K_1^{ind} of elliptically fibered $K3$ surfaces: a tale of two cycles

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Abstract We discuss two approaches to the computation of transcendental invariants of indecomposable algebraic K_1 classes. Both the construction of the classes and the evaluation of the regulator map are based on the elliptic fibration structure on the family of $K3$ surfaces. The first computation involves a Tauberian lemma, while the second produces a "Maass form with two poles".

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By a seminal result of Chen and Lewis [CL], one already knows that (for fixed lattice L) on a very general L-polarized $K3$ surface X, the indecomposable K_1-classes proliferate like loaves and fishes to span $H^{1,1}_{tr}(X,\mathbb{R})$ under the real regulator map. However, things in general are not settled for ever: the literature lacks a large class of concrete, nontrivial examples occurring in modular families (with the possible exception of Collino's examples obtained by degenerating the Ceresa cycle [Co][1]). The most natural source of such examples should be cycles supported on singular fibers of Kodaira type $I_{n\geq 1}$ in torically-induced or Weierstrass-type internal fibrations. In this paper we consider two families of higher Chow cycles of this type, and investigate properties of the transcendental functions produced by the real regulator map (and a variant reviewed in §1).

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[1] Though presented on Jacobians of genus 2 curves, these can be transferred (using a correspondence) to the corresponding family of Kummer $K3$ surfaces.

The first cycle is on the family of $H \oplus E_8 \oplus E_8 \oplus \langle -12 \rangle$-polarized Kummer $K3$'s studied by Beukers, Peters and Stienstra [BP, Pe, PS], which is parametrized by $\Gamma_1(6)^{+6} \backslash \mathfrak{H}$. By representing it as a family of toric hypersurfaces, one may produce an elliptic structure by restricting a fibration of the ambient toric Fano threefold constructed by appropriately "slicing" its reflexive polytope [AKMS, Ro]. In the spirit of mirror symmetry, we perform a power series computation of the transcendental regulator for our cycle (§2, with a technical detail resolved in §4). For our second example, we revisit the computation of [CDKL] for the Clingher-Doran $M := H \oplus E_8 \oplus E_8$-polarized 2-parameter family of $K3$'s, and prove the results (partially described there) relating the real regulator to higher Green's functions and the thesis of A. Mellit [Me].

While neither wise nor foolish, nor meriting any superlative degree of comparison, we hope these constructions lead to something far, far better (or just more general). The author would also like to thank Chuck Doran, James Lewis, Greg Pearlstein, Duco van Straten, and Stefan Müller-Stach for discussions related to this paper, and to acknowledge partial support from NSF Standard Grant DMS-1068974. We are especially grateful to Adrian Clingher for supplying Remark 3.8.

1 Real and transcendental regulators

We shall introduce only the groups and maps we require; for a more general treatment of cycle maps see Lewis's lectures in this volume [Le] (or §1 of [DK]). Let X be a smooth $K3$ surface over $\mathbb{C}$, and consider the abelian group of "empty rational equivalences"

$$\tilde{K}_1(X) := \frac{\left\{ \overset{\text{(finite sums)}}{\sum q_j.(f_j, D_j)} \;\middle|\; \begin{array}{c} q_j \in \mathbb{Q},\ D_j \subset X \text{ curves},\ f_j \in \mathbb{C}(\tilde{D}_j)^*; \\ \text{and } \sum q_j (\imath_j)_*((f_j)) = 0 \end{array} \right\}}{\langle (f, D) + (g, D) - (fg, D) \rangle}$$

where $\imath_j : \tilde{D}_j \to X$ is the composition of the inclusion of the curve with its desingularization. Algebraic K_1 is the quotient by Tame symbols

$$K_1(X) := \tilde{K}_1(X) \big/ \operatorname{Tame}\{K_2(\mathbb{C}(X))\},$$

with $\mathbb{Q}$-coefficients understood (here and throughout). There is a "formal" (but always zero) fundamental class map

$$cl : K_1(X) \to Hg^{2,1}(X) := F^2 H^3(X, \mathbb{C}) \cap H^3(X, \mathbb{Q}) = \{0\}$$

which is "computed" by sending

$$Z = \sum q_j.(f_j, D_j) \ \longmapsto \ \begin{cases} \Omega_Z := \frac{1}{2\pi i}\sum q_j(\imath_j)_*\frac{df_j}{f_j} \ \in \ F^2\mathcal{D}^3(X) \\ T_Z := \sum q_j(\imath_j)_*(f_j^{-1}(\mathbb{R}^-)) \ \in \ Z^3_{\text{top}}(X) \end{cases}.$$

Vanishing of $Hg^{2,1}(X)$ implies the existence of a $(2,0)$ current and piecewise $\mathcal{C}^\infty$ chain

$$\left.\begin{array}{c} \Xi \in F^2\mathcal{D}^2(X) \\ \Gamma \in C^2_{\text{top}}(X) \end{array}\right\} \text{ such that } \begin{cases} \Omega_Z = d\Xi \\ T_Z = \partial\Gamma \end{cases}.$$

The Abel-Jacobi map

$$AJ: K_1(X) \to J^{2,1}(X) := \frac{H^2(X,\mathbb{C})}{H^{2,0}(X,\mathbb{C}) \oplus H^2(X,\mathbb{Q})} \cong \frac{\{F^1H^2(X,\mathbb{C})\}^\vee}{H_2(X,\mathbb{Q})}$$

is the basic invariant, and a special case of the arithmetic Bloch-Beilinson conjecture says it should be injective. Writing $\log^-(\cdot)$ for the (discontinuous) branch with imaginary part $\in (-\pi,\pi]$ (thought of as a 0-current), and $\delta_{(\cdot)}$ for the current of integration over a chain, AJ is induced by

$$Z \ \longmapsto \ \tilde{R}_Z := \underbrace{\frac{1}{2\pi i}\sum q_j(\imath_j)_*\log^-(f_j)}_{R_Z} - \Xi + \delta_\Gamma \ \in \ \mathcal{D}^2(X).$$

To spell this out, evaluating the $\tilde{R}_Z$ against a d-closed smooth test form $\omega \in F^1A^2(X)$ gives

$$AJ(Z)(\omega) = \frac{1}{2\pi i}\sum q_j \int_{\tilde{D}_j} (\log^-(f_j))\imath_j^*\omega + \int_\Gamma \omega,$$

where Γ is defined "up to a cycle".

Now the group which interests us is the indecomposables

$$K_1^{\text{ind}}(X) := K_1(X)\,/\text{image}\,(\mathbb{C}^* \otimes \text{Div}(X)),$$

and it is conjecturally detected by

$$\overline{AJ}: K_1^{\text{ind}}(X) \to \left\{F^1H^2_{tr}(X,\mathbb{C})\right\}^\vee \Big/ H_2^{tr}(X,\mathbb{Q}).$$

Since $\overline{AJ}$ is hard to compute, one tends instead to compute one of two "quotients". The so-called **transcendental regulator**

$$\Psi: K_1^{\text{ind}}(X) \to \left\{\Omega^2(X)\right\}^\vee \Big/ \text{image}\left\{H_2^{tr}(X,\mathbb{Q})\right\}$$

is given (on $\omega^{2,0} \in \Omega^2(X)$) by

$$\Psi(Z)(\omega^{2,0}) = \int_\Gamma \omega^{2,0}.$$

Since image$\{H_2^{tr}(X,\mathbb{Q})\}$ is intractable for fixed X (except for Picard rank 20), this is primarily of use variationally: if X_t is a family of $K3$ surfaces over a Zariski open $U \subset \mathbb{P}^1$, carrying

- an algebraic family of cycles $Z_t \in K_1^{\text{ind}}(X_t)$,
- a smoothly varying (but possibly multivalued) family of chains Γ_t as above, and
- an algebraically family of holomorphic forms $\omega_t \in \Omega^2(X_t)$ with Picard-Fuchs operator D_{PF}^{ω} annihilating its periods,

then

$$D_{\text{PF}}^{\omega} \int_{\Gamma_t} \omega_t \in \mathbb{C}(t)$$

is an invariant of $\{Z_t\}$ [DM, Thm. 3.2]. We will compute this “inhomogeneous term” for the cycle in §2, with a small caveat (cf. Remark 2.6).

For the **real regulator**

$$r: \, K_1^{\text{ind}}(X) \to \left\{ H_{tr}^{1,1}(X,\mathbb{R}) \right\}^{\vee},$$

which is really the *imaginary* part of $\overline{AJ}$, the main difficulty is in producing appropriate test forms. It is defined by

$$r(Z)(\omega_{\mathbb{R}}) = \Re\left\{ 2\pi i \int \tilde{R}_Z \wedge \omega_{\mathbb{R}} \right\} = \sum q_i \int_{\tilde{D}_j} \log|f_j| \imath_j^* \omega_{\mathbb{R}},$$

on 2-forms $\omega_{\mathbb{R}}$ which must be *smooth, real, d-closed, of pure type* $(1,1)$*, and orthogonal to* $H_{alg}^{1,1}$. This approach is applied to a family of cycles in §3.

In both cases, the cycles of interest arise from an elliptic fibration of X

$$\begin{array}{ccccc} & & p & \leftarrow & 0,\infty \\ X & \supset & D & \twoheadleftarrow & \mathbb{P}_z^1 \\ \downarrow & & \downarrow & & \\ \mathbb{P}^1 & \ni & \{t_0\} & & \end{array}$$

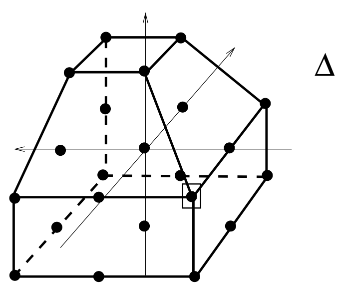
with a nodal rational (Kodaira type I_1) fiber. The class of $(z^a, D) \in K_1^{\text{ind}}(X)$ is independent of how we scale the coordinate z; it depends only on a. The **primitive class** associated to such a fiber, defined up to sign, is the one with $|a| = 1$. Note that its construction requires normalizing D, which can have implications for its minimal field of definition (or its monodromy). It has been known for a long time that similar constructions on I_n fibers in modular elliptic surfaces have trivial class in K_1^{ind}, being in the Tame image of Beilinson’s Eisenstein symbols [Be2]. More recent work of Asakura showed that this is not so on elliptic “Tate surfaces” [As], but did not compute the regulator. We defer to [CDKL] for further discussion of the context for these computations; our personal interest lies in the novel relationships between geometry and arithmetic they uncover through transcendental means.

