1(j)$ vanish for $j \geq 0$. Then $\mathrm{ANF}_\mathcal{T}(\jmath^*\mathcal{V}) = \{0\}$ for all $\mathcal{T} \overset{\jmath}{\to} S$ étale.

Accordingly, we shall say that $\mathcal{V}$ has an INF (infinitesimal normal function) if

$$\mathcal{H}^1(j) \neq 0 \text{ for some } j \geq 0.$$

Exercise: Any VHS of level 1, or level 3 CY type, has an INF.

Notice that this property makes sense for $\mathbb{R}$- or even $\mathbb{C}$-VHS (i.e. a varying Hodge flag plus $\mathbb{C}$-local system). So consider a Hermitian $\mathbb{C}$-VHS $\mathcal{V}^\lambda_{\mathbb{C}} \to \mathrm{X} = \Gamma \backslash D$ of weight -1 ($\mathtt{E}(\lambda)$ odd). To compute $\mathcal{H}^{*=0,1}_\lambda(j)$, fix $\varphi_0 \in D$ and set

$$\begin{aligned} W^0(k,j) &:= \left\{ w \in W^0(k) \,\middle|\, \tfrac{1}{2}(\mathtt{E}(w\cdot\lambda) - 1) = j \right\} \\ &= \left\{ w \in W \,\middle|\, \begin{array}{c} w(\Delta^+) \supseteq \Delta^+_0, \ |w| = k, \\ \text{and } \mathtt{E}(w\cdot\lambda) = 2j+1 \end{array} \right\} \end{aligned}$$

Proposition (K-K)

For any k, $\mathcal{H}^k_\lambda(j)|_{\varphi_0} \cong \oplus_{w\in W^0(k,j)} V_0^{w\cdot\lambda}$.

Sketch: Step 1 Commutativity of

$$\begin{array}{ccccccc}
V^\lambda & \longrightarrow & \mathfrak{n}^\vee \otimes V^\lambda & \longrightarrow & \wedge^2\mathfrak{n}^\vee \otimes V^\lambda & \longrightarrow & \cdots \\
\| & & \| & & \| & & \\
\left(\oplus_j Gr^j_{\mathcal{F}}\mathcal{V}\right)|_{\varphi_0} & \xrightarrow{\oplus_j \bar{\nabla}} & \left(\Omega^1_D \otimes (\oplus_j Gr^j_{\mathcal{F}}\mathcal{V})\right)|_{\varphi_0} & \xrightarrow{\oplus_j \bar{\nabla}} & \left(\Omega^2_D \otimes (\oplus_j Gr^j_{\mathcal{F}}\mathcal{V})\right)|_{\varphi_0} & \longrightarrow & \cdots
\end{array}$$

implies $\oplus_j \mathcal{H}^k(j) \cong H^k(\mathfrak{n}, V^\lambda)$.

Step 2 (e.g. k=1) Given $X^* \in \mathfrak{n}^\vee$, $v \in (V^\lambda)^{j-1,-j}$, the $\mathtt{E}$-eigenvalues of $X^*, v, X^* \otimes v$ are $2, 2j-1, 2j+1$ respectively. So

$$\mathrm{im}\{\mathcal{H}^1(j)|_{\varphi_0} \hookrightarrow H^1(\mathfrak{n}, V^\lambda)\} = \bigoplus_{\substack{\xi \in \Lambda \\ \mathtt{E}(\xi) = 2j+1}} H^1(\mathfrak{n}, V^\lambda)_\xi$$

which by Kostant

$$= \bigoplus_{\xi:\, \mathtt{E}(\xi)=2j+1} \left(\oplus_{w \in W^0(1)} V_0^{w\cdot\lambda}\right)_\xi .$$

Now use the fact that $\mathtt{E}$ is constant on each V_0^μ. $\square$

We turn to the consequences of the Proposition.

First, since $\mathtt{E}(\lambda) < 0$, we have $\frac{1}{2}(\mathtt{E}(\mathrm{id} \cdot \lambda) - 1) < 0$ (and of course $W^0(0) = \{\mathrm{id}\}$); so $\mathcal{H}^0_\lambda(j) = \{0\}$ $(\forall j \geq 0)$.

