On generalizations of the Shimura-Taniyama conjecture

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Given the zeta function of an algebraic variety,

what is the automorphic representation which corresponds to ?

We ask this questions in a more general setting.

Let E and F be number fields.

M: a pure motive of weight w over F with coefficients in E.

Take an embedding $E \hookrightarrow \mathbf{C}$. Then we ask:

Problem. Find a connected reductive algebraic group G over F, an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $G(F_A)$ and a representation r of the L-group ${}^L G$ such that $L(M, s) = L(s, \pi, r)$.

The answer (G, π, r) to this question is not unique in general.

It is a well-known conjecture that π exists on $GL(d)$ where d is the rank of M.

We will concentrate to find G as small as possible, for which we can formulate a natural generalization of the Shimura-Taniyama conjecture.

The other answers could be derived by the functoriality principle.

The importance to find a small G can be seen by the following classical example.

$\mathcal{E}$: an elliptic curve defined over $\mathbf{Q}$ with complex multiplication by an imaginary quadratic field K.

$\exists\psi$: a Hecke character of $K_A^\times$ such that $L(s,\psi) = L(s,\mathcal{E})$.

Let $T = R_{K/\mathbf{Q}}(\mathbf{G}_m)$ be a non-split torus over $\mathbf{Q}$.
We have $T(\mathbf{Q}_A) = K_A^\times$ and ψ can be regarded as an automorphic representation of $T(\mathbf{Q}_A) \subset \mathrm{GL}(2, \mathbf{Q}_A)$.

The motive M has three realizations.

$H_B(M)$: Betti realization, $\operatorname{rank} M = \dim_E(H_B(M))$

$H_\lambda(M)$: λ-adic realization, $\operatorname{rank} M = \dim_{E_\lambda}(H_\lambda(M))$

$H_{\mathrm{DR}}(M)$: de Rham realization

We have a representation $(V_\lambda = H_\lambda(M))$

$\rho_\lambda : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(V_\lambda)$

$$L(M,s) = \prod_{\mathfrak{p}} [\det(1 - (\rho_\lambda(\sigma_{\mathfrak{p}})|V_\lambda^{I_{\mathfrak{p}}})N(\mathfrak{p})^{-s}]^{-1}.$$

I am going to formulate a natural conjecture.

Three necessary ingredients are:

The Hodge group(Mumford Tate group)

λ-adic representations

L-groups and L-packets

§1. The Hodge group

V: a finite dimensional vector space over $\mathbf{Q}$.

w: an integer.

A decomposition

$$V \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_{p+q=w} V^{p,q}, \qquad \overline{V^{p,q}} = V^{q,p},$$

$V^{p,q}$ being a $\mathbf{C}$-subspace of $V \otimes_{\mathbf{Q}} \mathbf{C}$, is called a $\mathbf{Q}$-rational Hodge structure of weight w on V.

Let $S = R_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)$.

$\exists h$: a morphism $S \longrightarrow GL(V)$ of algebraic groups defined over $\mathbf{R}$ such that

$$h(z)v^{p,q} = z^{-p}\overline{z}^{-q}v^{p,q}, \qquad v^{p,q} \in V^{p,q}, \qquad z \in \mathbf{C}^{\times} = S(\mathbf{R}).$$

Definition. The Hodge group $\mathrm{Hg}(V)$ is the smallest algebraic subgroup defined over $\mathbf{Q}$ of $\mathrm{GL}(V)$ which contains the image of h.

Then $\mathrm{Hg}(V)$ is connected. If the Hodge structure is polarizable, then $\mathrm{Hg}(V)$ is reductive.

V: a finite dimensional vector space over E.

When we regard V as a vector space over $\mathbf{Q}$, we denote it by $\underline{V}$.

A decomposition

$$V \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_{p+q=w} V^{p,q}, \qquad \overline{V^{p,q}} = V^{q,p},$$

$V^{p,q}$ being an $E \otimes_{\mathbf{Q}} \mathbf{C}$-submodule of $V \otimes_{\mathbf{Q}} \mathbf{C}$, is called an E-rational Hodge structure of weight w.

Example.

A/F: an abelian variety defined over a number field F.

$F \hookrightarrow \mathbf{C}$.

Assume $\mathrm{End}(A) \otimes \mathbf{Q} \cong E$.