2 The Apéry family and an inhomogeneous Picard-Fuchs equation

Our first "mathematical short story" begins with the Laurent polynomial

$$\phi(u,v,w) := \frac{(u-1)(v-1)(w-1)(1-u-v+uv-uvw)}{uvw}$$

and the toric threefold $\mathbb{P}_\Delta$ attached to its Newton polytope



(whose singularity corresponding to the $\boxed{\bullet}$ shall not trouble us). The minimal resolution X_t of the Zariski closure of $\{1-t\phi=0\}$ in $\mathbb{P}_\Delta$ defines a $K3$ surface for $t \notin \{0, (\sqrt{2}\pm 1)^4, \infty\} =: \mathcal{L}$, which has Picard rank 19 for general t and is birational to the family considered in [Pe] (cf. [DK]). We shall work with $t \neq 0$ small, for which the singular fibers of the internal elliptic fibration[2]

$$\begin{array}{rcl} \pi: & X_t & \to \mathbb{P}^1 \\ & {\scriptstyle (u,v,w)} & \mapsto {\scriptstyle w} \end{array}$$

have Kodaira types

$w =$	0	1	∞	$\mathsf{w}(t)$	3 more near ∞
type	I_1^*	I_5	I_8	I_1	3 more I_1's

More precisely, a computation shows that the I_1 fibers occur at the solutions of

$$0 = (t^3)w^4 + (3t^2-2t^3)w^3 + (t^3+5t^2+3t)w^2 + (-8t^2-20t+1)w + (16t), \quad (1)$$

all of which but

$$\mathsf{w}(t) = -16tH(t) := -16t\{1+20t+456t^2+11280t^3+\cdots\}$$

[2] In the setup of [AKMS], π is induced by slicing Δ horizontally. This suggests a significant generalization of the computation carried out in this section. Also note that this particular π has Mordell-Weil rank 1.

are large (and asymptotic to $-t^{-1}+3e^{\frac{2\pi i}{3}j}t^{-\frac{2}{3}}+\cdots$ for $j=0,1,2$). With our running assumption of "t small", w is of course 1-to-1.

Remark 2.1. Globally speaking, (1) tells us how singular fibers swap and collide; taking the fixed fibers into account, a resultant calculation shows that all collisions occur for $t \in \mathcal{L}' := \mathcal{L} \cup \left\{1, \frac{27}{40}\right\}$.

On the family of $K3$ surfaces $\{X_t\}$, let φ_t represent a family of topological 2-cycles with class in $H^2_{tr}(X_t)$ and vanishing in homology at $t=0$, where X_t degenerates[3] to $X_0 = \mathbb{P}_\Delta \backslash (\mathbb{C}^*)^3$. Its class is invariant about $t=0$ and unique up to scale; we fix this by saying that its image under the map $Tube : H_2(X_t) \to H_3(\mathbb{P}_\Delta \backslash X_t)$ is the class of the torus $|u|=|v|=|w|=1$. We shall also need relative vanishing cycles for the internal fibration. Similarly, for w close to 1, we let $\varphi_{t,w}$ denote a family of 1-cycles on the elliptic curves $X_{t,w} := \pi^{-1}(w)$ vanishing in $H_1(X_{t,1})$; and let it also denote the multivalued family resulting from their topological continuation. The link between these cycles is via the Lefschetz thimble

$$\Phi_{t,w_0} := \bigcup_{w \in \overrightarrow{1.w_0}} \varphi_{t,w} \;\in\; C^2_{\text{top}}(X_t),$$

which has monodromy $\Phi_{t,w_0} \mapsto \Phi_{t,w_0} + \varphi_t$ as w_0 goes about the unit circle counterclockwise. (Here, "φ_t" is to be understood up to coboundary.) That w_0 is going around both 0 and $\mathsf{w}(t)$ is what is important here; that $X_{t,1}$ is singular is not an issue. The monodromy is the same on circles of radius less than 1 and $\geq \mathsf{w}(t)$.

Let $Z_t \in K_1^{\text{ind}}(X_t)$ be the primitive class supported on $X_{t,\mathsf{w}(t)}$, and note that $\Gamma_t := \Phi_{t,\mathsf{w}(t)}$ bounds on T_{Z_t}. Over $\mathbb{P}^1 \backslash \mathcal{L}'$, the continuation of Z_t has significant monodromy, which can be eliminated by lifting to (the preimage of $\mathbb{P}^1 \backslash \mathcal{L}'$ in) a double-cover of the curve (1). However, as long as t remains small, we need only that Z_t has no monodromy about $t=0$; one way to see this is by a limiting argument, cf. Remark 2.5 below. For the family of holomorphic 2-forms, take

$$\omega_t := \frac{1}{2\pi i} Res_{X_t} \hat{\omega}_t := \frac{1}{2\pi i} Res_{X_t} \left\{ \frac{\frac{du}{u} \wedge \frac{dv}{v} \wedge \frac{dw}{w}}{1 - t\phi(u,v,w)} \right\}$$

and write also

$$\omega_{t,w} := \frac{1}{2\pi i} Res_{X_{t,w}} \hat{\omega}_{t,w} := \frac{1}{2\pi i} Res_{X_{t,w}} \left\{ \frac{\frac{du}{u} \wedge \frac{dv}{v}}{1 - t\phi(u,v,w)} \right\}.$$

Then we are aiming to compute

[3] This degeneration is not semistable, which can be fixed by blowing up the components of X_0 at a few points; this need not trouble us.

$$\Psi(Z_t)(\omega_t) = \int_{\Phi_{t,\mathsf{w}(t)}} \omega_t = \frac{1}{2\pi i}\int_1^{\mathsf{w}(t)} \left(\int_{|u|=|v|=1} \frac{\frac{du}{u}\wedge\frac{dv}{v}}{1-t\phi(u,v,w)} \right) \frac{dw}{w}$$

in terms of power series in t.

Fix real numbers $0<\eta<\alpha\ll 1$. We will study the behavior of the (singular, multivalued) functions

$$\nu(t,t_0) := \int_{\Phi_{t,\mathsf{w}(t_0)}} \omega_t\ , \qquad \tilde{\nu}(t,t_0) := \nu(t,t_0) - \frac{\log(t_0)}{2\pi i}\int_{\varphi_t} \omega_t$$

on the set

$$\mathcal{S} := \{|t| \leq \alpha+\eta\} \times \{\alpha-\eta \leq |t_0| \leq \alpha+\eta\} \subset \mathbb{C}^2.$$

For fixed t, the previous remarks on Φ_{t,w_0} imply that $\tilde{\nu}$ has no monodromy in t_0 about a circle of radius $\geq |t|$. For fixed t_0, $\tilde{\nu}$ (or ν) has no monodromy in t about circles of radius $\leq |t_0|$, while remaining finite as $t\to 0$; and so we may write (uniquely)

$$\tilde{\nu}(t,t_0) = \sum_{n=0}^{\infty} \mathfrak{A}_n(t_0)t^n \quad \text{for } |t|<|t_0|.$$

As $w\to\mathsf{w}(\mathsf{t})$, $\int_{\varphi_{t,w}}\omega_{t,w}$ is asymptotic to (a constant multiple of) $\log(w-\mathsf{w}(t))$, which translates to $(w_0-\mathsf{w}(t))\log(w_0-\mathsf{w}(t))$-type behavior for $\int_{\Phi_{t,w_0}}\omega_t$ and thence to $(t_0-t)\log(t_0-t)$ for $\tilde{\nu}$ (or ν). More precisely, we must have on $\mathcal{S}$

$$\tilde{\nu}(t,t_0) = \left\{ (t-t_0)\log\left(\frac{t}{t_0}-1\right)\right\} F_0(t,t_0) + G_0(t,t_0) \tag{2}$$

and (therefore)

$$\delta_t\tilde{\nu}(t,t_0) = \log\left(\frac{t}{t_0}-1\right) F(t,t_0) + G(t,t_0) \tag{3}$$

where $F, G, F_0, G_0 \in \mathcal{O}(\mathcal{S})$ and $\delta_t := t\frac{\partial}{\partial t}$.

Clearly, the function we must compute is $\nu(t) := \nu(t,t)$. By the above formula, at least on the annulus $\mathcal{A} := \{\alpha-\eta\leq|t_0|\leq\alpha+\eta\}$, $\tilde{\nu}(t) := \tilde{\nu}(t,t)$ is monodromy-free about 0.

Lemma 2.2. *$\tilde{\nu}(t)$ extends to a holomorphic function on the disk $D := \{|t| < \alpha+\eta\}$, and so is representable by power series on $\mathcal{A}$, viz.*

$$\tilde{\nu}(t) = \sum_{m=0}^{\infty} \nu_m t^m.$$

Proof. Since the family $\{Z_t\}$ extends to a (global algebraic) higher Chow cycle on a cover of the total space $\cup_{t\in\mathbb{P}^1\backslash\mathcal{L}'}X_t$, the associated higher normal function is admissible on $D\backslash\{0\}$. (See for example [BPS].) One easily deduces (as in the proof of Prop. 5.28 in [SZ]) that its period $\nu(t)$, and hence $\tilde{\nu}(t)$, is of the form $\sum_{a,q} f_{a,q}(t)t^a\log^q(t)$ on $D\backslash\{0\}$, where $a\in\mathbb{Q}\cap[0,1)$, $q\in\mathbb{Z}_{\geq 0}$, and $f_{a,q}\in\mathcal{O}(D)$. Any function of this form with no monodromy is in $\mathcal{O}(D)$. $\square$

The proof of the following key "Tauberian lemma" is deferred to §4:

Lemma 2.3. *$\sum_{n=0}^{\infty}\mathfrak{A}_n(t)t^n$ converges uniformly on $\{|t|=\alpha\}$, to $\tilde{\nu}(t)$.*

The computations below will show (without using Lemma 2.3!) that the $\mathfrak{A}_n$ are given by Laurent series on $\mathcal{A}$ with poles of order n. Assuming this, we may apply Cauchy's theorem and Lemma 2.3 to obtain

$$\nu_m=\frac{1}{2\pi}\int_{|t|=\alpha}\frac{\tilde{\nu}(t)}{t^{m+1}}dt=\lim_{N\to\infty}\int_{|t|=\alpha}\frac{\sum_{n=0}^{N}\mathfrak{A}_n(t)t^n}{t^{m+1}}dt$$
$$=\lim_{N\to\infty}\sum_{n=0}^{N}[\mathfrak{A}_n(t)t^n]_m=\sum_{n=0}^{\infty}[\mathfrak{A}_n(t)t^n]_m$$
$$=\sum_{n=0}^{\infty}[\mathfrak{A}_n(t)]_{m-n}\,, \tag{4}$$

where $[\cdot]_m$ takes the m^{th} power series coefficient. (Notice that a corollary here is that the last sum itself is convergent.) This will justify the rearrangements we perform below.

