D-MODULES AND HODGE MODULES

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1. Smooth projective families

Given a family of complex algebraic varieties $f : X \to Y$, we can form the direct image sheaf $R^i f_* \mathbb{Z}$ which is the sheaf associated to

$$U \mapsto H^i(f^{-1}U, \mathbb{Z})$$

Note that we are using the classical rather than the Zariski topology here. At least when f is proper, this can be thought of as encoding the cohomology along the fibres.

When f is smooth (i.e. a submersion) and projective then a theorem of Ehresmann implies that it is a fibre bundle. It follows that $R^i f_* \mathbb{Z}$ is a *locally constant sheaf or local system*. We recall:

Theorem 1.1. *Let K be a commutative ring. There is an equivalence of categories between the category of local systems of K-modules and K-representations of the fundamental groupoid (paths modulo homotopy); when Y is connected these are equivalent to the category of K-representations $\pi_1(Y, y)$.*

Now suppose that Y is a manifold, and L is a local system of $\mathbb{C}$-vector spaces on it. Suppose that y_1 is infinitesimally close to $y_0 = y$, in other words $y_1 = \eta$ is a tangent vector to y. Then transport along the path is given by an operator

$$\nabla_\eta : V_y \to V_y$$

where $V = \mathcal{O}_Y \otimes_{\mathbb{C}} L$ is the associated locally free sheaf and $V_y \cong \mathcal{O}_y \otimes L_y$ is the stalk. This is a $\mathbb{C}$-linear operator satisfying the Leibnitz rule: if $f \in \mathcal{O}_y$ $\nabla_\eta(fv) = \eta(f)v + f\nabla_\eta v$. It is convenient to view this an operator $\nabla : V \to \Omega^1_Y \otimes V$ satisfying $\nabla(fv) = df \otimes v + f\nabla v$. An operator of this type is called a *connection*. The connection is uniquely determined by requiring that the sections of L are precisely solutions to $\nabla v = 0$. The fact that we have locally $n = rank(V)$ linearly independent solutions to $\nabla v = 0$ is called the *integrability* condition. We can make it explicit as follows. After choosing a basis for V_y, we can identify $\nabla_{\partial_i} = \partial_i - A_i$, where A_i are matrices of holomorphic functions and $\partial_i = \frac{\partial}{\partial x_i}$. A solution to $\nabla v = 0$ satisfies $\partial_i \partial_j v = \partial_j \partial_i v$, which yields

$$(\partial_j A_i - \partial_i A_j + [A_i, A_j])v = 0$$

Therefore

$$\partial_j A_i - \partial_i A_j + [A_i, A_j] = 0 \quad (1)$$

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gives a necessary, and it turns out sufficient condition, for integrability. This can be expressed in other ways such as the condition $\nabla^2 = 0$ for a suitable extension of ∇ to higher degree forms.

There is a further equivalence, which is a version of the Riemann-Hilbert correspondence.

Theorem 1.2 (Riemann-Hilbert I). *There is an equivalence of categories between the category of vector bundles with integrable connections and the category of local systems of complex vector spaces given by* $(V, \nabla) \mapsto \ker \nabla$.

Returning to the geometric situation of a smooth projective map $f : X \to Y$, we get a homomorphism

$$H^i(X_{y_0}) \to H^i(X_{y_1})$$

given by transporting cohomology classes along a path γ connecting two points $y_0, y_1 \in Y$. This depends only on the homotopy class of the path. Taking $y = y_0 = y_1$, we get the *monodromy representation* $\pi_1(Y, y) \to GL(H^i(X_y, \mathbb{Z}))$. The corresponding integrable connection on $V = \mathcal{O}_Y \otimes_{\mathbb{C}} R^i f_* \mathbb{C} = \mathcal{O}_Y \otimes_{\mathbb{Z}} R^i f_* \mathbb{Z}$ is called the *Gauss-Manin connection* ∇. This example carries a lot more structure. Each fibre carries a Hodge decomposition