The Betti cohomology $H^1(A, \mathbf{Q})$ carries an E-rational

Hodge structure of weight 1.

$H^1(A, \mathbf{Q}) \cong E^d$, $2 \dim A = d[E : \mathbf{Q}]$.

Define $h : S \longrightarrow GL(\underline{V})$ using the underlying $\mathbf{Q}$-rational Hodge structure $\underline{V}$. Since h is E-linear, we have

$$\mathrm{Im}(h) \subset \mathrm{GL}(V)(E \otimes_{\mathbf{Q}} \mathbf{C}) \subset \mathrm{GL}(\underline{V})(\mathbf{C}).$$

Definition. The Hodge group $\mathrm{Hg}(V)$ is the smallest algebraic subgroup defined over E of $GL(V)$ such that the group of $(E \otimes_{\mathbf{Q}} \mathbf{C})$-valued points contains $\mathrm{Im}(h)$.

We can give the relation between $\mathrm{Hg}(V)$ and $\mathrm{Hg}(\underline{V})$ by the concept E-envelope.

Let $H/\mathbf{Q} \subset \mathrm{R}_{E/\mathbf{Q}}(\mathrm{GL}(V)) \subset \mathrm{GL}(\underline{V})$.

The smallest algebraic subgroup G/E of $\mathrm{GL}(V)$ such that $\mathrm{R}_{E/\mathbf{Q}}(G) \supset H$ is the E-envelope of H.

Proposition. $\mathsf{Hg}(V)$ is the E-envelope of $\mathsf{Hg}(\underline{V})$.

The relation with the λ-adic representation. Let

$$\rho_\lambda : \mathsf{Gal}(\overline{F}/F) \longrightarrow \mathsf{GL}(H_\lambda(M)) \cong \mathsf{GL}(d, E_\lambda)$$

be the λ-adic representation.

Conjecture (Mumford-Tate-Serre). Let $E = \mathbf{Q}$, $\lambda = \ell$.

There exists an algebraic group H over $\mathbf{Q}$ such that

$H^0 = \mathsf{Hg}(M)$, (LHS is the identity componet of H) and that

$\mathrm{Im}(\rho_\ell)$ is Zariski dense in H. More strongly

$\mathrm{Im}(\rho_\ell)$ is open in $H(\mathbf{Q}_\ell)$ in ℓ-adic topology.

Let $V = H_B(M)$: $\mathbf{Q}$-rational Hodge structure.

We consider the tensor spaces:

$$W = V^{\otimes m} \otimes \check{V}^{\otimes n} \otimes T(l), \qquad M^{\otimes m} \otimes \check{M}^{\otimes n} \otimes T(l).$$

The tensors of type $(0,0)$ of W correspond to algebraic cycles (Hodge conjecture); hence after a finite extension $F \to F'$, $\mathrm{Im}(\sigma_\ell)$ fixes these tensors, so

$$\sigma_\ell(\mathrm{Gal}(\overline{F}/F')) \subset \mathrm{Hg}(M)(\mathbf{Q}_\ell).$$

On the other hand, the Tate conjecture says that the tensors fixed by $\sigma_\ell(\mathrm{Gal}(\overline{F}/F'))$ (for some F') are algebraic. Hence

$$\text{The Zariski closure of } \sigma_\ell(\mathrm{Gal}(\overline{F}/F')) = \mathrm{Hg}(M)(\mathbf{Q}_\ell).$$

In general, we have

Conjecture A. There exists an algebraic group H over E such that $H^0 = \mathrm{Hg}(M)$ and that $\mathrm{Im}(\rho_\lambda)$ is Zariski dense in H.

Two conjectures are compatible, because:

$H_\ell(M) = \oplus_{\lambda|\ell} H_\lambda(M)$, $\sigma_\ell(M) = \oplus_{\lambda|\ell} \rho_\lambda(M)$.

$\mathfrak{H}$: Zariski closure of $\mathrm{Im}(\sigma_\ell)$.

$\widetilde{\mathfrak{H}}$: E-envelope of $\mathfrak{H}$.

$$\mathrm{Im}(\sigma_\ell) = \mathrm{Im}(\oplus_{\lambda|\ell}\rho_\lambda) \subset \mathfrak{H}(\mathbf{Q}_\ell) \subset \mathsf{R}_{E/\mathbf{Q}}(\tilde{\mathfrak{H}})(\mathbf{Q}_\ell) = \prod_{\lambda|\ell} \tilde{\mathfrak{H}}(E_\lambda).$$

Hence

$$(*) \qquad \mathrm{Im}(\rho_\lambda) \subset \tilde{\mathfrak{H}}(E_\lambda).$$

H: the closure of $\mathrm{Im}(\rho_\lambda)$ w. r. t. E-Zariski topology.