Next, recalling that our choice of X implies a choice of $\sigma_{\mathtt{I}}$, it turns out that $W^0(1) = \{w_{\mathtt{I}}\}$. This leads to the

Corollary (K-K)

Assume that $\lambda = \tau(\lambda)$. Then $\tilde{\mathcal{V}}^\lambda$ has an INF $\iff$ $\mu(\lambda) := \frac{1}{2}\left(\mathtt{E}(w_{\mathtt{I}} \cdot \lambda) - 1\right) \geq 0$.

Example

$(\mathfrak{g} = \mathfrak{sp}_4,\ \mathtt{I} = 2,\ \lambda = \omega_1 + \omega_2)$ From previous Examples, we have $w_2 \cdot \lambda = 5\omega_1 - 3\omega_2$, $\mathtt{E}(\omega_1) = -1$, $\mathtt{E}(\omega_2) = -2$

$$\implies \frac{1}{2}\left(\mathtt{E}(w_2 \cdot \lambda) - 1\right) = \frac{1}{2}(-5 + 6 - 1) = 0$$

and $\tilde{V}^\lambda$ has an INF. In fact, $\mu(\lambda) = 0 \implies H^1(\mathrm{X}, \tilde{\mathbb{V}}^\lambda)$ is pure of type $(0, 0)$.

Theorem (K-K)

For D of tube type (and level$(\tilde{\mathcal{V}}^{\lambda}) > 1$), we have a complete classification, where $a \in \mathbb{Z}_+$ is arbitrary:

D	INF pairs (D, λ)
$\mathrm{I}_{p,p}\,(p \geq 2)$	$(\mathrm{I}_{2,2}, \{ {\omega_3 \atop \omega_1} \} + a\omega_2)$, $(\mathrm{I}_{3,3}, \omega_3)^*$
$\mathrm{II}_{2m\geq 4}$	$(\mathrm{II}_4, \omega_1 + a\{ {\omega_3 \atop \omega_4} \})$, $(\mathrm{II}_6, \omega_6)^*$
$\mathrm{III}_{n\geq 1}$	$(\mathrm{III}_1, {\scriptstyle (2a+1)\omega_1})^*$, $(\mathrm{III}_2, {\scriptstyle \omega_1 + a\omega_2})$, $(\mathrm{III}_3, \omega_3)^*$
$\mathrm{IV}_{2n-1\geq 5}$	$(\mathrm{IV}_{2n-1}, a\omega_1 + \omega_n)$
$\mathrm{IV}_{2n-2\geq 6}$	$(\mathrm{IV}_{2n-2}, a\omega_1 + \{ {\omega_{n-1} \atop \omega_n} \})$
EVII	$(\mathrm{EVII}, \omega_7)^*$

The starred items correspond to VHS (over X) of CY type. The case III_n was analyzed previously by Nori, and $(\mathrm{III}_3, \omega_3)$ corresponds to the Ceresa cycle on $\mathcal{A}_3$. Note that the type IV domains yield two infinite families of examples.

In the non-tube case, even to obtain the VHS appearing in the cohomology of an abelian family, or VHS of CY type, we have to generalize the $\tilde{\mathcal{V}}^{\lambda}$ construction via half-twists. Given an irrep V^{λ} of $\mathfrak{g}$ and $\mathrm{E} \in \mathfrak{t}$ as before, let $\tilde{\mathrm{E}} = (\mathrm{E}, 1) \in \mathfrak{g} \oplus \mathbb{C} = \tilde{\mathfrak{g}}$, and define irreps $V^{\lambda}\{\frac{a}{2}\}$ of $\tilde{\mathfrak{g}}$ by taking

$$V^{\lambda}\{\tfrac{a}{2}\}^{p,-p-1} := (V^{\lambda})^{p+\frac{a}{2},-p-\frac{a}{2}-1}$$

for the $(2p+1)$-eigenspaces of $\tilde{\mathrm{E}}$, and

$$\tilde{V}^{\lambda}\{\tfrac{a}{2}\} := V^{\lambda}\{\tfrac{a}{2}\} \oplus V^{\tau(\lambda)}\{-\tfrac{a}{2}\}$$

for the irreps of $\tilde{G}(\mathbb{R}) = U(1) \cdot G(\mathbb{R})$. For $\mathrm{I}_{p,n-p}$, we study the VHS $\tilde{\mathcal{V}}^{\lambda}_{\mathbb{R}}\{\frac{a}{2}\}$ occurring in H^* of k-Weil[6] abelian n-folds A, i.e. those with an imaginary quadratic field in $End(A)_{\mathbb{Q}}$, whose eigenspaces $H^1_{\pm} \subset H^1(A, \mathbb{C})$ have Hodge type $(\frac{n-k}{2}, \frac{n+k}{2})$. We also show that, for irreducible HSD of *any* type, the only "minimal-level" C-Y Hermitian VHS with an INF have level 3. (This includes examples over $\mathrm{I}_{1,n}$, $\mathrm{I}_{2,n}$, II_5, and EIII.)