$$H^i(X_y, \mathbb{C}) = \bigoplus_{p+q=i} H^{pq}(X_y)$$

As y varies, this only gives as decomposition of C^∞-bundles. What is more natural is to consider the Hodge filtration

$$F^p H^i(X_y) = \bigoplus_{p' \geq p} H^{p'q}(X_y)$$

These fit together as holomorphic subbundles of $F_p \subset V$. Differentiation doesn't quite preserve them, however, we have Griffith's transversality (aka the infinitesimal period relation)

$$\nabla(F^p) \subseteq \Omega^1_Y \otimes F^{p-1} \tag{2}$$

This can be abstracted to the notion of a *variation of Hodge structure* which is consists of a local system of abelian groups L, a filtration $F^\bullet$ by subbundles of $\mathcal{O}_Y \otimes L$ satisfying (2) such that $F^\bullet$ determines a pure Hodge structure on each fibre $V_y/m_y \cong \mathbb{C} \otimes L_y$. For almost any serious use of this notion, we also need to assume the existence of a *polarization* which is a family of quadratic forms on L which give polarizations of the Hodge structures on the fibres in the usual sense. We say that the variation is polarizable if a polarization exists. The geometric example $R^i f_* \mathbb{Z}$ is polarizable.

2. Decomposition theorem

Given a map $f : X \to Y$ of complex algebraic varieties, a natural question is how do we compute the singular cohomology of X in terms of the direct image sheaves? The classical answer, at least since the 1950's, is by the Leray spectral sequence.

$$H^p(Y, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q})$$

There is one important case, which is easy to understand.

Theorem 2.1 (Deligne)**.** *If f is smooth and projective, then Leray spectral sequence degenerates to an (noncanonical) isomorphism*

$$H^i(X,\mathbb{Q}) = \bigoplus_{p+q=i} H^p(X, R^q f_*\mathbb{Q})$$

This is deduced from the hard Lefschetz theorem. A readable account can be found in [GH]. Using this together with basic mixed Hodge theory, Deligne deduces:

Theorem 2.2 (Global invariant cycle theorem)**.** *Suppose that X is a smooth projective variety, and let $f : X \to Y$ be a projective map. Let $y \in U \subset Y$ be a Zariski open set such that $f|_{f^{-1}U}$ is smooth, then the restriction map*

$$H^i(X,\mathbb{Q}) \to H^i(X_y,\mathbb{Q})^{\pi_1(U,y)}$$

is onto.

Sketch. Since $H^i(X_y)$ is pure of weight i, mixed Hodge theory implies that the image of $H^i(f^{-1}U)$ in $H^i(X_y)$ is the image of the weight i part $W_iH^i(f^{-1}U) = im[H^i(X) \to H^i(U)]$ in $H^i(X_y)$. By the previous theorem, the "edge" map

$$H^i(f^{-1}U,\mathbb{Q}) \to H^0(U, R^i f_*\mathbb{Q}) = H^i(X_y)^{\pi_1(U,y)}$$

is onto. □

Actually, Deligne proved a stronger version of theorem 2.1 that we will recall. In rough terms, the derived direct image $\mathbb{R}f_*\mathbb{Q}$ in the derived category of Y can be thought of as

$$\mathbb{R}f_*\mathbb{Q} = f_*I^\bullet \quad \text{(up to quasi-isomorphism)}$$

where $I^\bullet$ is an acyclic (e.g. fine) resolution of $\mathbb{Q}$. Deligne proved that under the above hypothesis $\mathbb{R}f_*\mathbb{Q} \cong \bigoplus R^q f_*\mathbb{Q}[-q]$, or in other words it decomposes as a sum of the sheaves $R^q f_*\mathbb{Q}$ shifted into degree q. Since by general nonsense we have $H^i(X,\mathbb{Q}) = H^i(Y, \mathbb{R}f_*\mathbb{Q})$, this implies the above result concerning Leray.