Fix $w_0\in\bar{D}_1^*$ (i.e. $0<|w_0|\leq 1$), and assume $t\neq 0$ is "sufficiently small". Then we have

$$\int_{\Phi_{t,w_0}}\omega_t\;=\;2\pi i\int_1^{w_0}\left(\frac{1}{(2\pi i)^2}\int_{|u|=|v|=1}\frac{\mathrm{dlog}u\wedge\mathrm{dlog}v}{1-t\phi(u,v,w)}\right)\frac{dw}{w}\;=$$
$$2\pi i\int_1^{w_0}\sum_{n\geq 0}\frac{t^n(w-1)^n}{w^n}\left[\frac{(u-1)^n(v-1)^n(1-u-v+uv-uvw)^n}{u^nv^n}\right]_{(0,0)}\frac{dw}{w}$$

where $[\cdot]_{(\underline{0})}$ takes the coefficient of u^0v^0, which in this case equals

$$\sum_{k=0}^{n}(-w)^{n-k}\binom{n}{k}\binom{n+k}{n}^2.$$

With this substitution, the above integral

$$=\;2\pi i\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}^2t^n\int_1^{w_0}\frac{(w-1)^n}{w^{k+1}}dw$$

$$= 2\pi i \sum_{n=0}^{\infty} t^n \sum_{\substack{k,\,\ell\,=\,0\\ k\,\neq\,\ell}}^{n} (-1)^{\ell-k} \binom{n}{k}\binom{n}{\ell}\binom{n+k}{n}^2 \frac{w_0^{\ell-k}-1}{\ell-k}$$

$$+ 2\pi i \log w_0 \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2$$

in which we recognize the sum in the second term as $\sum_{n=0}^{\infty} t^n[\phi^n]_{(\underline{0})} = \frac{1}{(2\pi i)^2}\int_{\varphi_\lambda}\omega_t$.

Having carried out this calculation, the result can be continued in t to $|t| < |\mathsf{w}^{-1}(w_0)|$ as $\int_{\Phi_{t,w_0}}\omega_t$ is holomorphic there. We conclude that for $(t,t_0)\in\mathcal{S}$ with $|t|<|t_0|$,

$$\frac{\tilde{\nu}(t,t_0)}{2\pi i} = \frac{1}{2\pi i}\int_{\Phi_{t,\mathsf{w}(t_0)}}\omega_t - \frac{\log t_0}{(2\pi i)^2}\int_{\varphi_t}\omega_t$$

$$= \sum_{n=0}^{\infty} \left\{ \begin{array}{c} \sum_{\substack{k,\,\ell\,=\,0\\ k\,\neq\,\ell}}^{n} \binom{n}{k}\binom{n}{\ell}\binom{n+k}{n}^2 \frac{16^{\ell-k}t_0^{\ell-k}H(t_0)^{\ell-k}-(-1)^{\ell-k}}{\ell-k} \\ +\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}^2\log(-16H(t_0)) \end{array} \right\} t^n,$$

and the term in braces is our $\frac{1}{2\pi i}\mathfrak{A}_n(t_0)$ from above. Interpreting powers and log of $H(t_0)$ as power series in t_0, the claim below Lemma 2.3 is now verified. We may summarize what has been proved by saying that $\tilde{\nu}(t)$ may be computed *by substituting $t_0 = t$ in the last sum and rearranging by power of t*. Each coefficient becomes an infinite series (due to the terms with $k>\ell$) whose convergence is nontrivial and guaranteed by the preceding argument, as is the convergence of the resulting power series for small t. See Remark 2.6 below for the precise domain of convergence.

Performing this computation – that is, applying (4) – we find the first few power series coefficients:

$$\frac{\nu_0}{2\pi i} = \log 16 - \sum_{n\geq 1}\frac{\binom{2n}{n}^2}{16^n n},$$

$$\frac{\nu_1}{2\pi i} = 22 + 5\log 16 - 20\sum_{n\geq 2}\frac{\binom{2n}{n}^2}{16^n(n-1)},$$

$$\frac{\nu_2}{2\pi i} = \frac{1703}{4} + 73\log 16 - 8\sum_{n\geq 3}\frac{\binom{2n}{n}^2}{16^n}\frac{259n^2-258n+64}{(n-2)(2n-1)^2}.$$

In particular, we recognize[4] the first of these as $\frac{8}{\pi}G$, where G is Catalan's constant $\sum_{k\geq 0}\frac{(-1)^k}{(2k+1)^2}$; one naturally wonders if the others hold arithmetic

[4] cf. (for example) [DK, (6.15)ff]

interest. The sought-for function is, of course,

$$\frac{\nu(t)}{2\pi i} \;=\; \sum_{m\geq 0} \frac{\nu_m}{2\pi i} t^m \;+\; \frac{\log t}{(2\pi i)^2} \int_{\varphi_t} \omega_t .$$

The log term can be removed by tweaking our choice of Γ by a cycle (and hence ν by a period); this will simplify the computation with D_{PF}. Using results from [Pe], one can compute the periods of integral cycles φ, ξ, η (sent by monodromy about $t = 0$ to $\varphi, \xi - 12\varphi, \eta + \xi - 6\varphi$ resp.) to be

$$\int_{\varphi_t} \omega_t = (2\pi i)^2 \{1 + 5t + 73t^2 + 1445t^3 + \cdots\},$$

$$\int_{\xi_t} \omega_t = -12 \frac{\log t}{2\pi i} \int_{\varphi_t} \omega_t + (2\pi i)\{-144t - 2520t^2 - \cdots\},$$

$$\int_{\eta_t} \omega_t = 6 \frac{\log^2 t}{(2\pi i)^2} \int_{\varphi_t} \omega_t + \frac{\log t}{2\pi i} \int_{\varphi_\xi} \omega_t - 864t^2 - 25920t^3 - \cdots ,$$

the coefficients in the first of which are just $[\phi^n]_{(\underline{0})}$. Replacing Γ_t by $\hat{\Gamma}_t := \Gamma_t + \frac{1}{12}\xi_t$, changes $\frac{\nu}{2\pi i}$ to

$$\frac{\hat{\nu}(t)}{2\pi i} \;=\; \frac{\nu(t)}{2\pi i} + \frac{1}{12}\frac{1}{2\pi i} \int_{\xi_t} \omega_t \;=\; A + Bt + \cdots$$

where $A = \frac{\nu_0}{2\pi i}$, $B = \frac{\nu_1}{2\pi i} - 12$.

Using [op. cit.], one finds that the Picard-Fuchs operator killing periods of ω_t is

$$D^{\omega}_{\mathrm{PF}} = (t^2 - 34t + 1)\delta_t^3 + 2t(t-17)\delta_t^2 + 3t(t-9)\delta_t + t(t-5).$$

Applied to our "higher normal function", this gives

$$\left(D^{\omega}_{\mathrm{PF}} \frac{\nu}{2\pi i} = \right) \, D^{\omega}_{\mathrm{PF}} \frac{\hat{\nu}}{2\pi i} = (B - 5A)\, t + \text{h.o.t.}$$

where

$$\begin{aligned} B - 5A \;&=\; 10 + 5 \sum_{n\geq 1} \frac{\binom{2n}{n}^2}{16^n n} - 20 \sum_{n \geq 2} \frac{\binom{2n}{n}^2}{16^n (n-1)} \\ &=\; 10 + 5 \sum_{n\geq 1} \frac{\binom{2n}{n}^2}{16^n n(2n+2)} \;>\; 0. \end{aligned}$$

So $\hat{\nu}$ is not a period, and we conclude

Theorem 2.4. *For very general t, (the continuation of) Z_t has nontrivial class in $K_1^{ind}(X_t)$, detected by the transcendental regulator.*

Remark 2.5. The I_1 fiber $X_{t,\mathsf{w}(t)}$ supporting Z_t limits, as $t \to 0$, to the nodal rational curve $Y = \{16uv = (u-1)^2(v-1)^2\,,\ w=0\} \subset X_0$. It follows that $X_{t,\mathsf{w(t)}}$ admits a normalization over $\mathbb{C}[[t]]$, justifying the statement that Z_t has no monodromy about 0.

In fact, Z_t itself limits to a class $Z_0 \in H^2_{\mathcal{M}}(X_0, \mathbb{Q}(3))$ in motivic cohomology, and one can use this give an alternative proof of Z_t's nontriviality. More precisely, in the sense of [DK, sec. 6] Z_0 belongs to $W_{-2}H^2_{\mathcal{M}}(X_0, \mathbb{Q}(3)) \cong CH^2(\mathrm{Spec}(\mathbb{C}), 3) \cong K_3^{\mathrm{ind}}(\mathbb{C})$ with AJ map to $\mathbb{C}/\mathbb{Q}(2)$. Now the tangent vectors of Y at the singularity $(u,v) = (-1,-1)$ have slopes $\pm i$, which implies that Y admits a normalization – and hence that Z_0 is defined – over $\mathbb{Q}(i)$ (but not $\mathbb{Q}$). By Beilinson's variant of Borel's theorem (cf. [Be1, Ne], and especially [DK, Prop. 6.2]) AJ of any cycle in $K_3^{\mathrm{ind}}(\mathbb{Q}(i))$ has imaginary part a rational multiple of G.

With our choices above, $AJ(Z_0)$ must match up with $\lim_{t\to 0} \hat{\nu}(t)$ (cf. [DK, Prop. 6.3]). In a different guise, $AJ(Z_0)$ has been computed in [Ke, sec. 4][5] and comes out to exactly $16iG$. This proves immediately that $\hat{\nu}$ cannot have been a period, and satisfyingly explains the presence of $\frac{8}{\pi}G = \frac{16iG}{2\pi i}$ as the leading coefficient above.

Note that the computational method really requires little more than knowing ϕ, $\mathsf{w}(t)$, and D_{PF}, and is likely to work in greater generality than the approach outlined in the last Remark. For instance, uncovering Z_0 in general could require a nontrivial moving lemma calculation, and even here we did not discover Z_0 until the presence of G in $\hat{\nu}_0$ suggested it.

Remark 2.6. Because there are no collisions of internal singular fibers until $t_0 = (\sqrt{2}-1)^4$, Z_t remains well-defined and $\hat{\nu}(t) = \Psi(Z_t)(\omega_t)$ monodromy- and pole-free on D_{t_0}. Since Z_t (hence $\hat{\nu}$) has monodromy about t_0, this is precisely the radius of convergence of $\sum \nu_m t^m$.