By $(*)$, $H \subset \tilde{\mathfrak{H}}$. Assume H is independent of λ.

Then we have

$$\mathrm{Im}(\sigma_\ell) = \mathrm{Im}(\oplus_{\lambda|\ell}\rho_\lambda) \subset \prod_{\lambda|\ell} H(E_\lambda) = \mathsf{R}_{E/\mathbf{Q}}(H)(\mathbf{Q}_\ell).$$

Hence $\mathfrak{H} \subset \mathsf{R}_{E/\mathbf{Q}}(H)$, $\tilde{\mathfrak{H}} = H$.

§2. L-groups and L-packets

First we will review factor sets and the Weil group.

Factor sets

Let

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} F \longrightarrow 1$$

be an exact sequence of groups.

For $\sigma \in F$, we choose $\tilde{\sigma} \in G$ so that $\pi(\tilde{\sigma}) = \sigma$.

Then, for σ, $\tau \in F$, we have

$$f(\sigma,\tau)\widetilde{\sigma\tau} = \tilde{\sigma}\tilde{\tau}, \qquad f(\sigma,\tau) \in N.$$

We define $a(\sigma) \in \mathrm{Aut}(N)$ by

$$a(\sigma)n = \tilde{\sigma}n\tilde{\sigma}^{-1}, \qquad n \in N.$$

Then we have

$$i(f(\sigma,\tau))a(\sigma\tau) = a(\sigma)a(\tau),$$

$$f(\sigma,\tau)f(\sigma\tau,\rho) = (a(\sigma)f(\tau,\rho))f(\sigma,\tau\rho).$$

$i(f(\sigma,\tau))$: the inner automorphism of N defined by $f(\sigma,\tau)$.

The datum $\{a(\sigma), f(\sigma,\tau)\}$ is called

a factor set of F taking values in N.

We say two factor sets are *equivalent*

if there exists $\{\alpha_\sigma \in N\}_{\sigma \in F}$ such that

$$a'(\sigma) = i(\alpha_\sigma)a(\sigma),$$

$$f'(\sigma,\tau) = \alpha_\sigma(a(\sigma)\alpha_\tau)f(\sigma,\tau)\alpha_{\sigma\tau}^{-1}.$$

The transformation $\{a, f\} \longrightarrow \{a', f'\}$ is caused

when we change $\widetilde{\sigma}$ to $\alpha_\sigma\widetilde{\sigma}$.

Let $\varphi : \widetilde{F} \longrightarrow F$ be a group homomorphism.

Then $\{a(\varphi(\sigma)),\ f(\varphi(\sigma),\varphi(\tau))\}$ is a factor set of $\widetilde{F}$

taking values in N.

We call this process *the inflation* by φ.

We say a factor set *splits*

if it is equivalent to a factor set with $f(\sigma,\tau)=1$ for all σ, τ.

This is the case if and only if G is isomorphic to

the semi-direct product of N with F: $G = N \rtimes F$.

When N is abelian,

N is a left F-module by the action a of F on N,

and a factor set is a 2-cocycle. .

The Weil group

K/F: a finite Galois extension of fields, $n = [K : F]$.

$$1 \longrightarrow \mathrm{Gal}(K_{ab}/K) \longrightarrow \mathrm{Gal}(K_{ab}/F) \longrightarrow \mathrm{Gal}(K/F) \longrightarrow 1 \qquad \text{(exact)}$$

K_{ab}: the maximal abelian extension of K contained in a fixed $\overline{K}$.

We obtain a cohomology class

$$\eta \in H^2(\mathrm{Gal}(K/F), \mathrm{Gal}(K_{ab}/K)).$$

K: non-archimeden local field

$K^\times$ is a dense subgroup of $\mathrm{Gal}(K_{ab}/K)$.

The element ξ corresponding to η is a canonical generator (the fundamental class) of $H^2(\mathrm{Gal}(K/F), K^\times) \cong \mathbf{Z}/n\mathbf{Z}$.

