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An Elementary Proof of Suslin Reciprocity

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Abstract. We state and prove an important special case of Suslin reciprocity that has found significant use in the study of algebraic cycles. An introductory account is provided of the regulator and norm maps on Milnor K_2-groups (for function fields) employed in the proof.

Let X be a compact Riemann Surface. We define abelian groups

$$K_2(\mathbb{C}(X)) := \frac{\mathbb{C}(X)^* \bigwedge_{\mathbb{Z}} \mathbb{C}(X)^*}{\langle f \wedge (1-f) \rangle},$$

with elements written as products of "symbols", $\prod_\alpha \{f_\alpha, g_\alpha\}$. Here "$\bigwedge_{\mathbb{Z}}$" means that (i) $\{f, g\} = \{g, f\}^{-1}$ and (ii) $\{f^n, g\} = \{f, g^n\} = \{f, g\}^n$ ("multiplicative bilinearity" – this is the "$\mathbb{Z}$"). We also have (iii) $\{f, 1-f\} = 1$; these (sometimes together with (i) and (ii)) are called the *Steinberg relations* and the notation above means that we quotient out by the ideal they generate. Similarly set

$$K_2(\mathbb{C}) := \frac{\mathbb{C}^* \bigwedge_{\mathbb{Z}} \mathbb{C}^*}{\langle a \wedge (1-a) \rangle}.$$

Now let $f, g, h \in \mathbb{C}(X)^*$ with $h = 1$ on $|(f)| \cup |(g)|$, and write $\nu_p(h)$ for the order of vanishing of h at $p \in X$.

Theorem 1 (Suslin Reciprocity)

$$\prod_{p \in |(h)|} \{f(p), g(p)\}^{\nu_p(h)} = 1 \in K_2(\mathbb{C}),$$

that is, the expression can be rewritten as a product of Steinberg relations (i), (ii), *and* (iii).

The theorem originally is due to [S1] in a much more general form; [BT] is the standard reference for the proof. We felt it would be beneficial to have a more elementary (less general and technical) proof in the literature and hope this article can be useful and illuminating for algebraic geometers. It was written in 1999 as an initial mini-project for my advisor Phillip Griffiths, and, especially in Section II, owes much to his and Mark Green's ideas [GG2].

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One way of stating a more general result is as follows. There are Milnor K-groups $K_n^M(\mathbb{C}(X))$ and $K_n^M(\mathbb{C})$ generalizing the above[1] and "residue" homomorphisms

$$K_n^M(\mathbb{C}(X)) \longrightarrow \coprod_{x\in X} K_{n-1}^M(\mathbb{C})$$

which one may compose with "taking the product over all points"

$$\coprod_{x\in X} K_{n-1}^M(\mathbb{C}) \longrightarrow K_{n-1}^M(\mathbb{C}).$$

For $n = 1$, this composition is $\mathbb{C}(X)^* \to \coprod_{x\in X}\mathbb{Z} \to \mathbb{Z}$, which computes (by summing over all points) the degree of the divisor of a function, which is zero. Triviality of the composition for $n = 2$ is known as Weil reciprocity (see [GH, K] for two different proofs); while Suslin's theorem asserts the same for $n \geq 3$.

What we prove presently then, is a simplified version of his result for $n = 3$. It has shown itself (in this form) to be highly useful in the theory of algebraic cycles, being the nontrivial step in the proofs of the isomorphisms[2]

$$(1) \qquad CH^n(\mathrm{Spec}(\mathbf{F}), n) \cong K_n^M(\mathbf{F})$$

and

$$(2) \qquad Gr^i CH^n((\mathbb{P}^n, \mathbb{T}^{n-1})(\mathbf{F})) \cong \oplus^{\binom{n}{i}} K_i^M(\mathbf{F}),$$

where $\mathbb{T}^{n-1}$ = union of the $(n+1)$ coordinate hyperplanes in $\mathbb{P}^n$. (In the more general form, Suslin reciprocity is also the basis for Somekawa's interesting generalization of Milnor K-groups to products of abelian and semi-abelian varieties [So].) The isomorphisms (1) and (2) have led naturally to "toy models" for thinking about Abel–Jacobi and regulator maps on Bloch's higher Chow groups $CH^p(X, n)$ and higher AJ maps on $Gr^i CH^p(X(\mathbf{F}))_{\mathbb{Q}}$ $(i \geq 2)$, respectively. (Here Gr refers to a version of the Bloch–Beilinson filtration discussed in [L].)

We want to explain these statements briefly. Details can be found in [K] and forthcoming articles.

For a projective variety S defined over $k \supseteq \mathbb{Q}$, one may define by means of the $(n-1)$-current $R_{\mathbf{f}}$ associated to a symbol $\mathbf{f} = \{f_1, \ldots, f_n\} \in K_n^M(k(S))$, the *Milnor regulator*

[1] More generally, let $\mathbf{F}$ be any field containing $\mathbb{Q}$. For $n \geq 2$, one defines $K_n^M(\mathbf{F}) \cong \bigwedge_{\mathbb{Z}}^n \mathbf{F}^* / \langle \cdots \wedge f \wedge 1 - f \wedge \cdots \rangle$, which is to say with generators the "symbols" $\{f_1, \ldots, f_n\}$ $(f_i \in \mathbf{F}^*)$ and relations easy generalizations of (i), (ii), (iii) (from the case $n = 2$). For $n = 0, 1$ one sets $K_0^M(\mathbf{F}) := \mathbb{Z}$ and $K_1^M(\mathbf{F}) := \mathbf{F}^*$. We have omitted the superscript M in our discussions of K_2^M in view of the well-known isomorphism $K_2^M(\mathbf{F}) \cong K_2(\mathbf{F})$ (which does *not* hold for $n > 2$).

[2] Here $CH^n(\,\cdot\,, n)$ are Bloch's higher Chow groups [Bl], while in (2) $CH^n(\mathbb{P}^n, \mathbb{T}^{n-1})$ is a closely related relative (not higher) Chow group. Both (1) and (2) are in some sense due to [S2, Bl]; a nice proof of (1) can be found in [T] and of (2) in [GG1, K] (which between the two of them have the full details).

$$(3) \qquad K_n^M(k(S)) \to H^n_{\mathcal{D}}(\eta_S, \mathbb{Z}(n)) := \varinjlim_{V\subset S} H^n_{\mathcal{D}}(S \setminus V, \mathbb{Z}(n)).$$

(where the direct limit is over all codimension-1 subvarieties). Formulas for the maps $AJ\colon CH^p(X, n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(n))$ were arrived at in [K] by first considering the case $X = \eta_S$, $p = n$. The AJ map is obtained by composing (3) with the isomorphisms (1) [with $\mathbf{F} = k(S)$] and $CH^n(\eta_S(k), n) \cong CH^n(\mathrm{Spec}(k(S)), n)$ to get a map

$$CH^n(\eta_S(k), n) \longrightarrow H^n_{\mathcal{D}}(\eta_S, \mathbb{Z}(n));$$

once we understood how to "extend" this to $CH^n(S, n)$, the version for all p and n followed.