The monodromy of $\{Z_t\}$ means that the "inhomogeneous term" $D^{\omega}_{\mathrm{PF}}\hat{\nu}$ is algebraic rather than rational in t. It becomes single-valued upon pullback to the double cover of (1) which makes $\{Z_t\}$ globally well-defined (so that monodromy of the pullback of $\hat{\nu}$ is by periods alone). It was pointed out by D. van Straten that the curve (1) is in fact rational; it is not known whether this is so for the double cover.

3 M-polarized $K3$ surfaces and a higher Green's function

The cycle whose real regulator we shall study appeared in §6 of [CDKL], and we shall preface our second tale with a review of that construction,

[5] where G is incorrectly identified as a transcendental number; that is the conjecture, but its irrationality is still unproven. This has no bearing on nontriviality of $16iG$ modulo $\mathbb{Q}(2)$.

starting with a brief summary of material from [CD]. Let E_λ, for each $\lambda \in \mathbb{P}^1\backslash\{0,1,\infty\}$, denote the Legendre elliptic curve $\{y^2 = x(x-1)(x-\lambda)\}$. Given $(a,b) \in \mathbb{C}^2$, the minimal resolution of

$$\left\{Y^2Z - (4u^3 - 3au - b)W^2Z - \frac{1}{2}Z^2W - \frac{1}{2}W^3 = 0\right\} \subset \mathbb{P}^2_{[Y:Z:W]} \times \mathbb{P}^1_u$$

defines a $K3$ surface $X_{a,b}$ of *Shioda-Inose type*: that is, its Hodge structure $H^2_{tr}(X_{a,b})$ is integrally isomorphic to $H^2_{tr}(E_{\lambda_1} \times E_{\lambda_2})$ for certain Legendre parameters λ_1, λ_2. It turns out that these must satisfy $j(\lambda_1)j(\lambda_2) = a^3$, $j(\lambda_1) + j(\lambda_2) = a^3 - b^2 + 1$. The natural Weierstrass fibration $\theta : X_{a,b} \to \mathbb{P}^1_u$ has an I^*_{12} singular fiber over $u = \infty$, which gives the generic Picard rank 18. However, for a $K3$ we must have $\deg(\theta_*\omega_{X_{a,b}/\mathbb{P}^1}) = 2$, which implies the presence (for generic a, b) of 6 additional singular fibers, each of type I_1. Our cycle $Z_{a,b} \in K_1^{\mathrm{ind}}(X_{a,b})$ will be the primitive class supported on one of these, ignoring for the moment *which* one, as well as issues of sign and monodromy.

One of the achievements of [CD] was an explicit correspondence inducing the isomorphism of Hodge structures above. To produce this, notice that the I^*_{12} embeds a D^+_{16} lattice in $H^2(X_{a,b}, \mathbb{Z})$. This implies the existence of two sections of θ with 2-torsion difference, translating by which gives a Nikulin involution $\mathcal{N}$. This involution has a fixed point (the node) on each I_1 and 2 fixed points on the I^*_{12}, from which one deduces that that the minimal resolution of $X_{a,b}/\mathcal{N}$ has one I^*_6 and 6 I_2 fibers. This Kummer surface, which we denote $\mathcal{K}_{\lambda_1,\lambda_2}$, fits into a diagram of the form

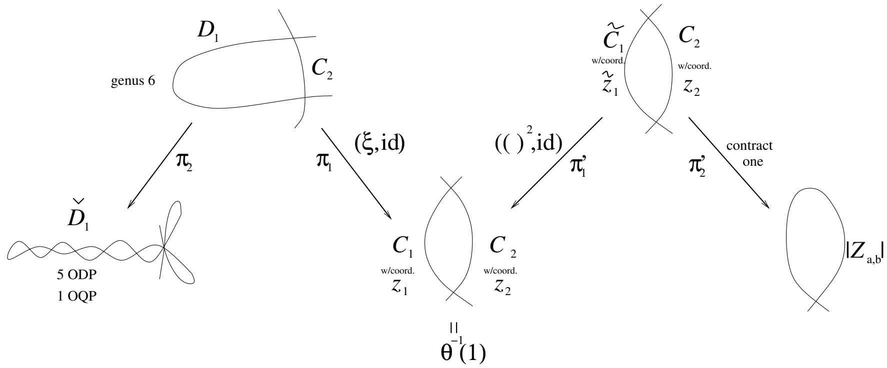
$$\begin{tikzcd}
& \tilde{\mathcal{K}}''_{\lambda_1,\lambda_2} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & & \tilde{\mathcal{K}}'_{\lambda_1,\lambda_2} \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
E_{\lambda_1} \times E_{\lambda_2} \arrow[dr, "/(\text{-id})^{\times 2}", "2:1"'] & & \mathcal{K} \arrow[dl, "\text{resolve}"'] \arrow[dr, "\text{resolve}"] \arrow[dd, "\rho"] & & X_{a,b} \arrow[dl, "/\mathcal{N}"', "2:1"] \arrow[dd] \\
& \check{\mathcal{K}}''_{\lambda_1,\lambda_2} & & \check{\mathcal{K}}'_{\lambda_1,\lambda_2} & \\
& & \mathbb{P}^1 & & \mathbb{P}^1 \arrow[ll, "\cong", "\kappa"']
\end{tikzcd} \tag{5}$$

– where for the moment we think of (a,b) and (λ_1, λ_2) as fixed and very general. Explicitly, $\mathcal{K}_{\lambda_1,\lambda_2}$ can be given as the minimal resolution of $\check{\mathcal{K}}_{\lambda_1,\lambda_2} :=$

$$\left\{U^2X_1X_2 = (X_1 - V)(X_1 - \lambda_1V)(X_2 - V)(X_2 - \lambda_2V)\right\} \subset \mathbb{P}^3_{[X_1:X_2:U:V]},$$

and the elliptic fibration ρ by $[\sum_{i=1}^{2}\left(-\frac{X_i^2}{\lambda_i}+\frac{\lambda_i+1}{\lambda_i}X_iV\right)-V^2 : X_1X_2]$, in terms of which the I_2 fibers lie over $1, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_1\lambda_2}, \frac{\lambda_1\lambda_2+1}{\lambda_1\lambda_2}, \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}$ and the I_6^* fiber over ∞.

The cycle $Z_{a,b}$ is taken to lie on the I_1 fiber over $\kappa^{-1}(1)$. With the aid of the diagram



of curves (rational except for D_1 and $\check{D}_1$) in the top half of (5), we may construct classes in K_1^{ind} of $\mathcal{K}_{\lambda_1,\lambda_2}$ and $E_{\lambda_1}\times E_{\lambda_2}$ with the same real regulator class as $Z_{a,b}$. Indeed, noting that

$$(\pi_2')_*\left\{(\tilde{C}_1,\tilde{z}_1)+(C_2,z_2)\right\}=Z_{a,b},$$

we can set

$$\mathfrak{Z}_{\lambda_1,\lambda_2}:=(\pi_1')_*\left\{(\tilde{C}_1,\tilde{z}_1)+(C_2,z_2)\right\}\equiv\frac{1}{2}(\pi_1)_*\left\{(D_1,z_1\circ\xi)+(C_2,z_2^2)\right\}$$

and $\mathfrak{W}_{\lambda_1,\lambda_2}:=\frac{1}{2}(\pi_2)_*\left\{(D_1,z_1\circ\xi)+(C_2,z_2^2)\right\}$. Explicit normalization of C_1 (or rather its image $\check{C}_1$ in $\check{\mathcal{K}}_{\lambda_1,\lambda_2}$) shows that $\mathfrak{Z}_{\lambda_1,\lambda_2}$ has $\pm$ monodromy in accordance with $\sqrt{\frac{\lambda_1(\lambda_2-1)}{(\lambda_1-1)\lambda_2}}$. Note that this function is constant on the diagonal.

For our test form, we now let[6]

$$\omega_{\mathbb{R},\underline{\lambda}}:=\Re\left\{\frac{dx_1}{y_1}\wedge\overline{\left(\frac{dx_2}{y_2}\right)}\right\}\in A^{1,1}(E_{\lambda_1}\times E_{\lambda_2}).$$

The maps π_i,π_i' are isomorphisms on $H^2_{tr,\mathbb{Q}}$, and we denote also by $[\omega_{\mathbb{R}}]_{\mathcal{K}}$, $[\omega_{\mathbb{R}}]_X$ two more classes, in $H^{1,1}_{tr,\mathbb{R}}$ of $\mathcal{K}_{\lambda_1,\lambda_2}$ resp. $X_{a,b}$, such that all three agree under pullback to $\tilde{\mathcal{K}}_{\lambda_1,\lambda_2},\tilde{\mathcal{K}}'_{\lambda_1,\lambda_2}$. Since AJ commutes with pushforward, we have $r(Z_{a,b})\{[\omega_{\mathbb{R}}]_X\}=r(\mathfrak{Z}_{\lambda_1,\lambda_2})\{[\omega_{\mathbb{R}}]_{\mathcal{K}}\}=r(\mathfrak{W}_{\lambda_1,\lambda_2})(\omega_{\mathbb{R}})=$

[6] we will usually drop the subscript $\underline{\lambda}$

$$\mathcal{R}(\lambda_1,\lambda_2) := \frac{1}{2}\int_{D_1}(\log|z_1\circ\xi|)\pi_2^*\imath_{\tilde{D}_1}^*\omega_{\mathbb{R}} = \frac{1}{2}\int_{C_1}\log|z_1|\underbrace{\xi_*\{\imath_{D_1}^*\pi_2^*\omega_{\mathbb{R}}\}}_{\in\mathcal{D}^{1,1}(C_1)}. \quad (6)$$