We can form an extension by ξ:

$$1 \longrightarrow K^\times \longrightarrow W_{F,K} \longrightarrow \mathrm{Gal}(K/F) \longrightarrow 1 \qquad \text{(exact)}$$

$W_{F,K}$: the relative Weil group

$W_F = \varprojlim W_{F,K}$: the absolute Weil group

$W'_F = \mathbf{G}_a \rtimes W_F$: Weil-Deligne group scheme

$W_\mathbf{C} = \mathbf{C}^\times$, $W_\mathbf{R} = W_{\mathbf{R},\mathbf{C}}$.

K: number field

$C_K = K_A^\times / K^\times$: idele class group.

D_K: the identity component of C_K

$\mathrm{Gal}(K_{ab}/K) \cong C_K/D_K$.

By using the canonical generator (fundamental class) of $H^2(\mathrm{Gal}(K/F), C_K) \cong \mathbf{Z}/n\mathbf{Z}$, we can construct the Weil group:

$$1 \longrightarrow C_K \longrightarrow W_{F,K} \longrightarrow \mathrm{Gal}(K/F) \longrightarrow 1 \qquad \text{(exact)}$$

$W_F = \varprojlim W_{F,K}$: the absolute Weil group

F: a field of characteristic 0.

G: a connected reductive algebraic group defined over F.

Consider G as a group defined over $\overline{F}$.

Take a maximal torus T and a Borel subgroup $B \supset T$.

Then we obtain a based root datum:

$$\mathcal{R}_0(G) = (X^*(T), \Delta, X_*(T), \check{\Delta}).$$

Here $X^*(T)$ is the character group of T.

Δ is the set of simple roots.

$X_*(T)$ is the group of cocharacters of T.

$\check{\Delta}$ is the set of simple coroots.

For $\alpha \in \Delta$, take an element $U_\alpha \ni u_\alpha \neq 1$.

Here U_α is the the root subgroup corresponding to α.

We call $(B, T, \{u_\alpha\}_{\alpha\in\Delta})$ *a splitting datum* for G.

Then we have

$$\mathrm{Aut}(\mathcal{R}_0(G)) \cong \mathrm{Aut}(G, B, T, \{u_\alpha\}_{\alpha\in\Delta}).$$

By the action of $\mathrm{Gal}(\overline{F}/F)$ on B and T,

we obtain a homomorphism

$$\mu_G : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{Aut}(\mathcal{R}_0(G)).$$

Take a connected reductive algebraic group ${}^L G^0$ over $\mathbf{C}$

such that

$$\mathcal{R}_0({}^L G^0) = (X_*(T), \check{\Delta}, X^*(T), \Delta)$$

and form a semi-direct product:

$${}^L G = {}^L G^0 \rtimes W_F.$$

(Variants: ${}^L G = {}^L G^0 \rtimes \mathrm{Gal}(\overline{F}/F)$, ${}^L G = {}^L G^0 \rtimes W_{F,K}$,

${}^L G = {}^L G^0 \rtimes \mathrm{Gal}(\overline{K}/F)$.)

Here, we have

$$\mathrm{Aut}(\mathcal{R}_0({}^L G^0)) \cong \mathrm{Aut}(\mathcal{R}_0(G)),$$

and $\mathrm{Gal}(\overline{F}/F)$ acts on ${}^L G^0$ through μ_G.

Assume that F is a local field.

If F is non-archimedean, put $W'_F = W'_F(\mathbf{C})$.

If F is archimedean, put $W'_F = W_F$ and regard every element as semisimple.

Definition. A homomorphism $\phi : W'_F \longrightarrow {}^LG$ is called a Langlands parameter if the diagram

$$\begin{array}{ccc} W'_F & \xrightarrow{\phi} & {}^LG \\ \downarrow & & \downarrow \\ W_F & = & W_F \end{array}$$

is commutative and the following conditions hold:

(i) ϕ is continuous, $\mathbf{G}_a$ is mapped to unipotent elements of ${}^LG^0$, and ϕ sends semisimple elements to semisimple elements.

(ii) If the image of ϕ is contained in a parabolic subgroup P of ${}^LG$, P is relevant.

If G is quasi-split, then the condition (ii) is automatically satisfied.

We say that two Langlands parameters equivalent if one is transformed to the other by an inner automorphism of ${}^L G^0$.