On the other hand, for any $\mathbf{F}/\mathbb{Q}$ where $\mathbf{F}$ is finitely generated, there is a variety $S/\mathbb{Q}$ such that $\mathbb{Q}(S) \cong \mathbf{F}$. Composing the regulator (3) [with $k = \mathbb{Q}$] with isomorphism (2) yields maps

$$Gr^i CH^n((\mathbb{P}^n, \mathbb{T}^{n-1})(\mathbf{F})) \longrightarrow \bigoplus^{\binom{n}{i}} H^i_{\mathcal{D}}(\eta_S, \mathbb{Z}(i))$$

given by explicit $i-1$-currents. Those currents have an analogue in the situation where $(\mathbb{P}^n, \mathbb{T}^{n-1})$ is replaced by a product of curves C_i defined over $\mathbb{Q}$; the result is an explicit recipe for maps

$$Gr^i CH^n((C_1 \times \cdots \times C_n)(\mathbf{F})) \longrightarrow \underline{H}^i_{\mathcal{D}}(\eta_S, \Lambda(i))$$

which turn out to be something like a quotient of the desired higher AJ maps. (See [K, §5.3] for the definition of the term on the right, which is somewhat involved.)

In the following two sections we develop the ideas of regulator and norm ("transfer" in [BT]) on (Milnor) K-theory which are employed in the proof of Suslin reciprocity (which is given at the end).

1 Regulator

Define a map[3]

$$R_X : K_2(\mathbb{C}(X)) \to \varinjlim_{Z\subset X} H^1(X - Z, \mathbb{C}^*) =: H^1(\eta_X, \mathbb{C}^*)$$

by sending

$$\{f, g\} \mapsto \left\{\gamma \in H_1(X - |(f)| \cup |(g)|, \mathbb{Z}) \mapsto e^{\frac{1}{2\pi i}\int_\gamma \log f \,\mathrm{dlog}\, g - \log g(p_0)\,\mathrm{dlog}\, f}\right\},$$

where p_0 is the base point from which we continue $\log f$, which is to say it will function (once) as the branch cut for $\log f$ along γ. (Since this is not a regulator on 1-forms but merely on 1-cycles, the choice of branch of $\log g$ does not matter). This map is extended "$\times$-linearly" to products of terms $\prod_\alpha \{f_\alpha, g_\alpha\}$ by using the multiplication induced on $\varinjlim H^1(\eta_X, \mathbb{C}^*)$ (by multiplication in $\mathbb{C}^*$) as its abelian group structure. We now show that it is well-defined. Some facts in this direction:

[3] Here η_X is the "generic point" of X: $Z \subseteq Z' \implies X - Z \supseteq X - Z' \implies H^1(X - Z, \mathbb{C}^*) \subseteq H^1(X - Z', \mathbb{C}^*)$, so the direct limit is of course highly nontrivial. Its prettiest strategic side-effect: in checking $R_X\{f, g\} = R_X\{f', g'\}$, we may have the paths *avoid* a finite point set, say $|(f)| \cup |(g)| \cup |(f')| \cup |(g')|$.

1.1 $\int_\gamma(\cdots)$ Is Independent of the Choice of $p_0 \in |\gamma|$, Branch of $\log f$ and "Branch" of $\log g(p_0)$.

Indeed, if p_0 and p_1 are two points on γ,

$$\int_{\gamma_{p_0}} \{(\log f)_0 \,\mathrm{dlog}\, g - \log g(p_0)\,\mathrm{dlog}\, f\} - \int_{\gamma_{p_1}} \{(\log f)_1 \,\mathrm{dlog}\, g - \log g(p_1)\,\mathrm{dlog}\, f\}$$

$$= -\int_\gamma \mathrm{dlog}\, f \int_{p_0}^{p_1} \mathrm{dlog}\, g + [\log g(p_1) - \log g(p_0)] \int_\gamma \mathrm{dlog}\, f = 0.$$

The first step uses the fact that $(\log f)_0$ and $(\log f)_1$ differ only from p_0 to p_1, where the difference is $-\int_\gamma \mathrm{dlog}\, f$, and the second follows from the bracketed quantity being equal to $\int_{p_0}^{p_1} \mathrm{dlog}\, g$.

1.2 $R(1-f, f) = 1$

On $\gamma \setminus \{p_0\}$ define a single-valued branch of the dilogarithm

$$\ln_2(f) := -\int_0^f \log(1-z)\,\mathrm{dlog}\, z.$$

Now $\ln_2$ has no monodromy about 0 ($=f$), while if f (on γ) goes counterclockwise $\nu = -\frac{1}{2\pi i}\int_\gamma \mathrm{dlog}(1-f)$ times around 1, then $\ln_2(f)$ changes by $2\pi i\nu \log f(p_0) = \int_\gamma \log f(p_0)\,\mathrm{dlog}(1-f)$.

For instance, if Γ is a path $\subset \mathbb{C}\setminus\{0,1\}$ based at $f(p_0)$ going around $\{1\}$ once counterclockwise, then $\int_\Gamma \log(1-z)\,\mathrm{dlog}\, z = \int_\Gamma \mathrm{d}\{\log(1-z)\log z\} - \int_\Gamma \log z\,\mathrm{dlog}(1-z)$. The second term is zero (mod $(2\pi i)^2\mathbb{Z}$) by the residue theorem; since $\log(1-z)\log z$ changes by $-2\pi i \log f(p_0)$, this is the value of the first term. (See [Ha] for a more complete discussion of monodromy of polylogarithms.)

We now have

$$\int_\gamma \log f(p_0)\,\mathrm{dlog}(1-f) = \int_\gamma \mathrm{d}\{\ln_2(f)\} = \int_\gamma \mathrm{d}\left\{-\int_0^f \log(1-z)\,\mathrm{dlog}\, z\right\}$$

$$= \int_\gamma \log(1-f)\,\mathrm{dlog}\, f,$$

and so $0 = \int_\gamma(\log(1-f)\,\mathrm{dlog}\, f - \log f(p_0)\,\mathrm{dlog}(1-f))$ as desired.