At this point, it is convenient to specialize to the Picard rank 19 locus $\lambda_1 = \lambda_2 =: \lambda$, along which the collisions of singular fibers do not affect $\rho^{-1}(1)$; this eliminates the monodromy in $\mathfrak{Z}$, hence that in $\mathcal{R}$. Writing $\mathsf{z} = \mathsf{x} + i\mathsf{y} = re^{i\phi}$, the computation of (6) carried out in [CDKL] specializes to

$$\mathcal{R}(\lambda) := \mathcal{R}(\lambda,\lambda) = -4|\lambda+1|\Re\int_{\mathbb{P}^1}\mathsf{z}\log\left|\frac{\mathsf{z}+i}{\mathsf{z}-i}\right|\frac{P_\lambda(\mathsf{z})\overline{Q_\lambda(\mathsf{z})}}{|S_\lambda(\mathsf{z})|}d\mathsf{z}\wedge d\bar{\mathsf{z}} \quad (7)$$

where

$$\begin{cases} P_\lambda(\mathsf{z}) := & (\lambda^2-\lambda-1)\mathsf{z}^4+2\mathsf{z}^2+(\lambda^3-\lambda^2-2\lambda+1) \\ Q_\lambda(\mathsf{z}) := & (\lambda^3-\lambda^2-2\lambda+1)\mathsf{z}^4+2\mathsf{z}^2+(\lambda^2-\lambda-1) \\ S_\lambda(\mathsf{z}) := & (\mathsf{z}^2-\lambda)(1-\lambda\mathsf{z}^2)(\mathsf{z}^2+1)\left(\mathsf{z}^2-(1+\lambda-\lambda^2)\right)\times \\ & \times\left((1+\lambda-\lambda^2)\mathsf{z}^2-1\right)\left(\mathsf{z}^4+(\lambda^3-3\lambda)\mathsf{z}^2+1\right). \end{cases}$$

An analytic argument [op. cit.] is required to show that $\lim_{\lambda\to 1}\mathcal{R}(\lambda,\lambda)$ agrees with[7]

$$\mathcal{R}(1) = -16\int_{\mathbb{P}^1}\frac{\log\left|\frac{\mathsf{z}+i}{\mathsf{z}-i}\right| r\sin\phi}{|\mathsf{z}^2+1||\mathsf{z}^2-1|^2}\,d\mathsf{x}\,d\mathsf{y} < 0, \quad (8)$$

whereupon we have

Theorem 3.1. *For very general (a,b) resp. (λ_1,λ_2), (the continuation of) $Z_{a,b}$ resp. $\mathfrak{Z}_{\lambda_1,\lambda_2}$, $\mathfrak{W}_{\lambda_1,\lambda_2}$ is real-regulator indecomposable.*

We now turn to the magnificent properties of the function $\mathcal{R}$. More precisely, writing

$$f(\lambda) := i\int_{E_\lambda}\frac{dx}{y}\wedge\overline{\left(\frac{dx}{y}\right)}, \quad \eta_\lambda := \frac{\omega_{\mathbb{R},\lambda}}{f(\lambda)} \in A_{\mathbb{R}}^{1,1}(E_\lambda\times E_\lambda),$$

and $\lambda : \Gamma(2)\backslash\mathfrak{H} \overset{\cong}{\to} \mathbb{P}^1\backslash\{0,1,\infty\}$ for the classical elliptic modular function, we will study

$$\Psi(\tau) := \frac{\mathcal{R}(\lambda(\tau))}{f(\lambda(\tau))} = r\left(\mathfrak{W}_{\lambda(\tau)}\right)\left(\eta_{\lambda(\tau)}\right)$$

for $\tau\in\mathfrak{H}$. As pointed out to the author by C. Doran, some of the general results below have also appeared in A. Mellit's thesis [Me]; we expect that a simple exposition of these matters is nevertheless of value.

Denote by $\mathcal{E}\overset{\pi}{\to}\mathfrak{H}$ the family of elliptic curves with fibers $\pi^{-1}(\tau) = \mathbb{C}/\mathbb{Z}\langle 1,\tau\rangle$, by $\mathcal{E}^{(2)}\overset{\pi^{(2)}}{\to}\mathfrak{H}$ its fiber-product with itself, and by $\mathcal{E}_U$ resp. $\mathcal{E}_U^{(2)}$ the restrictions to an analytic open neighborhood U in any fundamental do-

[7] since $\left|\log\left|\frac{\mathsf{z}+i}{\mathsf{z}-i}\right|\right| < C\left|\mathsf{z}^2-1\right|$ for z near ± 1, this clearly converges.

main for a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$. Let $\mathfrak{Y}$ denote a (complex-) analytic family of $\tilde{K}_1$-cycles on the fibres of $\pi^{(2)}$. We may regard this as an "analytic higher Chow cycle" on $\mathcal{E}_U^{(2)}$ – that is, as a formal sum $\sum q_i.(F_i, \mathcal{S}_i)$ of surfaces paired with meromorphic functions F_i on their analytic desingularizations, with sum of divisors $\sum q_i(F_i) = 0$ in $\mathcal{E}_U^{(2)}$. Therefore, $r_{\mathfrak{Y}} := \sum q_i \log|F_i|\delta_{\mathcal{S}_i}$ makes sense as a $(1,1)$ normal current on $\mathcal{E}_U^{(2)}$, of intersection type with respect to the fibers; likewise for the closed $(2,1)$ current $\Omega'_{\mathfrak{Y}} := (2\pi i \Omega_{\mathfrak{Y}} =) \sum q_i \frac{dF_i}{F_i}\delta_{\mathcal{S}_i}$.

Write $\tau = X + iY$. Let $z = x + iy$ resp. z_1, z_2 be the usual coordinates (modulo $\mathbb{Z}\langle 1, \tau\rangle$) on fibers of π resp. $\pi^{(2)}$. By abuse of notation, we have

$$\eta_\tau := \frac{dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2}{4Y} \tag{9}$$

which is in general Γ-invariant, and in case $\Gamma = \Gamma(2)$ matches up with the form $\eta_{\lambda(\tau)}$ under the isomorphism $(\pi^{(2)})^{-1}(\tau) \cong (E_{\lambda(\tau)})^{\times 2}$. We shall denote by $\mathcal{H}^k_{\pi^{(2)}}$, $\mathcal{H}^{p,q}_{\pi^{(2)}}$ the C^∞ relative cohomology sheaves on U, and by $\mathcal{L}^\bullet$ the Leray filtration on C^∞ forms $A^k(\mathcal{E}_U^{(2)})$. Calling $\alpha \in \mathcal{L}^a A^k(\mathcal{E}_U^{(2)})$ $\pi^{(2)}$*-closed* if $d\alpha \in \mathcal{L}^{a+1}$, we have natural maps

$$[\ ]_U^{\{a\}} : \mathcal{L}^a A^m(\mathcal{E}_U^{(2)})_{\pi^{(2)}\text{-cl}} \longrightarrow A^a(U; \mathcal{H}^{m-a}_{\pi^{(2)}})$$

to cohomology-sheaf valued forms.

Lemma 3.2. *There exists a smooth form $\tilde{\eta} \in A^{1,1}(\mathcal{E}_U^{(2)})$ pulling back to η_τ on fibers, and satisfying:*
(i) $[\partial\tilde{\eta}]_U^{\{1\}} \in A^{0,1}(U; \mathcal{H}^{2,0}_{\pi^{(2)}})$;
(ii) $[\bar{\partial}\tilde{\eta}]_U^{\{1\}} \in A^{1,0}(U; \mathcal{H}^{0,2}_{\pi^{(2)}})$; and
(iii) $\bar{\partial}\partial\tilde{\eta} = \frac{1}{2Y^2}\tilde{\eta} \wedge d\tau \wedge d\bar{\tau}$.

Proof. Consider the C^∞ uniformization $F : U \times (\mathbb{C}/\mathbb{Z}\langle 1, i\rangle) \xrightarrow{\cong} \mathcal{E}_U$ given by $F(\tau, w) := (\tau, \Re(w) + \tau\Im(w))$. According to the easy pullback computation

$$F^*\left(dz - \frac{y}{\Im(\tau)}d\tau\right) = \Re(dw) + \tau\Im(dw),$$

$\widetilde{dz} := dz - \frac{y}{\Im(\tau)}d\tau \in A^{1,0}(\mathcal{E}_U)$ is smooth and well-defined on $\mathcal{E}_U$ (whereas "dz" is not), while pulling back to dz on fibers. We compute

$$d(\widetilde{dz}) = \frac{d\bar{z} - dz}{\tau - \bar{\tau}} \wedge d\tau + \frac{z - \bar{z}}{(\tau - \bar{\tau})^2}d\tau \wedge d\bar{\tau},$$

from which it follows that

$$\partial(\widetilde{dz}) = \frac{\widetilde{dz} \wedge d\tau}{\bar{\tau} - \tau}\ , \quad \bar{\partial}(\widetilde{dz}) = \frac{d\bar{z} \wedge d\tau}{\tau - \bar{\tau}} + \frac{z - \bar{z}}{(\tau - \bar{\tau})^2}d\tau \wedge d\bar{\tau} = \frac{\overline{\widetilde{dz}} \wedge d\tau}{\tau - \bar{\tau}},$$

and then (by conjugation) $\partial(\overline{\widetilde{dz}}) = \frac{\widetilde{dz}\wedge d\bar{\tau}}{\bar{\tau}-\tau}$.

Swtiching to $\mathcal{E}_U^{(2)}$, since

$$\tilde{\eta} := \frac{i}{2}\frac{\widetilde{dz}_1 \wedge \overline{\widetilde{dz}_2} + \overline{\widetilde{dz}_1} \wedge \widetilde{dz}_2}{\tau - \bar{\tau}}$$

pulls back to η_τ on fibers, it is vertically closed. More concretely, we easily compute

$$\partial\tilde{\eta} = i\frac{\widetilde{dz}_1 \wedge \widetilde{dz}_2}{(\tau-\bar{\tau})^2} \wedge d\bar{\tau}\ , \quad \bar{\partial}\tilde{\eta} = -i\frac{\overline{\widetilde{dz}_1} \wedge \overline{\widetilde{dz}_2}}{(\tau-\bar{\tau})^2} \wedge d\tau,$$

which gives *(i)*-*(ii)* . At this point, *(iii)* is easy and left to the reader. □

The next few Lemmas deduce properties of the function

$$\Upsilon(\tau) := \mathcal{R}(\mathfrak{Y}_\tau)(\eta_\tau) = \pi_*^{(2)}\left(r_{\mathfrak{Y}} \wedge \tilde{\eta}\right)$$

on U; of course, we have the case $\mathfrak{Y} = \mathfrak{W}|_U$ and $\Upsilon = \Psi|_U$ (and $\Gamma = \Gamma(2)$) in mind.

Lemma 3.3. *We have $\Delta_{hyp}\Upsilon = -2\Upsilon$, where*

$$\Delta_{hyp} := -Y^2\Delta = -4Y^2\frac{\partial}{\partial\bar{\tau}}\frac{\partial}{\partial\tau}$$

is the hyperbolic Laplacian.