---

[6] Weil abelian varieties are the case $k = 0$ (corr. to tube domain $I_{p,p}$).

§4. Applications to algebraic cycles

Now the purpose of normal functions is to study algebraic cycles. The injectivity of δ has the following consequence:

Lemma

Let $\pi : \mathcal{X} \to S$ be a smooth proper family of varieties/$\mathbb{C}$, $\mathcal{V}$ the quotient of the VHS associated to $R^{2p-1}\pi_*\mathbb{Q}(r)$ by its maximal level-one sub-VHS. If $\mathcal{V}$ has $\mathcal{H}^0(j) = \{0\} = \mathcal{H}^1(j)$ for all $j \geq 0$, then the reduced Abel-Jacobi map

$$\overline{AJ}^p_{X_{s_0}} : \mathrm{Griff}^p(X_{s_0}) \to J^p(X_{s_0})/J_{\mathrm{alg}}$$

is zero for very general $s_0 \in S$.

Conversely, one might pose the

Conjecture

If $\mathcal{H}^0(j) = \{0\}$ $(\forall j \geq 0)$ and $\mathcal{H}^1(0) \neq \{0\}$, then for some étale neighborhood $\mathcal{T} \xrightarrow{\jmath} S$, $\mathrm{IH}^1(\mathcal{T}, \jmath^*\mathbb{V}) \neq \{0\}$.

Together with the classification, the Lemma yields the

Theorem (Nori; K-K)

(i) $\overline{AJ}^r = 0$ ($\forall r$) for a very general abelian, Weil-abelian or quaternionic-abelian variety of dim > 3, 6 resp. 8.
(ii) $\overline{AJ}^r = 0$ for a very general k-Weil abelian n-fold (with $k \leq n-6$) unless $r \in \left[\frac{n-k}{2}, \frac{n+k}{2}+1\right]$.

because these cases aren't on the list. Should we get excited about the cases that *are*?

Proposition

Assuming the Conjecture, each tube-type INF pair (except for $(\mathrm{III}_1, a\omega_1)$) arises from a normal function – and if the HC holds, from a family of cycles.

The last slide suggests the question: what about the Weil 4- and 6-folds ($\mathrm{I}_{2,2}$, $\mathrm{I}_{3,3}$), and quaternionic 8-folds (II_4), is special? Just as all abelian 1-, 2-, and 3-folds are (up to isogeny) Jacobians,

- Weil 4- and 6-folds are all $3:1$ Prym varieties, and
- quaternionic 8-folds are all "quaternionic Pryms".

(A dimension count shows this can't be true in higher dims.)

A $k:1$ Prym variety A is (an irreducible component of) the cokernel of an embedding $J(C) \hookrightarrow J(\tilde{C})$ associated to a $k:1$ étale morphism $\tilde{C} \twoheadrightarrow C$ of (smooth, proper, connected) curves. The Prym-Ceresa 1-cycle $Z_{\tilde{C}/C}$ on A is the push-forward of the Ceresa cycle on $J(\tilde{C})$.

Proposition

For $2:1$ Pryms, the Prym-Ceresa cycle is algebraically equivalent to zero.

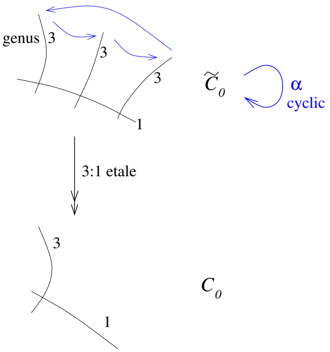
As a result, these cycles were overlooked for $k > 2$.