1.3 $R(f,g) = R(g,f)^{-1}$

As γ starts and ends at p_0, $\log\frac{g}{g(p_0)}$ and $\log\frac{f}{f(p_0)}$, which are zero at p_0, each change by a multiple of $2\pi i$. Hence

$$\int_\gamma \mathrm{d}\left\{\log\frac{g}{g(p_0)} \log\frac{f}{f(p_0)}\right\} \equiv 0 \pmod{(2\pi i)^2\mathbb{Z}},$$

and so

$$\int_\gamma \log g\,\mathrm{dlog}\, f - \log f(p_0)\,\mathrm{dlog}\, g \equiv -\int_\gamma \log f\,\mathrm{dlog}\, g - \log g(p_0)\,\mathrm{dlog}\, f.$$

Taking $e^{\frac{1}{2\pi i}(\text{each side})}$ gives the result.

1.4 $R(f'f,g) = R(f',g) \times R(f,g)$

This is obvious.

So R is well-defined and it makes sense to write $R_X\{f,g\}$, or more generally $R_X \prod_\alpha\{f_\alpha, g_\alpha\}$. Now if this yields 1 (*i.e.*, is trivial) on 1-cycles

$$\gamma \in \ker\left\{\varinjlim_{Z\subset X} H_1(X-Z,\mathbb{Z}) \to H_1(X,\mathbb{Z})\right\}$$

(loops around points), then we say $\prod_\alpha\{f_\alpha, g_\alpha\} \in K_2(X)$. Such elements constitute a *subgroup* of $K_2(\mathbb{C}(X))$, and we have the series of inclusions $K_2(\mathbb{C}(X)) \supseteq K_2(X) \supseteq \ker(R_X) \supseteq K_2(\mathbb{C})$. What if R_X is trivial on *all* 1-cycles?

Conjecture 1 $\ker(R_X) = K_2(\mathbb{C})$.

We prove this for $X = \mathbb{P}^1$. The interplay of (local) analysis and global algebra (on the function field) will show why this is so hard in general (for X of higher genus). We manage to get around this later (for the purposes of the "norm" algorithm) by working with "K_2 of meromorphic functions on branches of X" (since there we are only concerned with the information that the algorithm "commutes" with the local evaluation and regulator maps on K_2). But here we need a real *global* computation.

First of all, since $H^1(\mathbb{P}^1) = 0$, $K_2(\mathbb{P}^1) = \ker(R_{\mathbb{P}^1})$. So we will prove $K_2(\mathbb{P}^1) = K_2(\mathbb{C})$.

1.5 Local Analysis (for All Riemann Surfaces X)

Let $\beta \in X$ be some point and write $f, g \in \mathbb{C}(X)^*$ locally as $f = (z-\beta)^{\nu_\beta(f)}\tilde{f}$, $g = (z-\beta)^{\nu_\beta(g)}\tilde{g}$. We compute $R\{f,g\}(\gamma_\beta)$ where γ_β is a very small path about β and pick $p_0 \in X$ so that, in this local parametrization, $p_0 - \beta = 1$. (Note in particular that this implies $g(p_0) = \tilde{g}(p_0)$.) The integral is

$$\frac{1}{2\pi i}\int_{\gamma_\beta}\left\{\log((z-\beta)^{\nu_\beta(f)}\tilde{f})\,\mathrm{dlog}((z-\beta)^{\nu_\beta(g)}\tilde{g}) - \log g(p_0)\,\mathrm{dlog}((z-\beta)^{\nu_\beta(f)}\tilde{f})\right\}$$

$$= \frac{1}{2\pi i}\int_{\gamma_\beta}\Big\{\nu_\beta(g)\log\tilde{f}\,\mathrm{dlog}(z-\beta) - \nu_\beta(f)\log g(p_0)\,\mathrm{dlog}(z-\beta)$$

$$+ \nu_\beta(f)\log(z-\beta)\,\mathrm{dlog}\,\tilde{g} + \nu_\beta(f)\nu_\beta(g)\log(z-\beta)\,\mathrm{dlog}(z-\beta)$$

$$+ (\text{inessential terms})\Big\}.$$

Now use the residue theorem. Also, on the third term above use integration by parts to get the last term below, and in the last term above use $p_0 - \beta = 1$ plus integrating $d\{\log^2(z-\beta)\}$ to get the third term below:

$$
\begin{aligned}
&= \nu_\beta(g)\log \tilde{f}(\beta) - \nu_\beta(f)\log g(p_0) + \nu_\beta(f)\nu_\beta(g)\,\pi i \\
&\quad + \frac{\nu_\beta(f)}{2\pi i}\int_{\gamma_\beta}\left\{ d[\log(z-\beta)\log\tilde{g}] - \log\tilde{g}\,d\log(z-\beta)\right\} \\
&= \nu_\beta(g)\log \tilde{f}(\beta) - \nu_\beta(f)\log \tilde{g}(p_0) + \nu_\beta(f)\nu_\beta(g)\,\pi i \\
&\quad + \nu_\beta(f)\log\tilde{g}(p_0) - \nu_\beta(f)\log\tilde{g}(\beta) \\
&= \nu_\beta(g)\log \tilde{f}(\beta) - \nu_\beta(f)\log\tilde{g}(\beta) + \nu_\beta(f)\nu_\beta(g)\,\pi i.
\end{aligned}
$$

So, taking $e^{(\frac{1}{2\pi i}(\,\cdot\,))}$,

$$
R\{f,g\}(\gamma_\beta) = \lim_{z\to\beta}(-1)^{\nu_\beta(f)\nu_\beta(g)}\frac{f(z)^{\nu_\beta(g)}}{g(z)^{\nu_\beta(f)}} =: T_\beta\{f,g\},
$$

and we call $T_\beta\{f,g\}$ the "tame symbol of f and g (evaluated at β)". Now *Weil reciprocity* says that

$$
\prod_{\beta\in X} T_\beta\{f,g\} = 1,
$$

i.e., some kind of "global reciprocity" law *always* holds. Our computation implies, on the other hand, that if a pointwise "local reciprocity" $T_\beta\{f,g\} = 1$ holds at β for two functions, then the corresponding K-theory element must have trivial regulator around β. We restate this more generally in the following:

Proposition 1 $\prod_\alpha\{f_\alpha,g_\alpha\} \in K_2(X)$ *if and only if*

$$
\lim_{z\to\beta}(-1)^{\sum_\alpha \nu_\beta(f_\alpha)\nu_\beta(g_\alpha)}\left(\prod_\alpha \frac{f_\alpha(z)^{\nu_\beta(g_\alpha)}}{g_\alpha(z)^{\nu_\beta(f_\alpha)}}\right) =: T_\beta\prod_\alpha\{f_\alpha,g_\alpha\} = 1 \quad (\forall\beta\in X).
$$

This holds for all X. What follows does not.