Proof. We shall use the fact that the pairing

$$\pi_*^{(2)} : \mathcal{D}^{1,1}(\mathcal{E}_U^{(2)})_{\pi^{(2)}\text{-cl}} \otimes \mathcal{L}^1 A^{1,2}(\mathcal{E}_U^{(2)})_{\pi^{(2)}\text{-cl}} \longrightarrow \mathcal{D}^{0,1}(U)$$

factors, via $[\ ]_U^{\{0\}} \otimes [\ ]_U^{\{1\}}$, through $\mathcal{D}^0(U;\mathcal{H}^{1,1}_{\pi^{(2)}}) \otimes A^{0,1}(U;\mathcal{H}^{1,1}_{\pi^{(2)}})$. In particular, any components of type $A^{1,0}(U;\mathcal{H}^{0,2}_{\pi^{(2)}})$ in the right-hand factor are killed. (A similar observation applies to $\mathcal{L}^1 A^{2,1}(\mathcal{E}_U^{(2)})_{\pi^{(2)}\text{-cl}}$.) Moreover, $r_{\mathfrak{Y}}$ belongs to the left-hand factor, with $d[r_{\mathfrak{Y}}] = \frac{1}{2}\Omega'_{\mathfrak{Y}} + \frac{1}{2}\overline{\Omega'_{\mathfrak{Y}}}$.

From Lemma 3.2(ii), we have

$$\bar{\partial}\Upsilon = \bar{\partial}\pi_*^{(2)}(r_{\mathfrak{Y}} \wedge \tilde{\eta}) = \pi_*^{(2)}\left(\frac{1}{2}\overline{\Omega'_{\mathfrak{Y}}} \wedge \tilde{\eta}\right) + \pi_*^{(2)}\left(r_{\mathfrak{Y}} \wedge \bar{\partial}\tilde{\eta}\right) = \frac{1}{2}\pi_*^{(2)}\left(\overline{\Omega'_{\mathfrak{Y}}} \wedge \tilde{\eta}\right).$$

Since $\partial[\overline{\Omega'_{\mathfrak{Y}}}] = 0$,

$$\partial\bar{\partial}\Upsilon = -\frac{1}{2}\pi_*^{(2)}\left(\overline{\Omega'_{\mathfrak{Y}}} \wedge \partial\tilde{\eta}\right) = \pi_*^{(2)}\left(r_{\mathfrak{Y}} \wedge \bar{\partial}\partial\tilde{\eta}\right) - \pi_*^{(2)}\left(\bar{\partial}[r_{\mathfrak{Y}} \wedge \partial\tilde{\eta}]\right)$$

which by Lemma 3.2(i),(iii)

$$= \pi_*^{(2)}\left(r_{\mathfrak{Y}} \wedge \bar{\partial}\partial\tilde{\eta}\right) = \frac{1}{2Y^2}\pi_*^{(2)}\left(r_{\mathfrak{Y}} \wedge \tilde{\eta}\right) d\tau \wedge d\bar{\tau} = \frac{\Upsilon}{2Y^2} d\tau \wedge d\bar{\tau}.$$

□

Lemma 3.4. *Let $\tau_0 \in U$ be a CM point (i.e. quadratic irrationality), so that $\pi^{-1}(\tau_0)$ is a CM elliptic curve. Assume that $\mathfrak{Y}_{\tau_0}$ is defined over $\bar{\mathbb{Q}}$. Then $\Upsilon(\tau_0)$ is of the form $\sum \bar{\mathbb{Q}} \log \bar{\mathbb{Q}}$ (i.e., is a sum of algebraic multiples of logarithms of algebraic numbers).*

Proof. On $\left(\pi^{(2)}\right)^{-1}(\tau_0) =: E_0 \times E_0$, write $\mathfrak{D}_1 = E_0 \times \{0\}$, $\mathfrak{D}_2 = \{0\} \times E_0$, $\mathfrak{D}_3 = \Delta_{E_0}$, and $\mathfrak{D}_4$ for the graph of multiplication by τ_0. The presence of $\mathfrak{D}_4$ makes $H^{1,1}(E_0 \times E_0)$, and thus $[\eta_{\tau_0}]$, algebraic. In fact, a simple computation shows that

$$\eta_{\tau_0} \equiv \alpha_1 \delta_{\mathfrak{D}_1} + \alpha_2 \delta_{\mathfrak{D}_2} + \alpha_3 \delta_{\mathfrak{D}_3} + \alpha_4 \delta_{\mathfrak{D}_4}$$

in $H^{1,1}$, with $\alpha_1 := \frac{1-X_0}{2Y_0}$, $\alpha_2 := \frac{|\tau_0|^2 - X_0}{2Y_0}$, $\alpha_3 := \frac{X_0}{2Y_0}$, $\alpha_4 := -\frac{1}{2}$ (all obviously in $\mathbb{Q}(\Im(\tau_0))$). We may assume (by Bloch's moving lemma [Bl]) that $\mathfrak{Y}_{\tau_0} = \sum_i (g_i, D_i)$ with D_i and g_i $\bar{\mathbb{Q}}$-rational, and such that D_i intersects $\mathfrak{D}_j$ properly (with multiplicities all 1) away from $|(g_i)|$. This yields immediately $\Upsilon(\tau_0) =$

$$r(\mathfrak{Y}_{\tau_0})(\eta_{\tau_0}) = \sum_i \sum_{j=1}^{4} \alpha_j \int_{D_i} \log|g_i| \delta_{\mathfrak{D}_j}$$

$$= \sum_{i,j} \sum_{p \in D_i \cap \mathfrak{D}_j} \alpha_j \log|g_i(p)|,$$

with $g_i(p) \in \bar{\mathbb{Q}}$. □

Next, we allow $\mathfrak{Y}$ to fail to be a cycle over a point $\hat{\tau} \in U$; that is, suppose that the 1-cycle $\mathcal{C} := \sum q_i(F_i)$ is supported on $\left(\pi^{(2)}\right)^{-1}(\hat{\tau})$. We say $\mathfrak{Y}$ is singular at $\hat{\tau}$.

Lemma 3.5. *If $\hat{\tau}$ is a CM point, then either (i) Υ is smooth at $\hat{\tau}$ or (ii) $\Upsilon \sim c \log|\tau - \hat{\tau}|$ as $\tau \to \hat{\tau}$, for some $c \in \mathbb{Q}(\Im(\hat{\tau}))^*$. If $\hat{\tau}$ is not a CM point, then the singularity is apparent; that is, Υ remains smooth at $\hat{\tau}$.*

Proof. Since $\sum q_i \int_{\mathcal{S}_{i,\tau}} |\eta_\tau|$ is bounded by a constant and $F_i|_\tau$ depends algebraically on τ, $\Upsilon(\tau) = \sum q_i \int_{\mathcal{S}_{i,\tau}} (\log|F_i|_\tau|)\eta_\tau$ (resp. $\frac{\partial \Upsilon}{\partial \bar{\tau}}$) is bounded by a multiple of $\log|\tau - \hat{\tau}|$ (resp. $\frac{1}{\tau - \hat{\tau}}$). As in the proof of Lemma 3.3, we have $\bar{\partial}\Upsilon = \frac{1}{2}\pi_*^{(2)}\left(\overline{\Omega'_{\mathfrak{Y}}} \wedge \tilde{\eta}\right)$, but $\partial[\overline{\Omega_{\mathfrak{Y}}}] = -2\pi i \delta_{\mathcal{C}}$ instead of 0. Hence

$$\partial\bar{\partial}\Upsilon = -\frac{1}{2}\pi_*^{(2)}\left(\overline{\Omega'_{\mathfrak{Y}}} \wedge \partial\tilde{\eta}\right) - \frac{2\pi i}{2}\pi_*^{(2)}\left(\delta_{\mathcal{C}} \wedge \tilde{\eta}\right)$$

$$= \frac{\Upsilon}{2Y^2} d\tau \wedge d\bar{\tau} - \pi i c \delta_{\{\hat{\tau}\}},$$

where $c := \int_{\mathcal{C}} \eta_{\hat{\tau}}$ belongs to $\mathbb{Q}(\Im(\hat{\tau}))$ by the proof of Lemma 3.4 if $\hat{\tau}$ is CM. From $\Delta_{\text{hyp}}(\Upsilon - c\log|\tau - \hat{\tau}|) = -2\Upsilon$ it now follows that $\frac{\Upsilon - c\log|\tau-\hat{\tau}|}{\log|\tau-\hat{\tau}|} \to 0$.

If $\hat{\tau}$ is not CM, or if $c = 0$ above, then $[\mathcal{C}]$ extends to a section of $\mathcal{H}^{1,1}_{\pi^{(2)},\text{alg}}$; indeed, $\mathcal{C} = \mathcal{M}\cdot(\pi^{(2)})^{-1}(\hat{\tau})$ for some surface $\mathcal{M} = \sum q_i'\mathcal{M}_i \subset \mathcal{E}_U^{(2)}$. Subtracting the decomposable cycle $\sum q_i'.(\tau - \hat{\tau}, \mathcal{M}_i)$ from $\mathfrak{Y}$ (and applying Bloch's moving lemma to make it properly intersect the fiber $(\pi^{(2)})^{-1}(\hat{\tau})$) removes the singularity without affecting Υ. □

Finally, let $\bar{\mathcal{E}}_{\Gamma}^{(2)} \xrightarrow{\bar{\pi}} X(\Gamma)$ be Shokurov's smooth compactification [Sh] of the Kuga modular variety $\Gamma\backslash\mathcal{E}^{(2)} \to Y(\Gamma)$. (In case $\Gamma = \Gamma(2)$, it has fibers $E_\lambda \times E_\lambda$ for $\lambda \in Y(\Gamma)$.) Consider a (higher Chow) precycle $\bar{\mathfrak{Y}}_\Gamma = \sum q_i.(F_i, \mathcal{S}_i)$ on $\bar{\mathcal{E}}_{\Gamma}^{(2)}$, with "boundary" $\sum q_i(F_i)$ supported on $\bar{\pi}^{-1}(\Xi)$ for some finite set $\Xi \subset X(\Gamma)$. Let η_Γ be the (nonholomorphic, real) section of the logarithmically extended Hodge bundle $\mathcal{H}^{1,1}_{\bar{\pi},e} \to X(\Gamma)$ provided by (9). Asymptotics of $\Upsilon_\Gamma(x) := r(\bar{\mathfrak{Y}}_{\Gamma,x})(\eta_{\Gamma,x})$ at points in $\Xi \cap Y(\Gamma)$ are clear from Lemma 3.5, so let $\kappa \in X(\Gamma)\backslash Y(\Gamma)$ be a cusp (with local holomorphic coordinate q).