Proposition (K-K)

For the $3:1$ Prym 6-folds associated to an étale cover with $g_C = 4$ and $g_{\tilde{C}} = 10$, AJ of the Prym-Ceresa cycle yields an nontrivial admissible normal function ν, so that $\delta\nu$ recovers the INF in the case $(\mathrm{I}_{3,3}, \omega_3)$.

Sketch: To see this, we can degenerate to the picture shown, where the subscript "0" means "at the degenerate fiber". Accordingly, we have

$$A_0 = \frac{J(E) \oplus \bigoplus_{i=1}^{3} J(C_i')}{J(E) \oplus J(C')_\Delta}.$$



The main points in the argument are now:

- For general A, $\mathbb{C}\langle\omega,\omega'\rangle = H^{3,0}(A)^{\alpha} \subset H^3(A)^{\alpha}_{\mathbb{C}} = (\mathcal{V}^{\omega_3})^{\oplus 2}$, with each $\mathcal{V}^{\omega_3}$ of type $(1,9,9,1)$ and defined over $\mathbb{Q}$.

- Upon degeneration, writing $\Omega^1(J(C'_i)) = \mathbb{C}\langle\omega_i^1,\omega_i^2,\omega_i^3\rangle$, ω pulls back to $\wedge_{j=1}^3(\omega_1^j + \zeta_3\omega_2^j + \bar{\zeta}_3\omega_3^j) \in \Omega^3(J(\tilde{C}_0))^{\alpha}$.

- The projection of $C_1'^{+} - C_1'^{-} = \partial\Gamma_1 \in Z_1(J(C_1'))$ to A_0 has $\int_{\mathrm{pr}(\Gamma_1)}\omega = \int_{\Gamma_1}\omega_1^1\wedge\omega_1^2\wedge\omega_1^3 \not\equiv 0$ (i.e. not a period) generically, by Ceresa's result for C_1'.

- The degeneration of the Prym-Ceresa cycle is
$$\sum \pi_A(C_i'^{+} - C_i'^{-}) = \partial\Gamma.$$
Since this is α-invariant, $\int_\Gamma \omega = 3\int_{\mathrm{pr}(\Gamma_1)}\omega \not\equiv 0$.

- So the image in the limit, hence generically, of the P-C cycle is nonzero under $J(H^3(A)^\vee) \twoheadrightarrow J((H^3(A)^{\alpha})^\vee)$. $\square$

These 3:1 Pryms dominate a locally symmetric family $\mathcal{A} \to \mathrm{X}$ of abelian varieties called Faber-Weil 6-folds.

Using Nori's trick of pulling $Z_{\tilde{C}/C}$ and ν back under Hecke correspondences, together with Raghunathan, we obtain (i) of:

Proposition

For a very general Faber-Weil 6-fold $A/\mathbb{C}$ and $2 \leq r \leq 5$:
(i) $\mathrm{Griff}^r(A)$ and $\mathrm{im}(\overline{AJ}^r_A)$ are countably ∞-dim'l; and
(ii) $|\mathrm{CH}^r(A)/\ell| = \infty$ for all primes ℓ.

Similar results are expected for each INF one is able to geometrically realize, provided $\mathrm{rk}_{\mathbb{Q}} G > 1$ and $\tilde{\mathcal{V}}_\lambda$ is "abelian".

Example

The INF $(\mathrm{III}_2, \omega_1 + \omega_2)$ would correspond to $\mathrm{Griff}^3(A \times A)$, for A a very general abelian surface. I am not aware of a geometric realization.

To predict (or rule out) *higher* normal functions arising from indecomposable *higher* cycles in K_n^{alg} of our family, one can try to classify INF pairs for $\tilde{\mathcal{V}}^{\lambda}$ of weight $-1-n$.

For tube domains, one obtains (with $a \in \mathbb{Z}_+$ arbitrary):

$n = 1$

$(\mathrm{I}_{2,2}, a\omega_2)$,$(\mathrm{II}_4, a\omega_4)$,$(\mathrm{III}_1, 2a\omega_1)$,$(\mathrm{III}_2, a\omega_2)$,$(\mathrm{IV}_{m\geq 5}, a\omega_1)$

$n \geq 2$

$(\mathrm{III}_1, (2a+n-3)\omega_1)$ (that's it!)

For instance, K_1^{ind} of a $K3$ shows up as $(\mathrm{IV}_{19}, \omega_1)$, but the dearth of other cases is striking!

– Thank You –