1.6 Global Arithmetic in $K_2(\mathbb{C}(\mathbb{P}^1))$

We establish yet another:

Proposition 2 $T_\beta\prod_\alpha\{f_\alpha,g_\alpha\} = 1$ $(\forall\beta\in\mathbb{P}^1)$ *if and only if*

$$
\prod_\alpha\{f_\alpha,g_\alpha\} \in K_2(\mathbb{C}) \left(\subseteq K_2(\mathbb{C}(\mathbb{P}^1))\right).
$$

Combined with the previous result, this will prove $K_2(\mathbb{P}^1) = K_2(\mathbb{C})$.

The implication "$\Leftarrow$" is of course trivial since constants have no poles or zeroes (and so the $\nu_\beta(\,\cdot\,)$ are all 0). We shall begin the other direction with a single term

$$
\{f,g\} = \left\{\prod_i (z-a_i)^{m_i}, \prod_i (z-b_j)^{n_j}\right\},
$$

where a_i and b_j are all distinct, and the following:

Lemma 1 $\{z-a, z-b\} = \{z-a, a-b\}\{b-a, z-b\}$.

Proof Put $A = z-a$, $B = z-b$, $C = a-b$. We have $B = A + C$, *i.e.*, $1 = \frac{A}{B} + \frac{C}{B}$, which by the Steinberg relations implies that

$$
1 = \left\{\frac{A}{B}, \frac{C}{B}\right\} = \{A,C\}\{A,B\}^{-1}\{B,C\}^{-1}\{B,B\},
$$

and so

$$
\{A,B\} = \{A,C\}\{C,B\}\{B,B\}.
$$

Now $\{B,B\} = \{B,B\}^{-1} = \frac{\{1-B,B\}}{\{B,B\}} = \{\frac{1-B}{B},B\} = \{\frac{1}{B}-1,B\} = \{\frac{1}{B}-1,\frac{1}{B}\}^{-1} = \{-1,\frac{1}{B}\}^{-1}\{1-\frac{1}{B},\frac{1}{B}\}^{-1} = \{-1,B\}$. So

$$
\{A,B\} = \{A,C\}\{C,B\}\{-1,B\} = \{A,C\}\{-C,B\},
$$

which is the desired equality. ∎

Case 1 Assume one term, f and g monic with $|(f)|\cap|(g)| = \varnothing$ or $\{\infty\}$. (We are assuming $T_\beta\{f,g\} = 1$ for all $\beta\in\mathbb{P}^1$.)

$$
\begin{aligned}
\{f,g\} &= \prod_{i,j}\{z-a_i, z-b_j\}^{m_in_j} = \prod_{i,j}\left(\{z-a_i, a_i-b_j\},\{b_j-a_i, z-b_j\}\right)^{m_in_j} \\
&= \prod_i\{z-a_i, \prod_j (a_i-b_j)^{n_j}\}^{m_i} \Big/ \prod_j\{z-b_j, \prod_i (b_j-a_i)^{m_i}\}^{n_j} \\
&= \prod_i\{z-a_i, g(a_i)^{m_i}\} \Big/ \prod_j\{z-b_j, f(b_j)^{n_j}\} \\
&= \prod_i\{z-a_i, 1\} \Big/ \prod_j\{z-b_j, 1\} = 1,
\end{aligned}
$$

where the second-to-last step comes from local reciprocity, since a_i and b_j are distinct. Two quick proofs that $\{A,1\} = 1$: either use $\{A,1\} = \{A,1^0\} = \{A,1\}^0 = 1$ or $\{A,1\} = \frac{\{A,A\}}{\{A,A\}} = 1$. Trivially $1 \in K_2(\mathbb{C})$ so we are done.

Case 2 Remove the assumption on divisors. Assume, with all a_i, b_j, c_k distinct, that

$$f = \prod_k (z - c_k)^{q_k} \prod_i (z - a_i)^{m_i} \quad \text{and} \quad g = \prod_\ell (z - c_\ell)^{r_\ell} \prod_j (z - b_j)^{n_j}$$

satisfy local reciprocity at each β. Then $\{f, g\}$

$$\begin{aligned}
&= \prod_k \Big(\{z - c_k, z - c_k\}^{q_k r_k} \prod_{\ell \neq k} \{z - c_k, z - c_\ell\}^{q_k r_\ell} \\
&\quad \times \prod_j \{z - c_k, z - b_j\}^{q_k n_j} \prod_i \{z - a_i, z - c_k\}^{m_i r_k} \Big) \times \prod_{i,j} \{z - a_i, z - b_j\}^{m_i n_j} \\
&= \prod_k \{z - c_k, -1\}^{q_k r_k} \times \prod_{k, \ell \neq k} \{z - c_k, c_k - c_\ell\}^{q_k r_\ell} \times \prod_{\ell, k \neq \ell} \{c_\ell - c_k, z - c_\ell\}^{q_k r_\ell} \\
&\quad \times \prod_{k,j} \left(\{z - c_k, c_k - b_j\}\{b_j - c_k, z - b_j\} \right)^{q_k n_j} \\
&\quad \times \prod_{k,i} \left(\{z - a_i, a_i - c_k\}\{c_k - a_i, z - c_k\} \right)^{m_i r_k} \\
&\quad \times \prod_{i,j} \left(\{z - a_i, a_i - b_j\}\{b_j - a_i, z - b_j\} \right)^{m_i n_j}
\end{aligned}$$

[now switch k and ℓ in the third factor above]

$$\begin{aligned}
&= \prod_k \left\{ z - c_k, (-1)^{q_k r_k} \frac{\left(\prod_{\ell \neq k} (c_k - c_\ell)^{r_\ell} \prod_j (c_k - b_j)^{n_j} \right)^{q_k}}{\left(\prod_i (c_k - a_i)^{m_i} \prod_{\ell \neq k} (c_k - c_l)^{q_\ell} \right)^{r_k}} \right\} \\
&\quad \times \prod_i \left\{ z - a_i, \left(\prod_j (a_i - b_j)^{n_j} \prod_k (a_i - c_k)^{r_k} \right)^{m_i} \right\} \\
&\quad \times \left(\prod_j \left\{ z - b_j, \left(\prod_i (b_j - a_i)^{m_i} \prod_k (b_j - c_k)^{q_k} \right)^{n_j} \right\} \right)^{-1} \\
&= \prod_k \{z - c_k, 1\} \prod_i \{z - a_i, 1\} \Big/ \prod_j \{z - b_j, 1\} = 1 \in K_2(\mathbb{C}).
\end{aligned}$$

Case 3 Separate $|(f)|$ and $|(g)|$ again but remove the requirement that f and g be monic. That is, let $f = \xi \tilde{f}$ and $g = \eta \tilde{g}$, where $\tilde{f} = \prod_i (z - a_i)^{m_i}$ and $\tilde{g} = \prod_j (z - b_j)^{n_j}$.