$\Phi(G/F) = \Phi(G)$: the set of equivalence classes of the Langlands parameters.

$\Pi(G(F))$: the set of equivalence classes of irreducible admissible representations of $G(F)$.

Conjecture. (Langlands) For every $\phi \in \Phi(G)$, there exists a finite subset $\Pi_\phi = \Pi_\phi(G(F))$ of $\Pi(G(F))$ which partitions $\Pi(G(F))$:

$$\Pi(G(F)) = \sqcup_{\phi \in \Phi(G)} \Pi_\phi.$$

We call Π_ϕ the L-packet associated to ϕ.

When $F = \mathbf{C}$, $\mathbf{R}$, Langlands proved the conjecture.

When $G = GL(n)$, F is non-archimeden, the result of Harris-Taylor-Henniart establishes the Conjecture.

For $G = GL(n)$, Π_ϕ is a singleton.

In general, the structure of Π_ϕ can be very complicated.

The existence of Π_ϕ remains conjectural in general.

F: local or global

G, H: connected reductive groups defined over F.

G: quasi-split over F

$\varphi : {}^L H \longrightarrow {}^L G$: L-homomorphism

There is a natural relation of (automorphic) representations of H to those of G. (functoriality principle)

§3. Local parameters attached to a motive

Put $V = H_B(M)$, $\mathsf{Hg}(M) = \mathsf{Hg}(V)$.

$\mathsf{Hg}(M)$ is a connected reductive algebraic group defined over E.

For a finite place λ of E, we have the λ-adic realization $H_\lambda(M)$ and the λ-adic representation

$$\rho_\lambda : \mathsf{Gal}(\overline{F}/F) \longrightarrow \mathsf{GL}(H_\lambda(M)) \cong \mathsf{GL}(d, E_\lambda).$$

Hereafter we assume Conjecture A.

We are going to construct a homomorphism

$$\psi_v : W'_{F_v} (= W'_{F_v}(\mathbf{C})) \longrightarrow H(\mathbf{C})$$

for every place v of F.

Let v be non-archimedean.

Take a finite place λ of E which is prime to v.

By restriction, we have a λ-adic representation

$$\rho_{\lambda,v} : \mathsf{Gal}(\overline{F_v}/F_v) \longrightarrow H(E_\lambda).$$

Take an embedding $E_\lambda \hookrightarrow \mathbf{C}$.

By the procedure due to Deligne, we obtain a representation

$$\psi_v : W'_{F_v} \longrightarrow H(\mathbf{C}),$$

which sends semisimple elements to semisimple elements.

Remark. The equivalence class of ψ_v may depend on the choice of λ. If M is attached to the cohomology of a projective smooth algebraic variety X defined over F, the independence is known when either X is an abelian variety or X has good reduction at v.

Let v be archimedean. Take an embedding $E \hookrightarrow \mathbf{C}$. We have a homomorphism

$$h : S(\mathbf{R}) = \mathbf{C}^\times \longrightarrow H(\mathbf{C})$$

attached to the Hodge structure on $H_B(M)$.

If v is imaginary, then $W_\mathbf{C} = \mathbf{C}^\times$. We take $\psi_v = h$.

Suppose that v is real.

We have the non-split exact sequence

$$1 \longrightarrow \mathbf{C}^\times \longrightarrow W_{\mathbf{R}} \longrightarrow \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \longrightarrow 1.$$

We can realize $W_{\mathbf{R}}$ inside $\mathbf{H}^\times$, the group of nonzero Hamilton quaternions, as

$$W_{\mathbf{R}} = \langle \mathbf{C}^\times, j \rangle, \qquad j^2 = -1, \quad jzj^{-1} = \bar{z}, \ z \in \mathbf{C}^\times.$$

Put $\tau' = \rho_\lambda(c) \in H(E_\lambda) \subset H(\mathbf{C})$, where $c \in \mathrm{Gal}(\overline{F}_v/F_v) \cong \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ is the complex conjugation. Then we can show that

$$\tau' h(z) = h(\bar{z})\tau', \qquad z \in S(\mathbf{R}) = \mathbf{C}^\times.$$

If w is even, we put $\tau = \tau'$.