Lemma 3.6. *(a) $|\Upsilon_\Gamma|$ is bounded by a constant near κ. (b) If $\kappa \notin \Xi$, then this bound is improved to a constant multiple of $\frac{1}{\log|q|}$.*

Proof. Assume for simplicity κ is unipotent, so that $\bar{\mathcal{E}}_{\Gamma}^{(1)}$ has a Néron N-gon over $q = 0$. Writing ω_q for a local generator of the extended relative canonical sheaf, $\log|q|\eta_{\Gamma,q} = \Re(\omega_{q,1} \wedge \bar{\omega}_{q,2})$ limits to a nonzero homology class on $\bar{\pi}^{-1}(\kappa)$. If $\kappa \notin \Xi$, then $r(\bar{\mathfrak{Y}}_\Gamma)$ restricts to a cohomology class on $\bar{\pi}^{-1}(\kappa)$, which pairs with the former to give a (finite) number. Dividing by $\log|q|$ gives *(b)*. For *(a)*, the beginning of the proof of Lemma 3.5 shows that when $x \in \Xi$, the bound is worse than that in *(b)* by a factor of $\log|q|$. □

For our purposes, a *higher Green's function* $G(\tau)$ of weight $2k$ and level Γ on $\mathfrak{H}$ will be defined by the following properties:

- G is smooth and real-valued on $\mathfrak{H}^\circ := \mathfrak{H}\backslash\{\Gamma.\hat{\tau}\}$ for some $\hat{\tau} \in \mathfrak{H}$;
- G is Γ-invariant;
- $\Delta_{\text{hyp}} G = k(1-k)G$ (on $\mathfrak{H}^\circ$);
- G tends to zero at all cusps; and
- $G(\tau) \sim c\log|\tau - \hat{\tau}|$ (as $\tau \to \hat{\tau}$) for some $c \in \bar{\mathbb{Q}}^*$.

Uniqueness is clear given $c, k, \hat{\tau}, \Gamma$: the difference of two distinct such functions would be a Maass form with eigenvalue -2, which is impossible since Δ_{hyp} is a positive definite operator. Existence is explained in [Me].

Under the conditions that $\hat{\tau}$ is CM and $S_{2k}(\Gamma) = \{0\}$, Gross and Zagier [GZ] conjectured (roughly) that

$$\textit{for any CM point } \tau_0,\ G(\tau_0) \textit{ is of the form } \sum \bar{\mathbb{Q}} \log \bar{\mathbb{Q}}. \tag{10}$$

(Clearly, its validity is independent of $c \in \bar{\mathbb{Q}}^*$.) Mellit was able to prove this for the case $k = 2$, $\hat{\tau} = i$, $\Gamma = PSL_2(\mathbb{Z})$ using the above ideas together with an explicit family of cycles. Noting that $S_4(\Gamma(2)) = \{0\}$, $\lambda(1+i) = -1$ and $\lambda(\frac{1+i}{2}) = 2$, our cycle leads to another case:

Theorem 3.7. *Ψ is a $\bar{\mathbb{Q}}$-linear combination of two higher Green's function of weight 4 and level $\Gamma(2)$ with $\hat{\tau} = 1+i$ and $\frac{1+i}{2}$; moreover, it verifies conjecture (10).*

Proof. This follows from Lemmas 3.3-3.6, once we verify that $\mathfrak{W}$ extends to a cycle on $\bar{\mathcal{E}}_\Gamma^{(2)} \backslash \bar{\pi}^{-1}(\{-1,2\}) \overset{\bar{\pi}}{\to} X(\Gamma) \backslash \{-1,2\}$.[8] Equivalently, we may check this for $\mathfrak{Z}$ on the (1-parameter) Kummer family, for which the following analysis on the singular model $\check{\mathcal{K}}_{\lambda,\lambda}$ will suffice. Referring to (6) and (7), the function on the nodal rational curve $\check{C}_1 \subset \check{\mathcal{K}}_{\lambda,\lambda} \subset \mathbb{P}^3$ whose zero and pole cancel at the node is $z_1 = \frac{\mathsf{z}+i}{\mathsf{z}-i}$. By a computation in [CDKL], $\check{C}_1$ is a double cover of a rational curve with parameter

$$u = \frac{1-\lambda \mathsf{z}^2}{\mathsf{z}^2 - \lambda}, \tag{11}$$

in terms of which its equation is

$$U^2(u^2 + \lambda u + 1)^2 = V^2(u+1)^2(u+\lambda)(\lambda u + 1). \tag{12}$$

We need to determine the values of λ for which (12) acquires a component where $z_1 \equiv 0$ or ∞ ($\Leftrightarrow \mathsf{z}^2 \equiv -1$), i.e. where $\check{\mathfrak{Z}} := (z_1, \check{C}_1)$ has boundary.

Inverting the parameter (11) to $\mathsf{z}^2 = \frac{1+\lambda u}{u+\lambda}$, this happens when $\lambda = -1$, and also if (12) has a component with $u \equiv -1$, which occurs when $\lambda = 2$. (In spite of singular fiber collisions or degeneration of the $K3$ at $\lambda = 0, 1, \infty$, the cycle extends.) It follows that $\check{\mathfrak{Z}}$ has boundary only at $\lambda = -1, 2$. In fact, z_1 limits to both 0 and ∞ on components of $\check{C}_1$ in each case, and so the boundary cannot be corrected by adding a decomposable cycle. □

So for example this shows that $\Psi(\zeta_6) = \mathcal{R}(\zeta_6)/f(\zeta_6)$ (made quite explicit by (7)) satisfies (10). More generally, one might optimistically view (10) as predicting (for each $k, \hat{\tau}, \Gamma$ as above) the existence of a family of indecomposable K_1-classes. The moral of this story is perhaps that (generalized) algebraic cycles are far more ubiquitous, and useful, than the Hodge or Bloch-Beilinson conjectures would suggest on their own.

Remark 3.8. In fact, we can say precisely what the linear combination in Theorem 3.7 is. A computation by A. Clingher [Cl] shows that there is a rational involution of the family $\{\mathcal{K}_{\lambda,\lambda}\}_{\lambda\in\mathbb{P}^1}$ over $\lambda \mapsto 1-\lambda$ sending the alternate fibration $\rho \mapsto \frac{1-2\lambda+\lambda^2\rho}{(1-\lambda)^2}$ (hence $\rho^{-1}(1) \to \rho^{-1}(1)$) and restricting to

[8] It is possible, but tedious, to instead check the asymptotics for Ψ at $0, \pm 1, 2, \infty$ directly from the formula (7), cf. the appendix to section 6 of [CDKL].

the identity on $\mathcal{K}_{\frac{1}{2},\frac{1}{2}}$. Since the cycle family $\{\mathfrak{Z}_{\lambda,\lambda}\}$ is preserved by this involution, Ψ is invariant under $\lambda \leftrightarrow 1-\lambda$, and so we get that the $\bar{\mathbb{Q}}$-coefficients of $\log|\lambda+1|$ and $\log|\lambda-2|$ in Ψ are equal.

4 Proof of the Tauberian lemma 2.3

Though we could not find this result in the literature, what follows makes substantial use of ideas from [Ko]. We will give a fairly detailed proof, since those working in cycles may not be familiar with this part of complex analysis. We retain the notation of §2, with α fixed throughout.

Since F, G, F_0, G_0 are holomorphic on $\mathcal{S}$, they are uniformly continuous there, hence also in

$$S := \{(t,t_0) \in \mathcal{S} \,||t_0| = \alpha,\ |t| \leq \alpha\}.$$

It is clear from §2 that $\tilde{\nu}$ is (uniformly) continuous on S. Defining also

$$S_0 := \left\{(t,t_0) \in \mathcal{S} \,\middle|\, |t_0| = \alpha,\ \frac{t}{t_0} \in [0,1]\right\},$$

we have $S_0 \subset S \subset \mathcal{S}$.

We work first on S_0, writing $t = \beta t_0$ $(\beta \in [0,1])$ with $t_0 = \alpha e^{i\lambda_0}$. Set

$$V(\beta,\lambda_0) := \tilde{\nu}\left(\beta\alpha e^{i\lambda_0}, \alpha e^{i\lambda_0}\right),$$

$a_n(\lambda_0) := \mathfrak{A}_n(\alpha e^{i\lambda_0})\alpha^n e^{in\lambda_0}$, and $s_N(\lambda_0) := \sum_{n=0}^N a_n(\lambda_0)$, so that $V(\beta,\lambda_0) = \sum_{n=0}^\infty a_n(\lambda_0)\beta^n$ for $\beta \in [0,1)$. We will show that as $N \to \infty$,

$$s_N(\lambda_0) \text{ converges uniformly to } V(1,\lambda_0) \tag{13}$$

in λ_0. On $\{|t| = \alpha\}$, $\tilde{\nu}(t) = V(1,\lambda_0)$ and $\sum_{n=0}^N \mathfrak{A}_n(t)t^n = s_N(\lambda_0)$, so (13) is equivalent to Lemma 2.3.

The first step is to break this problem into three pieces ((i)-(iii) below). Using[9] $1-\beta^n \leq n(1-\beta)$ for $\beta \in [0,1]$,

$$|s_N(\lambda_0) - V(\beta,\lambda_0)| = \left|\sum_{n=1}^N a_n(\lambda_0)(1-\beta^n) - \sum_{n=N+1}^\infty a_n(\lambda_0)\beta^n\right|$$

$$\leq \sum_{n=1}^N n(1-\beta)\,|a_n(\lambda_0)| + \frac{1}{N}\sum_{n=N+1}^\infty n\,|a_n(\lambda_0)|\,\beta^n$$

[9] To see this, examine the function $(n-1) - n\beta + \beta^n$.

$$\leq (1-\beta)\sum_{n=1}^{N}|na_n(\lambda_0)| + \frac{1}{N(1-\beta)}\sup_{n>N}|na_n(\lambda_0)|,$$

and so

$$\left|s_N(\lambda_0) - V\left(1-\frac{1}{N},\lambda_0\right)\right| \leq \frac{1}{N}\sum_{n=1}^{N}|na_n(\lambda_0)| + \sup_{n>N}|na_n(\lambda_0)|.$$

Noting $V(1,\lambda_0) = \tilde{\nu}\left(\alpha e^{i\lambda_0}, \alpha e^{i\lambda_0}\right)$, we therefore have the bound

$$|s_N(\lambda_0) - \tilde{\nu}(t_0,t_0)|$$

$$\leq \left|s_N(\lambda_0) - V\left(1-\frac{1}{N},\lambda_0\right)\right| + \left|V\left(1-\frac{1}{N},\lambda_0\right) - V\left(1-\lambda_0\right)\right|$$

$$\leq \underbrace{\frac{1}{N}\sum_{n=1}^{N}|na_n(\lambda_0)|}_{(i)} + \underbrace{\sup_{n>N}|na_n(\lambda_0)|}_{(ii)} + \underbrace{\left|V\left(1-\frac{1}{N},\lambda_0\right) - V(1,\lambda_0)\right|}_{(iii)}. \quad (14)$$

To prove (13), we need to bound (i),(ii),(iii) uniformly in λ_0 (by taking N sufficiently large). In fact, (iii) is obvious by uniform continuity of V on S_0, and so we turn to (ii).