Then

$$\begin{aligned}
\{f, g\} &= \{\xi, \eta\} \prod_j \{\xi, z - b_j\}^{n_j} \times \prod_i \{z - a_i, \eta\}^{m_i} \times \prod_{i,j} \{z - a_i, z - b_j\}^{m_i n_j} \\
&= \{\xi, \eta\} \prod_j \{\xi^{n_j}, z - b_j\} \times \prod_i \{z - a_i, \eta^{m_i}\} \\
&\quad \times \left(\prod_i \{z - a_i, \tilde{g}(a_i)^{m_i}\} \Big/ \prod_j \{z - b_j, \tilde{f}(b_j)^{n_j}\} \right) \\
&= \{\xi, \eta\} \frac{\prod_i \{z - a_i, (\eta \tilde{g}(a_i))^{m_i} [= 1]\}}{\prod_j \{z - b_j, (\xi \tilde{f}(b_j))^{n_j} [= 1]\}} = \{\xi, \eta\} \in K_2(\mathbb{C}).
\end{aligned}$$

Combining the Cases. (Remove all assumptions on f and g.) So we have essentially $f = \xi \prod_i (z - a_i) \prod_k (z - c_k)^{q_k}$ and $g = \eta \prod_j (z - b_j) \prod_\ell (z - c_\ell)^{r_\ell}$. Defining for every $\beta \in X$

$$\tilde{g}_\beta := \frac{g}{(z - \beta)^{\nu_\beta(g)}} \text{ and } \tilde{f}_\beta := \frac{f}{(z - \beta)^{\nu_\beta(f)}},$$

from the previous computations it is clear that

$$\{f, g\} = \Big(\prod_{\beta \in |(f)| \cup |(g)| \setminus \infty} \Big\{ z - \beta, (-1)^{\nu_\beta(f)\nu_\beta(g)} \frac{\tilde{g}_\beta(\beta)^{\nu_\beta(f)}}{\tilde{f}_\beta(\beta)^{\nu_\beta(g)}} \Big\} \Big) \times \{\xi, \eta\}.$$

For a product $\prod_\alpha \{f_\alpha, g_\alpha\}$ we have therefore in $K_2(\mathbb{C}(\mathbb{P}^1))$

$$\begin{aligned}
&\prod_{\beta \in |(f_\alpha)| \cup |(g_\alpha)| \setminus \infty} \Big\{ z - \beta, \prod_\alpha (-1)^{\nu_\beta(f_\alpha)\nu_\beta(g_\alpha)} \frac{\tilde{g}_{\alpha_\beta}(\beta)^{\nu_\beta(f_\alpha)}}{\tilde{f}_{\alpha_\beta}(\beta)^{\nu_\beta(g_\alpha)}} \Big\} \times \{\xi_\alpha, \eta_\alpha\} \\
&\qquad = \prod_\alpha \{\xi_\alpha, \eta_\alpha\} \in K_2(\mathbb{C}),
\end{aligned}$$

since the big product over α is just $T_\beta \prod_\alpha \{f_\alpha, g_\alpha\}$ (= 1 by assumption). This completes the proof that $K_2(\mathbb{P}^1) = K_2(\mathbb{C})$.

2 Norm

From the Riemann–Roch theorem follows the existence of a "primitive pair" of meromorphic functions $h, x \colon X \to \mathbb{P}^1$. What we mean by "primitive" is the following:

(i) Geometrically, they give an embedding $X \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ (*not* $\mathbb{P}^2$: there you get at least normal crossings in general; $\mathbb{P}^1 \times \mathbb{P}^1$ has a bit "more" structure, being the compactification of $\mathbb{C}^* \times \mathbb{C}^*$ by four $\mathbb{P}^1$s rather than three). We write (z, w) for coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, and think of $X \xrightarrow{h} \mathbb{P}^1$ as giving a branched covering of the z-sphere. When convenient we write z in lieu of h to denote the function on X (an exception would be "$h^{-1}(z)$").

(ii) Algebraically, they generate the function field: sending $w \mapsto x$ gives an isomorphism $\mathbb{C}(z)[w]\,/(\Phi(z,w)) \xrightarrow{\cong} \mathbb{C}(X)$, where $\Phi(z,w) = w^n + w^{n-1}\mathcal{R}_1(z) + \cdots + \mathcal{R}_n(z)$ is the minimal polynomial of x, and $\mathcal{R}_i(z) \in \mathbb{C}(z)$ are rational functions. The "graph" of (i) is the solution set $X_\Phi = \{(z,w) \mid \Phi(z,w) = 0\}$. (Every Riemann surface is algebraic!) Since $\mathbb{C}(z) \cong \mathbb{C}(\mathbb{P}^1)$, this expresses $\mathbb{C}(X)$ as an extension of $\mathbb{C}(\mathbb{P}^1_z)$.

2.1 Galois K_2-Norm for Splitting Field Extensions. Preliminary Remarks on Strategy in the Function-Field Case

We will describe an algorithm similar to the Galois norm which maps $K_2(\mathbb{C}(X)) \to K_2(\mathbb{C}(\mathbb{P}^1))$. Simplifying for a moment to subfields of $\mathbb{C}$, suppose we have a splitting field extension $\mathcal{L}/\mathcal{K}$, $\mathcal{L} = \mathcal{K}(x)$ $(x \in \mathbb{C})$, with $\Phi(x)[=0]$ the mimimal polynomial of x over $\mathcal{K}$, with roots $\{x = x_1\}, x_2, \ldots, x_n$. Sending $w \mapsto x$ gives an isomorphism $\mathcal{K}[w]/(\Phi(w)) \xrightarrow{\cong} \mathcal{L}$, and so we may write $F, G \in \mathcal{L}$ as $f(x), g(x)$, where $\deg(f(w)), \deg(g(w)) < n$. ($F = f(x)$ and $G = g(x)$ are numbers, $f(w)$ and $g(w)$ are polynomials.) Define

$$N_{\mathcal{L}/\mathcal{K}}\{F,G\} := \prod_{i=1}^{n} \{f(x_i), g(x_i)\}.$$

Notice that while the extension $\mathcal{L}/\mathcal{K}$ has degree n, $\left(\mathcal{K}[w]/(f(w))\right)/\mathcal{K} = \mathcal{K}(\sigma)/\mathcal{K}$ and $\left(\mathcal{K}[w]/(g(w))\right)/\mathcal{K} = \mathcal{K}(\tau)/\mathcal{K}$ are *lower*-degree extensions *not* contained in $\mathcal{L}$ and with degrees *not* necessarily dividing n. (Here σ and τ are complex numbers satisfying $f(\sigma) = 0$ and $g(\tau) = 0$, *i.e.*, $g(w)$ and $f(w)$ are their minimal polynomials, with conjugates σ_j and τ_k.) So, if we could somehow exchange the role of Φ with that of f and/or g, we could pass from terms $\in K_2(\mathcal{L})$ to terms $\in K_2$ (lower degree extensions) (or so I claim). We will work this out completely in the function field case below.