Suppose w is odd. Then the scalars $\mathbf{C}^\times$ is contained in the center Z of $H(\mathbf{C})$. Taking $t \in Z$ such that $t^2 = h(-1) = (-1)^w$, we put $\tau = \tau' t$. Now τ satisfies

$$\tau^2 = h(-1), \qquad \tau h(z) = h(\bar{z})\tau, \qquad z \in S(\mathbf{R}).$$

Therefore we can define a representation ψ_v of $W_{\mathbf{R}}$ into $H(\mathbf{C})$ by

$$\psi_v | \mathbf{C}^\times = h, \qquad \psi_v(j) = \tau.$$

(Two choices of t amount to give equivalent parameters.)

§4. Main Conjecture

$$\rho_\lambda : \mathsf{Gal}(\overline{F}/F) \longrightarrow GL(H_\lambda(M))$$

We fix an embedding $E_\lambda \hookrightarrow \mathbf{C}$.

There exists the unique finite Galois extension K of F such that the diagram

$$\text{(R)}\quad \begin{tikzcd} 1 \arrow[r] & \rho_\lambda(\mathsf{Gal}(\overline{F}/K)) \arrow[r] \arrow[d] & \rho_\lambda(\mathsf{Gal}(\overline{F}/F)) \arrow[r] \arrow[d] & \mathsf{Gal}(K/F) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H^0(\mathbf{C}) \arrow[r] & H(\mathbf{C}) \arrow[r] & \mathsf{Gal}(K/F) \arrow[r] & 1 \end{tikzcd}$$

is commutative. Here $H^0 = \mathsf{Hg}(M)$, the vertical arrows are inclusion maps and both rows are exact. In fact, K is uniquely determined by the condition

$$H^0(\mathbf{C}) \cap \rho_\lambda(\mathsf{Gal}(\overline{F}/F)) = \rho_\lambda(\mathsf{Gal}(\overline{F}/K)), \qquad \mathsf{Gal}(\overline{F}/K) \supset \mathsf{Ker}(\rho_\lambda).$$

The exact sequence in the second row of (R) defines a homomorphism

$$\mathrm{Gal}(K/F) \longrightarrow \mathrm{Aut}(H^0)/\mathrm{Inn}(H^0).$$

We have a split exact sequence

$$1 \longrightarrow \mathrm{Inn}(H^0) \longrightarrow \mathrm{Aut}(H^0) \longrightarrow \mathrm{Aut}(\mathcal{R}_0(H^0)) \longrightarrow 1.$$

Here

$$\mathcal{R}_0(H^0) = (X^*(T), \Delta, X_*(T), \check{\Delta}).$$

is a based root datum. Thus we have a homomorphism

$$\mu_{H^0} : \mathrm{Gal}(K/F) \longrightarrow \mathrm{Aut}(\mathcal{R}_0(H^0)).$$

Proposition. *There exists a connected reductive algebraic group G defined over F such that*

(i) G is quasi-split over F.

(ii) ${}^L G^0 = H^0(\mathbf{C})$.

(iii) $\mu_G = \mu_{H^0}$. (Here we regard μ_{H^0} as a homomorphism of $\mathrm{Gal}(\overline{F}/F)$ into $\mathrm{Aut}(\mathcal{R}_0(H^0))$.)

We are going to compare ${}^L G$ with the group extension

$$1 \longrightarrow H^0(\mathbf{C}) \longrightarrow H(\mathbf{C}) \longrightarrow \mathrm{Gal}(K/F) \longrightarrow 1$$

of (R).

$\{a(\sigma), f(\sigma,\tau)\}$: a factor set of $\mathsf{Gal}(K/F)$ taking values in $H^0(\mathbf{C})$ defined by the exact sequence (R).

The exact sequence

$$1 \longrightarrow \mathsf{Inn}(H^0) \longrightarrow \mathsf{Aut}(H^0) \xrightarrow{\pi} \mathsf{Out}(H^0) \longrightarrow 1$$

splits. The splitting is given by

$$\mathsf{Out}(H^0) \cong \mathsf{Aut}(\mathcal{R}_0(H^0)) \cong \mathsf{Aut}(H^0, B, T, \{u_\alpha\}_{\alpha\in\Delta}).$$

Let

$$s : \mathrm{Out}(H^0) \longrightarrow \mathrm{Aut}(H^0)$$

be a homomorphism such that $\pi \circ s = \mathrm{id}$.