For general maps, the Leray spectral sequence does not degenerate, nor does the above decomposition hold. However, Beilinson, Bernstein, Deligne and Gabber have shown that $\mathbb{R}f_*\mathbb{Q}$ will decompose in a weaker sense into simpler parts called "perverse sheaves" . Since it will take some time to what explain these words mean, for the moment we record the theorem along with a topological application.

Theorem 2.3 (Decomposition theorem)**.** *Let $f : X \to Y$ be a projective map of complex algebraic varieties, with X smooth, then $\mathbb{R}f_*\mathbb{Q}$ decomposes into a sum of translates of simple perverse sheaves.*

As a corollary it possible to deduce a local invariant cycle theorem (which is deeper than the global invariant cycle theorem). With the same assumptions and notation as theorem 2.2 let B be a sufficiently small ball centered at $0 \in Y$ and $y \in B \cap U$. Then $f^{-1}B$ is homotopy equivalent to X_0. Thus we have a natural restriction map

$$H^i(X_0,\mathbb{Q}) \to H^i(X_y,\mathbb{Q})^{\pi_1(B\cap U,y)} \tag{3}$$

Corollary 2.4. *The map* (3) *is onto*

Sketch. Let $L = R^i f_{\mathbb{Q}}[-i]$ be extended as a sum of translated simple perverse sheaves on Y. (In notation to be explained later on, $L = IC(R^i f_* \mathbb{Q})[-i - \dim Y]$.) By analyzing the decomposition, one can see that L is a summand of $\mathbb{R}f_*\mathbb{Q}$. Therefore $H^i(B, \mathbb{R}f_*\mathbb{Q}) \to H^i(B, L)$ is onto. This map can be shown to coincide with (3). □

When $\dim Y = 1$, the above corollary was first proved by Schmid using the existence and properties of limit mixed Hodge structures. The original proof of the decomposition theorem made use of the fact, that since the map is algebraic, it can be reduced mod p and then the statement can be deduced from a strong form of the Weil conjectures. There is a second proof of the decomposition theorem due to Saito [S1] which doesn't involve a characteristic p methods, but instead uses Hodge theory. Our goal is explain some of the ideas behind Saito's work. A third proof was found by de Cataldo and Migliorini also using Hodge theory.

3. D-modules and perverse sheaves

The first ingredient of Saito's theory is the theory of D-modules. A D-module can be thought of as a generalization of the notion of an integrable connection. Given a complex manifold (respectively nonsingular complex algebraic variety) X, let D_X be the sheaf of rings of holomorphic (respetively algebraic) differential operators. In the simplest case, $X = \mathbb{A}^n$, we let $\Gamma(D_X)$ is the so called n dimensional Weyl algebra given by generator $x_1, \ldots x_n, \partial_1, \ldots \partial_n$ and relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0,\ [\partial_i, x_j] = \delta_{ij}$$

A left module D_X-module (or D-module if X is understood) is a sheaf of abelian groups M on X, where D_X acts on the left. In particular, it is an $\mathcal{O}_X$-module, with locally an action by ∂_i subject to the above relations. We have already seen examples:

Example 3.1. *Let V be a vector bundle with an integrable connection ∇ is naturally a D-module, where $\partial_i v = \nabla_{\partial_i} v$. Note that the integrability is essential to guaranteeing that ∂_i and ∂_j commute.*

In general, a D-module need not be locally free or even coherent as an $\mathcal{O}_X$-module. However, we will need to impose some finiteness conditions in order to get a manageable theory. A D-module is coherent if it is locally finite presented. So locally such a module is given by the cokernel of matrix $p_{ij}(x_1 \ldots, \partial_1 \ldots)$, so it can be identified with corresponding system of PDE.