Passing back to function fields, the *roots* x_i get replaced with the *branches* x_i of x over the z-sphere, which are no longer $\in \mathcal{L} = \mathbb{C}(X)$. (That is the only real difference, likewise for σ and τ.) So the computations which follow are not really in $K_2(\mathbb{C}(X))$; they merely constitute an algorithm. However, they are "correct" locally and pointwise almost everywhere, enough to preserve (commute with) the regulator and $K_2(\mathbb{C})$-evaluation at z, in a sense to be described later.

2.2 The Norm Algorithm

This is based on an idea in [GG2]. Let

$$f(z,w) = \prod_{j=1}^{\ell(<n)} (w - \sigma_j(z)), g(z,w) = \prod_{k=1}^{m(<n)} (w - \tau_k(z))$$

be general (monic, for simplicity) elements $\in \mathbb{C}(z)[x]/(\Phi(z,x))$, and of course

$$\Phi(z,w) = \prod_{i=1}^{n} (w - x_i(z)).$$

(The "functions" σ_j, τ_k, x_i all have branch cuts and so are not meromorphic over the z-sphere.) It is important in what follows that $\ell, m < n$. Omitting the z-variable (writing, for instance, $f(x_i)$ for $f(z, x_i(z))$), we write "$\widetilde{N}_\Phi\{f,g\}$" $:= \prod_i \{f(x_i), g(x_i)\}$ (where $\widetilde{N}_\Phi\{f,g\}$ is really a formal placeholder, since what follows is not, strictly speaking, a quantity)

$$= \prod_i \Big\{ \prod_j (x_i - \sigma_j), \prod_k (x_i - \tau_k) \Big\}.$$

Now formally "use" the Lemma 1.

$$= \prod_{i,j,k} \{x_i - \sigma_j, \sigma_j - \tau_k\}\{\tau_k - \sigma_j, x_i - \tau_k\}$$

$$= \prod_j \Big\{ \prod_i (x_i - \sigma_j), \prod_k (\sigma_j - \tau_k) \Big\} \times \prod_k \Big\{ \prod_j (\tau_k - \sigma_j), \prod_i (x_i - \tau_k) \Big\}$$

$$= \prod_j \{(-1)^n \Phi(\sigma_j), g(\sigma_j)\} \times \prod_k \{f(\tau_k), (-1)^n \Phi(\tau_k)\}.$$

Now we reduce, *e.g.*, in the first factor, $(-1)^n\Phi(w)$ and $g(w)$ modulo $f(w)$ to get, respectively, $\bar{\Phi}(w)$ and $\bar{g}(w)$, both of degrees $< \ell$. Since $f(\sigma_j) = 0$, $\bar{\Phi}(\sigma_j) = \bar{\Phi}(\sigma_j) + \phi(\sigma_j) f(\sigma_j) = (-1)^n \Phi(\sigma_j)$. Similarly $\bar{g}(\sigma_j) = g(\sigma_j)$.

$$= \prod_j \{\bar{\Phi}(\sigma_j), \bar{g}(\sigma_j)\} \times \prod_k \{\bar{f}(\tau_k), \tilde{\bar{\Phi}}(\tau_k)\}$$

$$= \widetilde{N}_f\{\bar{\Phi}, \bar{g}\} \times \widetilde{N}_g\{\bar{f}, \tilde{\bar{\Phi}}\}.$$

These should be thought of as norms on $K_2(\mathbb{C}(X_f))$ and $K_2(\mathbb{C}(X_g))$ relative to the extensions $\mathbb{C}(X_f)/\mathbb{C}(\mathbb{P}^1_z)$ and $\mathbb{C}(X_g)/\mathbb{C}(\mathbb{P}^1_z)$ (rather than $\mathbb{C}(X = X_\Phi)/\mathbb{C}(\mathbb{P}^1_z)$), where *e.g.*, $X_f = \{(z,w) \mid f(z,w) = 0\} \xrightarrow{(x,\sigma)} \mathbb{P}^1 \times \mathbb{P}^1$.[4]

Continuing this process, we reach degree 0 (in w, corresponding to a degree 1 (trivial) field extension of $\mathbb{C}(\mathbb{P}^1_z)$) so that everything is rational functions of z. Thus we land in $K_2(\mathbb{C}(\mathbb{P}^1))$, and define by abuse of notation "$N_h\{f,g\}$"$:= N_\Phi\{f,g\}$:=the element so obtained. So in retrospect, this can formally be seen as a recursive definition of an element in $K_2(\mathbb{C}(\mathbb{P}^1))$.

[4] X_f and X_g are *not* intermediate in the covering $X \to \mathbb{P}^1_z$. Rather all three are intermediate in some covering $Y \to \mathbb{P}^1_z$. Here is the full "dictionary" of meromorphic functions on these Riemann surfaces:

$$X = X_\Phi \leftrightarrow (z = h, f, g, w = x), \quad X_f \leftrightarrow (z = h_f, g = \bar{g}, \Phi = \bar{\Phi}, w = \sigma),$$

$$X_g \leftrightarrow (z = h_g, f = \bar{f}, \Phi = \tilde{\bar{\Phi}}, w = \tau), \quad Y \leftrightarrow (z = h_Y, w, \sigma, \tau, x);$$

σ, τ, x, together with z, give maps from Y to (the embedded images in $\mathbb{P}^1 \times \mathbb{P}^1$ of) X_f, X_g, X, respectively. The σ_j, τ_k, x_i are just the branches of w on X_f, X_g, X over $\mathbb{P}^1_z$, respectively. On Y one may write the branches of σ as $\sigma_{ijk}(z)$ (resp. τ as $\tau_{ijk}(z)$, x as $x_{ijk}(z)$), where changing i or k (resp. i or j, j or k) has no effect.