Take $\sigma \in \mathrm{Gal}(K/F)$. Since $\pi(s(\pi(a(\sigma)))) = \pi(a(\sigma))$, there exists $\alpha_\sigma \in H^0(\mathbf{C})$ such that

$$s(\pi(a(\sigma))) = i(\alpha_\sigma)a(\sigma).$$

Now we consider an equivalent factor set to $\{a(\sigma), f(\sigma, \tau)\}$.

$$a_Z(\sigma) = i(\alpha_\sigma)a(\sigma),$$

$$f_Z(\sigma, \tau) = \alpha_\sigma(a(\sigma)\alpha_\tau)f(\sigma, \tau)\alpha_{\sigma\tau}^{-1}.$$

The mapping $\mathrm{Gal}(K/F) \ni \sigma \longrightarrow a_Z(\sigma) \in \mathrm{Aut}(H^0(\mathbf{C}))$ is a homomorphism. Hence $i(f_Z(\sigma, \tau)) = 1$. This implies that $f_Z(\sigma, \tau) \in Z(H^0(\mathbf{C}))$.

Theorem. The cohomology class of f_Z in $H^2(\text{Gal}(K/F), Z(H^0(\mathbf{C}))$ is uniquely determined by ρ_λ, i.e. doesn't depend on the choices of s, $\widetilde{\sigma}$, α_σ.

f_Z does not split in general.

Definition. A finite Galois extension L of F containing K is called a splitting field for ρ_λ if the cohomology class of f_Z in $H^2(\text{Gal}(L/F), Z(H^0(\mathbf{C})))$ becomes trivial after inflation to $\text{Gal}(L/F)$ by the canonical map $\text{Gal}(L/F) \longrightarrow \text{Gal}(K/F)$. (Or by the canonical map $W_{F,L} \longrightarrow \text{Gal}(K/F)$.)

Assume that L is a splitting field for ρ_λ. Put

$$^LG = {}^LG^0 \rtimes W_{F,L}.$$

Then (R) is embedded into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & ^LG^0 & \longrightarrow & ^LG & \longrightarrow & W_{F,L} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H^0(\mathbf{C}) & \longrightarrow & H(\mathbf{C}) & \longrightarrow & \mathrm{Gal}(K/F) & \longrightarrow & 1. \end{array}$$

We define a homomorphism r of ^LG into $\mathrm{GL}(H_B(M))(\mathbf{C})$ as the composite of the homomorphism $^LG \longrightarrow H(\mathbf{C})$ and the canonical injection. The mapping $\psi_v : W'_{F_v} \longrightarrow H(\mathbf{C})$ can be lifted to the mapping $\phi_v : W'_{F_v} \longrightarrow {}^LG$ so that it is the Langlands parameter.

Now we can formulate a generalized Shimura-Taniyama conjecture.

Main Conjecture. Assume that a splitting field L for f_Z exists. Then there exists an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $G(F_A)$ such that $L(s, \pi, r) = L(M, s)$. Moreover
(i) π_v belongs to the L-packet $\Pi_{\phi_v}(G/F_v)$.
(ii) π is cuspidal if ρ_λ is absolutely irreducible .
(iii) π is tempered.
(iv) π is essentially unitary. In other words, there exists a morphism $\nu : G \longrightarrow \mathbb{G}_m$ defined over F and a quasicharacter χ of $F_A^\times$ such that $\pi \otimes (\chi \circ \nu)$ is unitary.

The discussions above apply also to the case

$H \subset \widetilde{H} \subset \mathrm{GL}(V)$, $\widetilde{H}/F$ is reductive.

§5. On splitting fields

Theorem (Tate-Langlands). Assume that $Z(H^0(\mathbf{C}))$ is connected. Then a splitting field for ρ_λ exists (for the Weil group formalism).

Assume that ρ_λ is absolutely irreducible and $w \neq 0$.

Suppose that $\rho_\lambda|\mathrm{Gal}(\overline{F}/K)$ is isotypic.

Then the center of H^0 is connected. Hence the splitting field exists.

Next assume that $\rho_\lambda | \mathrm{Gal}(\overline{F}/K)$ is not isotypic.

By Clifford's theorem, there exists a field $F \subsetneq K' \subset K$ and a λ-adic representation

$$\tau_\lambda : \mathrm{Gal}(\overline{F}/K') \longrightarrow \mathrm{GL}(W)$$

such that $\rho_\lambda \cong \mathrm{Ind}_{\mathrm{Gal}(\overline{F}/K')}^{\mathrm{Gal}(\overline{F}/F)} \tau_\lambda$.