Example 3.2. *Any vector bundle with integrable connection is coherent. Locally $\mathcal{O}_X = D_X / \sum \partial_i D_X$ and general vector bundle is sum of such.*

Example 3.3. *Although $M = \mathbb{C}[x_1, \ldots, x_n, \frac{1}{f}]$ is finitely generated as a module over the Weyl algebra, it is clearly not over the polynomial ring. Since the Weyl algebra is left noetherian, M is necessarily coherent.*

Given a coherent module M, we define a subset $char(M) \subset T^*X$ of the cotangent bundle, called the characteristic variety, as follows. Given coordinates $x_1, \ldots x_n$ on X, $x_i, y_j = dx_j$ gives coordinates on T^*X Let $(p_{ij}(x_1, \ldots, \partial_1, \ldots))$ be a presentation matrix for M, then $char(M)$ is defined by setting the highest order part of$p_{ij}(x_1, \ldots, y_1, \ldots)$ to zero. We say that M is *holonomic* if $\dim char(M) = \dim X$.

Although there are certainly natural examples of nonholonomic modules such as D_X itself, all examples of interest to us have this property.

Example 3.4. *Any vector bundle V with integrable connection is holonomic because $char(M)$ is the zero section of T^*X.*

The significance of holonomicity can be partly explained by the following:

Theorem 3.5. *The category of holomonomic D-modules is abelian. Furthermore, it is artinian in the sense that an object is built of out simple ones by successive extensions. Any simple holonomic module is generically a vector bundle with integrable connection.*[1]

Thus we can think of a holonomic D-module as an integrable connection with singularities. We will also need to impose conditions on these singularities. Suppose that $\bar{X}$ is smooth and projective with a divisor E with normal crossings. Let $X = \bar{X} - E$. Recall that an integrable connection (V, ∇) is *regular* if it admits an extension to a vector bundle $\bar{V}$ with an operator $\bar{\nabla} : \bar{V} \to \Omega^1_{\bar{X}}(\log E) \otimes \bar{V}$. For example, Schmid proved that the connection underlying a polarized variation of Hodge structure is regular.

Definition 3.6. *A holonomic module is regular if all simple subquotients are generically regular integrable connections.*

We now indicate the generalized form of the Riemann-Hilbert correspondence due to Kashiwara and Mekbout. Fix a complex manifold X. Given a D-module with a presentation

$$D^m \xrightarrow{P} D^n \to M \to 0$$

We see easily that there is a bijection between the set of solutions and the Hom

$$\{\vec{f} \in \mathcal{O}^n \mid P\vec{f} = 0\} \cong Hom_D(M, \mathcal{O}_X)$$

Thus it make the sense to call

$$Sol(M) = \mathcal{H}om_D(M, \mathcal{O}_X)$$

the sheaf of solutions for any M (here $\mathcal{H}om$ denotes the sheaf Hom). When M is a vector bundle with integrable connection, $Sol(M)$ is locally constant. In general, Kashiwara proved a weaker property holds.

Definition 3.7. *A sheaf F of vector spaces, over some field, on X is constructible if there exists a partition $\{U_i\}$ of X into (analytic) Zariski locally closed sets such that $F|_{U_i}$ is locally constant.*

Theorem 3.8. *When M is holonomic, $Sol(M)$ is a constructible sheaf of $\mathbb{C}$-vector spaces.*

Thus we have a functor

$$Sol : Mod_{rh}(D_X) \to Constr(X, \mathbb{C})$$

between the abelian categories of regular holonomic modules and constructible sheaves of $\mathbb{C}$-vector spaces. One might hope for an equivalence, but the story turns out to be more complicated. First we need to work on the level derived categories. Given an abelian category $\mathcal{A}$, the bounded derived category $D^b(\mathcal{A})$ is the

[1] More precisely, there exists a nonempty smooth Zariski open subset of the support, such that the restriction of the module to this is given by an integrable connection.

category of the bounded complexes in $\mathcal{A}$ where quasi-isomorphisms are formally turned into isomorphisms. Then we can form the derived categories $D^b_{rh}(X)$ and $D^b_c(X,\mathbb{C})$ of complexes of D-modules with bounded regular holonomic cohomology and complexes of sheaves of $\mathbb{C}$-vector spaces with bounded constructible cohomology respectively.