2.3 Behavior with Respect to Evaluation and Regulator Maps

For any Riemann surface Y one may verify that pointwise evaluation $\Theta_Y\{f,g\}(p) := \{f(p),g(p)\}$ induces a well-defined map (*cf.* the Appendix).

$$\Theta_Y : K_2(\mathbb{C}(Y)) \longrightarrow \{\eta_Y \to K_2(\mathbb{C})\}.$$

Somewhat more exotically, we would like to be able to hit $\tilde{N}_\Phi$ and N_Φ (the beginning and end of the norm algorithm) both with Θ to obtain

$$(*) \qquad \prod_{p_i \in h^{-1}(z)} \{f(p_i),g(p_i)\} = \left[\Theta_{\mathbb{P}^1}(N_h\{f,g\})\right](z)$$

where the p_i are counted with multiplicity if z is a branch point. Unfortunately this is true only *almost everywhere*: while the norm algorithm commutes with evaluation (in the sense that the same manipulations would be correct in $K_2(\mathbb{C})$ over a fixed z_0), the introduction of $\sigma_j - \tau_k$ in the norm algorithm (via the Lemma 1) produces zeroes (and poles) where there were none.

On the other hand, if we knew *a posteriori* that $N_h\{f,g\}$ were of the form $K_2(\mathbb{C}) \subseteq K_2(\mathbb{P}^1)$, then we would know that these zeroes (and poles) had been removed either (i) in the remainder of the norm algorithm, or (ii) in the use of the Steinberg relations in $\otimes^2\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(\mathbb{P}^1)} \setminus \{0,\infty\}]$ to reach $\otimes^2\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty\}]$. In either case $(*)$ holds for all $z \in \mathbb{P}^1$ for which the right hand term makes sense, *i.e.*, for which $h^{-1}(z) \cap (|(f)| \cup |(g)|) = \varnothing$. To see this, one can simply repeat the algorithm of the well-definedness argument (extended to include fractional powers of ϵ) locally on $\mathbb{P}^1$.

So towards this objective we show that the norm algorithm commutes with something which (a) yields local information and (b) does not flinch at the sight of zeroes: the regulator (whose paths may avoid going *through* any specified number of points, because of the direct limit). We claim the following "projection formula":

$$(**) \qquad [R_{\mathbb{P}^1}(N_h\{f,g\})](\gamma) = \left(R_X\{f,g\}\right)(h^{-1}\gamma),$$

where $h^{-1}\gamma$ is a path in X with (possibly non-closed) branches γ_i over the z-sphere. There is absolutely no problem with the meaning of the left-hand side, because $N_h\{f,g\}$ is an element of $K_2(\mathbb{C}(\mathbb{P}^1))$.

Next, because we are going to break the path $h^{-1}\gamma$ into pieces, we need an equivalent form of the regulator that does *not* involve continuing $\log f$ along a path. So henceforth "log" will always mean the branch[5] with argument $\in (-\pi,\pi]$, whether it is being applied to f or even f_i. To compensate for this in the expression for the regulator, we must also replace $-\int_\gamma \log g(p_0)\,\mathrm{dlog} f$ by $2\pi i \sum_{q\in\gamma\cap T_f} \pm \log g(q)$, where $T_f = f^{-1}(\mathbb{R}^-)$ is shorthand for the branch cuts of $\log f$ on X, and the sign is positive for a jump (along γ) from 0 to $2\pi i$ and negative for the opposite.

[5] "branch" here has nothing to do with branches of X over $\mathbb{P}^1$ (unlike, say, "branches of f and g").

Now, writing $f_i = f(z,x_i(z))$ and $g_i = g(z,x_i(z))$ for branches of f and g and working on the right-hand side of $(**)$, we have:

$$\begin{aligned}(R_X\{f,g\})(h^{-1}\gamma) &= \exp\Big[\Big(\frac{1}{2\pi\sqrt{-1}}\int_{h^{-1}\gamma} \log f\,\mathrm{dlog}\, g\Big) + \sum_{q\in\gamma\cap T_f} \log g(q)\Big]\\ &= \prod_i \exp\Big[\Big(\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_i} \log f_i\,\mathrm{dlog}\, g_i\Big) + \sum_{q\in\gamma\cap T_{f_i}} \log g_i(q)\Big]\\ &=: \tilde{R}_{\mathbb{P}^1}\Big(\prod_i \{f_i,g_i\}\Big)(\gamma) = \tilde{R}_{\mathbb{P}^1}\big(\tilde{N}_h\{f,g\}\big)(\gamma).\end{aligned}$$

Again, $\tilde{R}$ is just a formal placeholder rather than an actual regulator, although we do have a well-defined quantity here. Our claim is that the norm algorithm, applied to the expression $\prod_i\{f_i,g_i\}$ in parentheses (to obtain $N_h\{f,g\}$), preserves the value of this quantity while gradually turning it into an expression which *is* a regulator of something (on $\mathbb{P}^1$). We outline how to see this. If one backtracks through the proof of our algebraic Lemma, one finds that the formal Steinberg relations (iii), forgotten in the stage of the norm algorithm which we have written out, are

$$\prod_{i,j,k}\Big(\Big\{\frac{x_i-\sigma_j}{x_i-\tau_k}, 1-\frac{x_i-\sigma_j}{x_i-\tau_k}\Big\} \times \{-(x_i-\tau_k), x_i-\tau_k\}\Big).$$

We want to show that $[\tilde{R}_{\mathbb{P}^1}(\,\cdot\,)](\gamma)$ applied to this gives 1. Referring to the discussion of Y in footnote 4, this is (a power of)

$$\Big[R_Y\Big(\Big\{\frac{x-\sigma}{x-\tau}, 1-\frac{x-\sigma}{x-\tau}\Big\} \times \{-(x-\tau), x-\tau\}\Big)\Big](h_Y^{-1}\gamma),$$

which clearly *is* 1. One deals with the (far more numerous) formal relations (i) and (ii) in the same way.

The upshot is that this alternate form of the regulator is compatible with the *formal* operations of the norm algorithm. So the right-hand side of $(**)$ becomes

$$\tilde{R}_{\mathbb{P}}\big(\tilde{N}_f\{\bar{\Phi},\bar{g}\} \times \tilde{N}_g\{\bar{f},\tilde{\bar{\Phi}}\}\big) = \big[R_{X_f}\{\bar{\Phi},\bar{g}\}\big](h_f^{-1}\gamma) \times \big[R_{X_g}\{\bar{f},\tilde{\bar{\Phi}}\}\big](h_g^{-1}\gamma)$$

by essentially the same computation as above in reverse. In this way we gradually "descend to $\mathbb{P}$" and the left-hand side of $(**)$.