Now $\delta_t\tilde{\nu}(t,t_0) = \sum_{n=0}^{\infty} n\mathfrak{A}_n(t_0)t^n$ in $\mathcal{S}$ for $|t| < |t_0|$; moreover, for *fixed* t_0 with $|t_0| = \alpha$, the function $\delta_t\tilde{\nu}(t,t_0)$ on $\{|t| = \alpha\}$ is both L^1 and L^2 (as log, $\log^2$ are integrable). Working on S, with $t = \gamma t_0 = \beta t_0 e^{i\lambda} = \beta\alpha e^{i(\lambda+\lambda_0)}$ ($|\gamma| \leq 1$), we now define

$$V_t(\gamma,\lambda_0) := (\delta_t\tilde{\nu})\left(\gamma\alpha e^{i\lambda_o}, \alpha e^{i\lambda_0}\right),$$

which for $|\gamma| < 1$

$$= \sum_{n=0}^{\infty} na_n(\lambda_0)\gamma^n.$$

By the Cauchy integral formula, for $0 < |\beta| < 1$,

$$n\mathfrak{A}_n(t_0) \;=\; \frac{1}{2\pi i}\int_{|t|=\alpha\beta}\frac{(\delta_t\tilde{\nu})(t,t_0)}{t^{n+1}}dt$$

$$=\; \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{(\delta_t\tilde{\nu})(\beta t_0 e^{i\lambda},t_0)}{(\beta\alpha e^{i(\lambda+\lambda_0)})^{n+1}} i\beta\alpha e^{i(\lambda+\lambda_0)}d\lambda$$

$$=\; \frac{1}{2\pi\alpha^n\beta^n e^{in\lambda_0}}\int_{-\pi}^{\pi} V_t\left(\beta e^{i\lambda},\lambda_0\right)e^{-in\lambda}d\lambda. \quad (15)$$

For fixed (small) $\epsilon > 0$, the reader will readily verify that

$$\lim_{\beta\to 1^-}\int_{-\epsilon}^{\epsilon}\left|\log(e^{i\lambda}-1)-\log(\beta e^{i\lambda}-1)\right|d\lambda=0.$$

In conjunction with (3), and the uniform continuity of $V_t(\gamma,\lambda_0)$ (in γ) on $\{|\gamma|\leq 1\}\backslash\{\arg(\gamma)\in(-\epsilon,\epsilon)\}$, this implies

$$\lim_{\beta\to 1^-}\int_{-\pi}^{\pi}\left|V_t(e^{i\lambda},\lambda_0)-V_\lambda(\beta e^{i\lambda},\lambda_0)\right|d\lambda=0.$$

Therefore, taking the limit of (15) as $\beta\to 1$, we obtain

$$na_n(\lambda_0)\;=\;n\mathfrak{A}_n(t_0)\alpha^n e^{in\lambda_0}\;=\;\frac{1}{2\pi}\int_{-\pi}^{\pi}V_t\left(e^{i\lambda},\lambda_0\right)e^{-in\lambda}d\lambda \tag{16}$$

and then

$$na_n(\lambda_0+\delta)-na_n(\lambda_0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{V_t\left(e^{i\lambda},\lambda_0+\delta\right)-V_t\left(e^{i\lambda},\lambda_0\right)\right\}e^{-in\lambda}d\lambda$$

$$=\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{(\delta_t\tilde{\nu})\left(e^{i\lambda}\cdot\alpha e^{i(\lambda_0+\delta)},\alpha e^{i(\lambda_0+\delta)}\right)-(\delta_t\tilde{\nu})\left(e^{i\lambda}\cdot\alpha e^{i\lambda_0},\alpha e^{i\lambda_0}\right)\right\}e^{-in\lambda}d\lambda$$

$$=\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\begin{array}{c}\log(e^{i\lambda}-1)\left[F\left(\alpha e^{i(\lambda_0+\lambda+\delta)},\alpha e^{i(\lambda_0+\delta)}\right)-F\left(\alpha e^{i(\lambda_0+\lambda)},\alpha e^{i\lambda_0}\right)\right]\\ +\left[G\left(\alpha e^{i(\lambda_0+\lambda+\delta)},\alpha e^{i(\lambda_0+\delta)}\right)-G\left(\alpha e^{i(\lambda_0+\lambda)},\alpha e^{i\lambda_0}\right)\right]\end{array}\right\}e^{-in\lambda}d\lambda.$$

By uniform continuity of F and G, the differences in square brackets can be bounded $<\varepsilon$ by taking δ sufficiently small. Together with L^1 integrability of $\log(e^{i\lambda}-1)$, this gives (uniform) continuity of $a_n(\lambda_0)$. Similar reasoning shows that $\int_{-\pi}^{\pi}\left|V_t(e^{i\lambda},\lambda_0)\right|^2 d\lambda$ is (uniformly) continuous in λ_0.

As $V_t(e^{i\lambda},\lambda_0)$ is L^2, Parseval gives

$$\sum_{n=0}^{\infty}\left|na_n(\lambda_0)\right|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|V_t(e^{i\lambda},\lambda_0)\right|^2 d\lambda.$$

The right-hand side minus the N^{th} partial sums of the left yields a decreasing sequence of continuous, non-negative functions limiting to 0 pointwise. A standard argument using compactness of the circle shows this limit must be uniform. This proves that

$$|na_n(\lambda_0)|\to 0 \text{ uniformly in } \lambda_0,$$

which takes care of (14)(ii).

To treat (14)(i), let $\epsilon>0$ be given, and let $N\in\mathbb{N}$ be such that $n\geq N \implies |na_n(\lambda_0)|<\frac{\epsilon}{2}$ $(\forall\lambda_0)$. For all $n\leq N$ (and hence for all n), there exists $M\in\mathbb{N}$ such that $|na_n(\lambda_0)|\leq M$. Now, taking $m\geq\frac{2NM}{\epsilon}$, we have

$$\frac{1}{m}\sum_{n=0}^{m}|na_n(\lambda_0)| \leq \frac{\epsilon}{2}\cdot\frac{1}{NM}\sum_{n=0}^{N}|na_n(\lambda_0)| + \frac{1}{m}\sum_{n=N+1}^{m}|na_n(\lambda_0)|$$

$$< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon,$$

uniformly in λ_0, which completes the proof.

References

[As] M. Asakura, *On dlog image of K_2 of elliptic surface minus singular fibers*, preprint, available at arXiv:math/0511190v4.

[AKMS] A. Avram, M. Kreuzer, M. Mendelberg, H. Skarke, *Searching for $K3$ fibrations,* Nucl. Phys. B494 (1997), 567-589.

[Be1] A. Beilinson, *Higher regulators and values of L-functions*, J. Soviet Math. 30 (1985), 2036-2070.

[Be2] ———, *Higher regulators of modular curves*, in "Applications of algebraic K-theory to Algebraic Geometry and Number Theory (Boulder, CO, 1983)", Contemp. Math. 55, AMS, Providence, RI, 1986, 1-34.

[BP] F. Beukers and C. Peters, *A family of $K3$ surfaces and $\zeta(3)$*, J. Reine Angew. Math. 351 (1984), 42-54.

[Bl] S. Bloch, *The moving lemma for higher Chow groups*, J. Algebraic Geom. 3 (1993), no. 3, 537-568.

[BPS] P. Brosnan, G. Pearlstein, and C. Schnell, *The locus of Hodge classes in an admissible variation of mixed Hodge structure*, C. R. Math. Acad. Sci. Paris 348 (2010), no. 11–12, 657–660.

[CL] X. Chen and J. Lewis, *The Hodge-D-conjecture for K3 and Abelian surfaces*, J. Algebraic Geom. 14 (2005), no. 2, 213-240.

[Cl] A. Clingher, private communication.

[CDKL] X. Chen, C. Doran, M. Kerr, and J. Lewis, Normal functions, Picard-Fuchs equations, and elliptic fibrations on $K3$ surfaces, preprint, available at arXiv:1108.2223v2.

[CD] A. Clingher and C. Doran, *Modular invariants for lattice polarized $K3$ surfaces*, Michigan Math. J. 55 (2007), no. 2, 355-393.

[Co] A. Collino, *Griffiths's infinitesimal invariant and higher K-theory on hyperelliptic Jacobians*, J. Algebraic Geom. 6 (1997), no. 3, 393-415.

[DM] P. del Angel and S. Müller-Stach, *Differential equations associated to families of algebraic cycles*, Annales de l'institut Fourier 58 (2008), no. 6, 2075-2085.

[DK] C. Doran and M. Kerr, *Algebraic K-theory of toric hypersurfaces*, CNTP 5 (2011), no. 2, 397-600.

[GZ] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. 84 (1986), no. 2, 225–320.

[Ke] M. Kerr, *A regulator formula for Milnor K-groups*, K-Theory 29 (2003), no. 3, 175-210.

[Ko] J. Korevaar, "Tauberian theory: a century of developments", Springer, 2004.

[Le] J. Lewis, *Transcendental methods in the study of algebraic cycles with a special emphasis on Calabi-Yau varieties*, this volume.

[Me] A. Mellit, "Higher Green's functions for modular forms", Univ. Bonn Ph.D. Thesis, 2008, available at arXiv:0804.3184.

[Ne] J. Neukirch, *The Beilinson conjecture for algebraic number fields*, in "Beilinson's conjectures on special values of L-functions" (Rapoport et al, Eds.), Academic Press, 1988.

[Pe] C. Peters, *Monodromy and Picard-Fuchs equations for families of $K3$ surfaces and elliptic curves*, Ann. ENS 19 (1986), 583-607.
[PS] C. Peters and J. Stienstra, *A pencil of $K3$ surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero*, in "Arithmetic of complex manifolds (Erlangen, 1988)", Springer Lect. Not. Math. 1399, 1989, 110-127.
[Ro] F. Rohsiepe, *Fibration structure in toric Calabi-Yau fourfolds*, hep-th/0502138.
[Sh] S. Shokurov, *Holomorphic forms of highest degree on Kuga's modular varieties.* (Russian) Mat. Sb. (N.S.) 101 (143) (1976), no. 1, 131-157, 160.
[SZ] J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure. I*, Invent. Math. 80 (1985), 489-542.