Theorem 3.9 (Riemann-Hilbert II). *The higher solution module*

$$\mathbb{R}\mathcal{H}om(-,\mathcal{O}_X) : D^b_{rh}(X) \to D^b_c(X,\mathbb{C})^{op}$$

induces an equivalence.

This still leaves the question, if we all look at the image of $Mod_{rh}(D_X)$ under the functor, where does it go? The answer is that it goes to the category of perverse sheaves. For the sake of expedience, we take this as a definition rather than a theorem. To simplify notion, we write $RH(M) = \mathbb{R}\mathcal{H}om(M,\mathcal{O}_X)[\dim X]$. The shift $[\dim X]$ is added to make the theory self dual, but it is not something to be overly concerned about for now.

Definition 3.10. *An object of $D^b_c(X,\mathbb{C})$ is called perverse if it isomorphic to $RH(M)$ for some regular holonomic module M. An object $L \in D^bConstr(X,\mathbb{Q})$ is perverse if $L\otimes\mathbb{C}$ is perverse.*

We have a parallel to theorem 3.5

Theorem 3.11. *The full subcategory of perverse sheaves (over $\mathbb{C}$ or $\mathbb{Q}$) is abelian and artinian.*

Example 3.12. *Suppose that $j : U \to X$ is the complement of a divisor E with normal crossings in an n dimensional manifold X. Let L be a local system on U. Then $\mathbb{R}j_*L[n]$ is perverse. In particular, $L[n]$ is perverse when $E = \emptyset$. The corresponding D-module is given by a taking the subsheaf of $j_*(\mathcal{O}_U \otimes L)$ of sections with meromorphic singularities along E.*

In order to construct a good homology theory for singular spaces, called intersection (co)homology, Goresky and Macpherson [GM] modified the usual definition of homology by only allowing chains which meet the singular strata with an appropriate codimension. They even allow cochains with coefficients in a generically defined local system L. It turns out that a sheaf of such cochains with values in L is a perverse sheaf that we denote by $IC(L)$. There is an abstract characterization which is more convenient for us than the original:

Theorem 3.13. *Given a local system L defined on a Zariski open subset U of an irreducible closed set Z. There exists a unique perverse sheaf $IC(L)$ on X with support Z such that $IC(L)|_U \cong L[\dim Z]$ and $IC(L)$ has no simple subquotients supported on the complement of U.*

Corollary 3.14. *A simple perverse sheaf is isomorphic to $IC(L)$ for some irreducible local system L.*

Example 3.15. *Continuing the notation from example 3.12, let $\bar{V}$ be the Deligne extension of $\mathcal{O}_U \otimes L$ to X. This is the unique extension so that the connection ∇ has logarithmic singularities along E with residues having eigenvalues in $[0,1)$. $IC(L) \subseteq \mathbb{R}j_*L[n]$ corresponds to the D-submodule of $j_*(\mathcal{O}_U \otimes L)$ generated by $\bar{V}$.*

4. Hodge modules

Let X be a smooth algebraic variety. A rational variation of Hodge structure consists of filtered vector bundle with integrable connection together with a compatible local system of $\mathbb{Q}$-vector spaces. A Hodge module is an extension of this notion to the D-module/perverse sheaf world. A section of D_X has order at most k if it contains no more than k derivatives. Let F_kD_X be the subsheaf of these.

Definition 4.1. *Let M be a holonomic module. A good filtration on M is an exhaustive increasing filtration by coherent $\mathcal{O}_X$-submodules $F_\bullet M \subseteq M$ such that $F_pD \cdot F_qM \subseteq F_{p+q}M$.*

Here is the key example.