The Proof of Suslin

This is now slick: suppose $h = 1$ on $|(f)| \cup |(g)|$. Then $(R_X\{f,g\})(h^{-1}\gamma) = 1$ for all γ on $\mathbb{P}^1$ avoiding 1 (simply slide γ to $\{0\}$ on $\mathbb{P}^1\setminus\{1\}$). By $(**)$, $N_h\{f,g\} \in \ker R_{\mathbb{P}^1}$, which by our work in §1 is $K_2(\mathbb{C})$. So $N_h\{f,g\}$ consists of constants, and

$$\frac{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(0)}{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(\infty)} = 1.$$

Moreover, since $N_h\{f,g\} \in K_2(\mathbb{C})$ and only $h^{-1}(1[=z])$ intersects $|(f)|$ and $|(g)|$, it follows from the discussion following $(*)$ that we may use $(*)$ at $z = 0, \infty$. That is,

$$1 = \frac{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(0)}{\Theta_{\mathbb{P}^1}(N_h\{f,g\})(\infty)} = \frac{\prod_{p\in h^{-1}(0)}\{f(p),g(p)\}}{\prod_{q\in h^{-1}(\infty)}\{f(q),g(q)\}} = \prod_{p\in|(h)|}\{f(p),g(p)\}^{\nu_p(h)}. \blacksquare$$

A Appendix: Evaluation Map

We want to prove that Θ_Y is "well-defined", *i.e.*, for each fixed element of $K_2(\mathbb{C}(Y))$, taking two different representatives and evaluating them at p (for all but finitely many p) should not give two different elements of $K_2(\mathbb{C})$. For a finite number of points it *may* happen that evaluation (for one or the other, or both, of the representatives) does not produce an element of $K_2(\mathbb{C})$ (because there is a zero or pole in the way). To say that this is the *only* way equality can fail is a stronger statement than "Θ_Y is well-defined", and we shall prove the stronger statement.

So one needs to prove the following fact:

$$\prod\{A_i,B_i\}^{m_i} = \prod\{'A_j,'B_j\}^{'m_j} \text{ in } K_2(\mathbb{C}(Y))$$

$$\text{and } A_i,\ B_i,\ 'A_j,\ 'B_j \text{ all } \neq 0,\infty \text{ at } p$$

implies

$$\prod\{A_i(p),B_i(p)\}^{m_i} = \prod\{'A_j(p),'B_j(p)\}^{'m_j} \text{ in } K_2(\mathbb{C}).$$

The nontrivial thing to show here is that it does not matter if the Steinberg relation *by which the $K_2(\mathbb{C}(Y))$ equivalence is accomplished*, contains terms with zeroes or poles at p.

Rewrite the hypothesis in $\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}(Y)} \setminus \{0,\infty\}]$ as a term-for-term equality

$$(\#)\quad \sum m_iA_i\otimes B_i - \sum {'m_j}\,'A_j\otimes {'B_j} = \sum_* (\omega_*\otimes\xi_*\eta_* - \omega_*\otimes\xi_* - \omega_*\otimes\eta_*) + \sum_* ('\omega_*\otimes{'\xi_*} + {'\xi_*}\otimes{'\omega_*}) + \sum_* {'\eta_*}\otimes(1-{'\eta_*}).$$

Fix a function ϵ with a first order zero at p; if $Y = \mathbb{P}^1$ then it could be $(z-p)$. For $\alpha \in \mathbb{C}(Y)$ we will write $\alpha = \epsilon^a\tilde{\alpha}$ where $\tilde{\alpha}(p) \neq 0,\infty$. Some terminology: if $a = 0$ then α is "reduced"; if both α and β are reduced then $\alpha\otimes\beta$ is; and if all (type (i), (ii) or (iii)) terms in a Steinberg are reduced then that Steinberg is. Furthermore, for any $\alpha\otimes\beta = \epsilon^a\tilde{\alpha}\otimes\epsilon^b\tilde{\beta}$ there is a fixed algorithm to produce a (very lengthy) sum of Steinbergs $\mathcal{S}(\alpha\otimes\beta)$ such that (term-for-term)

$$(\#\#)\qquad \alpha\otimes\beta = \mathcal{S}(\alpha\otimes\beta) + \tilde{\alpha}\otimes\tilde{\beta} + \epsilon\otimes(-1)^{ab}\frac{\tilde{\alpha}^b}{\tilde{\beta}^a}.$$

If $\alpha\otimes\beta$ is already reduced then $\mathcal{S}(\alpha\otimes\beta) = 0$.

Now develop the right-hand side of (#) as follows.

(i) Set aside the Steinbergs that are reduced to begin with, and apply the fixed algorithm (##) to *every term* of each remaining Steinberg (the reduced terms among these will be unaffected).

(ii) The resulting (nonzero) $\mathcal{S}$'s are in one-to-one correspondence with all unreduced *terms* from the original right-hand side of (#). Since these terms had to cancel to give the (entirely reduced) terms of the left-hand side, by the same cancellation scheme the $\mathcal{S}$-terms all cancel (oddly enough some of these will be reduced).

(iii) Since the only remaining terms containing ϵ are now of type $\epsilon\otimes(\cdots)$, and (obviously) none of these is reduced, they also neatly cancel out.

The upshot is that we have rewritten the right-hand side of (#) (after some pair creation/annihilation):

$$\sum_*(\tilde{\omega}_*\otimes\tilde{\xi}_*\tilde{\eta}_* - \tilde{\omega}_*\otimes\tilde{\xi}_* - \tilde{\omega}_*\otimes\tilde{\eta}_*) + \sum_*('\tilde{\omega}_*\otimes{'\tilde{\xi}_*} + {'\tilde{\xi}_*}\otimes{'\tilde{\omega}_*}) + \sum_* {'\tilde{\eta}_*}\otimes(\widetilde{1-{'\eta_*}}).$$

The first two sums are of reduced Steinbergs and therefore evaluate to Steinbergs at p. On the other hand, ${'\tilde{\eta}_*}\otimes(\widetilde{1-{'\eta_*}})$ may *not* be a Steinberg. For example, if $a > 0$ and $\alpha = \epsilon^a\tilde{\alpha}$ then

$$\tilde{\alpha}\otimes\widetilde{1-\alpha} = \tilde{\alpha}\otimes(1-\epsilon^a\tilde{\alpha})$$

is not a Steinberg, while if instead $\alpha = \epsilon^{-a}\tilde{\alpha}$ then

$$\tilde{\alpha}\otimes\widetilde{1-\alpha} = \tilde{\alpha}\otimes(\epsilon^a - \tilde{\alpha})$$

is not either! However, evaluating them at p (since $\epsilon(p) = 0$) yields respectively

$$\tilde{\alpha}(p)\otimes 1 \quad\text{and}\quad \tilde{\alpha}(p)\otimes -\tilde{\alpha}(p)$$

which *are* Steinberg relations in $\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}} \setminus \{0,\infty\}]$. Therefore we have expressed

$$\sum m_iA_i(p)\otimes B_i(p) - \sum {'m_j}\,'A_j(p)\otimes{'B_j}(p)$$

as a sum of Steinbergs, the corresponding element of $K_2(\mathbb{C})$ is zero, and we are done.

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