Here W denotes a finite dimensional vector space over E_λ.

By the theorem just quoted, a splitting field for τ_λ exists.

By Main Conjecture, there exist a connected reductive algebraic group $\widetilde{G}$ defined over K', an irreducible automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(K'_A)$ and a representation $\widetilde{r}$ of ${}^L\widetilde{G}$ such that $L(\widetilde{M}, s) = L(M, s) = L(s, \widetilde{\pi}, \widetilde{r})$.

Put $G' = R_{K'/F}(\widetilde{G})$. The L-group of G' is the induced group of ${}^L\widetilde{G}$. Since $G'(F_A) = \widetilde{G}(K'_A)$, we can find an irreducible automorphic representation π' of $G'(F_A)$ and a representation r' of ${}^LG'$ such that $L(M, s) = L(s, \pi', r')$. The Langlands parameter to which π' corresponds can be described explicitly. In this way, the problem is "almost solved".

However the group G' is slightly "bigger" than G. In this sense, this construction may not be the best.

Theorem. We assume the Hodge conjecture. If ρ_λ is absolutely irreducible and $w \neq 0$, then a splitting field of f_Z as a factor set exists.

We can prove the existence of the splitting field under a natural additional hypothesis.

Question. Are there phenomena of functoriality which cannot be explained by the L-group formalism?

The answer is very probably yes.

§6. Examples.

(1) We may take

$\widetilde{H} = \mathrm{GL}(d)$, $\tilde{G} = \mathrm{GL}(d)$.

By the main conjecture, there exists an automorphic reperesentation π of $\tilde{G}(F_A)$ such that $L(M, s) = L(s, \pi, r)$.

Here r is a standard representation of $\mathrm{GL}(d)$.

Since the L-packets of $\mathrm{GL}(d)$ are singletons, π is unique.

(2) A: an elliptic curve defined over F.

$M = H^1(A)$: a motive over F of rank 2, with coefficients in $\mathbf{Q}$.

As a special case of (1), we have $L(s, A) = L(s, \pi)$ with an automorphic representation π of $\mathrm{GL}(2, F_A)$.

If F is totally real, π corresponds to a Hilbert modular form of weight $(2, 2, \cdots, 2)$.

A: an abelian variety defined over F, $\dim A = n$.

Assume $E \subseteq \mathrm{End}(A) \otimes \mathbf{Q}$, $[E : \mathbf{Q}] = n$.

$M = H^1(A)$: a motive over F of rank 2, with coefficients in E.

As a special case of (1), we have $L(s, A) = L(s, \pi)$ with an automorphic representation π of $\mathrm{GL}(2, F_A)$.

(3) A: an abelian variety defined over F, $\dim A = n$.

$\mathrm{Hg}(A) \subset \mathrm{GSp}(V, \psi) \cong \mathrm{GSp}(n)$.

We take $\widetilde{H} = \mathrm{GSp}(n)$. As $\tilde{G}$, we can take

$$\tilde{G} = (\mathrm{GL}(1) \times \mathrm{Spin}(2n+1))/C, \qquad C = \{1, a\}.$$

Here $a = (a_1, a_2)$, $a_1 = -1 \in \mathrm{GL}(1, \mathbf{C})$,

a_2: the element of order 2 in the center of $\mathrm{Spin}(2n+1)$.

There exists an automorphic representation π of $\tilde{G}(F_A)$ such that $L(s, A) = L(s, \pi, r)$. Here r is a $2n$ dimensional representation of ${}^L G = {}^L G^0 \times W_F$.

If $n = 2$, then $\widetilde{G} = \mathrm{GSp}(2)$.

If $n = 2$, $F = \mathbf{Q}$. Then π is expected to corresponds to a holomorphic Siegel modular form of genus 2, weight 2.

Conjecture. $\exists F$**: a holomorphic Siegel modular form of genus** 2**, weight** 2 **such that** $L(s, A) = L(s, F)$**.**

Here $L(s, F)$ is the spinor L-function of F.

I stated this conjecture about 30 years ago (Inv. Math. 60 (1980).)

Recently J. Tilouine, Brumer, Poor are studying this conjecture and its refinements.