Example 4.2. *Let V be a variation of Hodge structure with Hodge filtration $F^\bullet$. Set $F_kV = F^{-k}$. Then this a good filtration. The inclusion $F_pD \cdot F_qV \subseteq F_{p+q}V$ follows from Griffiths transversality.*

Given a holonomic module M, a rational lattice is perverse sheaf K of $\mathbb{Q}$-vector spaces such that $K \otimes \mathbb{C} \cong RH(M)$. (It would more correct to the use the dual of K, given the way we set things up, but we shall ignore this.) Let $MF(D_X, \mathbb{Q})$ denote the category of regular holonomic D_X-modules with a good filtration together with a rational lattice.

Example 4.3. *Any polarizable variation of Hodge structure gives an object of $MF(D_X, \mathbb{Q})$.*

Saito [S1] defines the full subcategory $MH(X) \subset MF(D_X, \mathbb{Q})$ of polarizable Hodge with the following properties.

1. $MH(X)$ is abelian and semisimple i.e. every object is a direct sum of simple objects.
2. The objects are generically given by variations of Hodge structures. More precisely, if $(M, F, K) \in MH(X)$ is simple, then there exists a Zariski open subset U of the support of K such that $(M, F, K)|_U$ is a polarizable variation of Hodge structure. Furthermore K is $IC(L)$, where L is the underlying local system of the VHS.
3. Conversely, any polarizable variation of Hodge structure $(L, \dots)$ defined on a Zariski open subset of X extends to an object of $MH(X)$ such that the underlying perverse sheaf is $IC(L)$.

Note that these statements are theorems proved in the union of both papers [S1, S2]. The actual construction of MH is by induction on the dimension of the support. A Hodge module with zero dimensional support is defined to be a sum of Hodge structures placed at points. The induction step is extremely delicate. It involves requiring these objects can be specialized, in the appropriate sense[2], to any local hypersurface of the form $f = 0$, and that after doing so, the result lands in MH.

We finally come to the key theorem.

[2]The buzzword is "nearby/vanishing cycles". The technical issues, for the curious, are (1) to lift this to the filtered D-module world; (2) to impose stability of Hodge modules and their polarizations under this construction.

Theorem 4.4 (Saito)**.** *Polarizable Hodge modules are stable under direct images and the decomposition holds for them. In particular, if $(M, F, K) \in MH(X)$ and $f : X \to Y$ is projective, then $\mathbb{R}f_*K$ decomposes into a sum of intersection cohomology complexes associated to polarized variations of Hodge structures.*

Corollary 4.5. *Intersection cohomology with coefficients in a variation of Hodge structure carries a pure Hodge structure.*

5. Further developments

We barely scratched the surface of Saito's theory. He also gave a mixed version which generalizes the notion variation of mixed Hodge structures [S2]. One consequence of the this theory is the following:

Theorem 5.1. *Ordinary cohomology with coefficient is a variation of Hodge structure carries a mixed Hodge structure.*

In another direction, Sabbah [S] and Mochizuki [M] have been developing a theory of twistor D-modules which further extends Saito's ideas in an entirely different direction.

References

[BBD] Beilinson, Bernstein, Deligne, *Faisceux Pervers* (1982)
[dM] M. de Cataldo, L. Migliorini, *The decomposition theorem, perverse sheaves, and the topology of algebraic maps,* BAMS 2009
[GM] Goresky, MacPherson, *Intersection homology II*, Inventiones (1983)
[GH] Griffiths, Harris, *Principles of algebraic geometry*, Wiley (1978)
[HTT] Hotta, Takeuchi, Tanasaki, *D-modules, perverse sheaves, and representation theory*, Birkauser (2008)
[M] Mochizuki, *Asymptotic behaviour of tame harmonic bundles ...*, Mem AMS (2007)
[PS] Peters, Steenbrink, *Mixed Hodge structures*, Springer (2008)
[S] Sabbah, *Polarizable twistor D-modules* Asterisque (2005)
[S1] Saito, *Module Hodge Polarizables* RIMS (1988)
[S2] Saito, *Mixed Hodge modules* RIMS (1990)

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