Elliptic billiards

Setup: Construct elliptical pool table with
{ 5 pts. of the rim
ladder & shovel
poured concrete }

idiotic. ellipse = $\cup$ of pts in $\mathbb{E}$ of whose distances to 2 foci = fixed constant, sent buddy out to buy 2 string rope, but he said s.t. about Pascal's mystic hexagon to checkout lady & she wouldn't sell.

Game: I choose starting point (close to rim) & $2 < N < 10$; you have to shoot ball so returns to starting point (after bouncing off rim N times) traveling in same direction, using
{ practice
foci
sidewalk chalk + rope }

Solution to these problems will highlight the work of 2 French mathematicians, 2 centuries apart, who did fundamental work on conics (which of course include ellipses) and were both mechanically inclined.

Pascal (1623–1662):

- Home-schooled by father, who had the idea that mathematics ed. should be delayed → 15 [when a friend of mine who does AG in Canada named his son Blaise I was amazed...) But when BP started doing geometry at 12 the dad relented & bought him copy of Elements.
- Father tax collector – BP invented 1st digital calculator to help
- fundamental work on combinatorics & probability, solids of revolution
- physics: observed that pressure in atmosphere decreased w/height & deduced $\exists$ of vacuum above it; which led Descartes (who disagreed) to remark that "Pascal has too much vacuum in his head).

Poncelet (1788–1867):

- Educated at École polytechnique by likes of Monge, Ampère, Brianchon
- Joined Napoleon's army in early 20's & had horse shot dead under him by Russians who took him as a POW. During imprisonment (in Siberia?) he couldn't properly recall what he had learned so invented projective geometry instead.
- The solutions we'll discuss to these 2 problems in real planar geometry will highlight the need for introducing notions of <u>genericity</u>, <u>points at ∞</u>, and <u>complex</u> coordinates which are partly due to him (also: linear/bilinear algebra, RS's)
  doing geometry in
- major treatise on projective properties of conics (pub. 1822) was subtitled "A work of utility for those studying the applications of descriptive geometry and geometric operations on land", which highlights the fact that this was a person who was in fact an <u>army engineer</u> building bridges. & later a prof. of mechanics who doubled the efficiency of waterturbines

I) Curves through points

($\mathcal{P}^n$ = polynomials of degree $\leq n$ in x,y ; $\mathcal{P}^n_{\mathcal{S}}$ = polynomials vanishing on $\mathcal{S}$)

$\overbrace{\underbrace{A, B, C, D, E}_{\mathcal{S}_0}, P_1, P_2, P_3}^{\mathcal{S}} \in \mathbb{R}^2$ distinct

$Q \subset \mathbb{R}^2$ conic : general equation has form

basis for $\mathcal{P}^2$

$$0 = f_Q(x,y) = a_1 \cdot 1 + a_2 \cdot x + a_3\, y + a_4\, xy + a_5\, x^2 + a_6\, y^2$$

For Q to pass through $\mathcal{S}_0$, need $f_Q \in \mathcal{P}^2_{\mathcal{S}_0} \iff$

vector $\vec{a}$

$$0 = f_Q(A) = (1, x_A, y_A, x_A y_A, x_A^2, y_A^2) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix}$$
$$\vdots$$
$$0 = f_Q(E) = (1, x_E, \ldots, y_E^2) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} \qquad \iff M\vec{a} = \vec{0}$$

5 by 6 matrix M

If $\mathcal{S}_0$ "sufficiently general" (no 4 pts. collinear $\Rightarrow$ rank $M = 5$), $\dim \mathcal{P}^2_{\mathcal{S}_0} = 6 - 5 = 1 \Rightarrow$ all "solutions" multiples of a single polynomial $\Rightarrow$ all define same conic Q.

<u>Upshot</u>: $\exists !\ Q \ni A, \ldots, E$.

$\mathcal{C} \subset \mathbb{R}^2$ cubic : general equation has form

$$0 = f_{\mathcal{C}}(x,y) = b_1 \cdot 1 + \cdots + b_6 \cdot y^2 + b_7 \cdot x^3 + b_8\, x^2 y + b_9\, x y^2 + b_{10}\, y^3$$

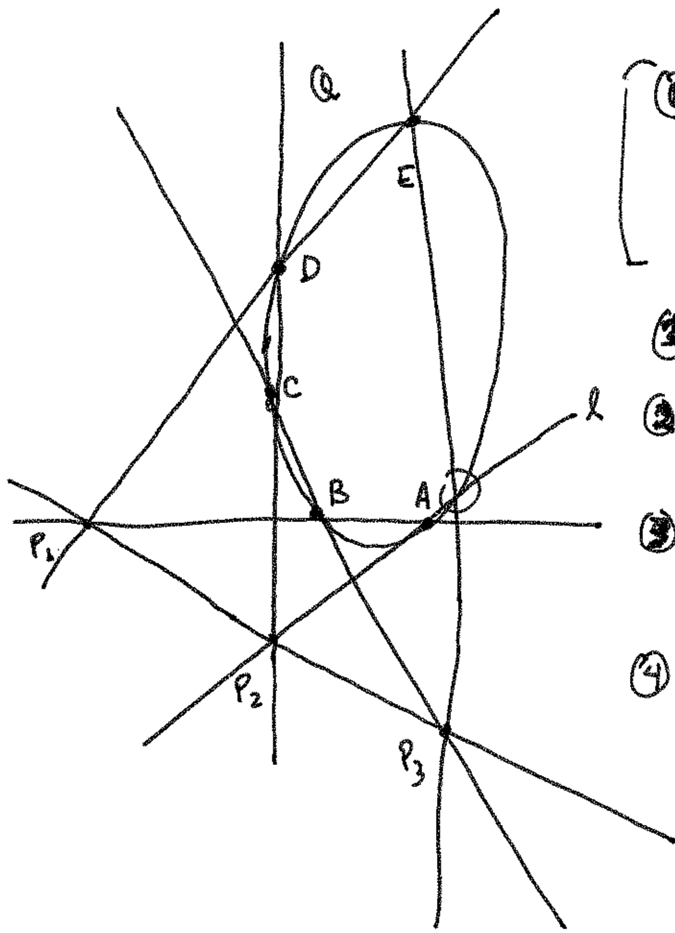
$f_{\mathcal{C}} \in \mathcal{P}^3_{\mathcal{S}} \iff \underset{8\times 10}{\mathcal{M}} \cdot \underset{10\text{-vector}}{\vec{b}} = \vec{0}$, so

$\mathcal{S}$ "sufficiently general" (no 4 pts. collinear, no 7 pts. conconic) $\Rightarrow \dim \mathcal{P}^3_{\mathcal{S}} = 10 - 8 = 2$.

$\Rightarrow$ equations of 3 cubics through $\mathcal{S}$ have a linear dependency.

II) Mystic hexagram ← (Mersenne had a math club; presented by Pascal 1639 at age of 16)

Take A, B, C, D, E no $\underline{\underline{3}}$ collinear ($\Rightarrow$ Q smooth; assume ellipse). Objective: construct points on Q.



⓪ [plot A, B, C, D, E (white)
draw Q (dotted yellow)
AB, CD — blue
BC, DE — red]

① $P_1 := AB \cap DE$ (white)

② draw ℓ through A but $\underline{\underline{\text{not}}}$ thru B, C, D, E (red)

③ [draw $P_1 P_2$ (yellow)]
$P_3 := P_1 P_2 \cap BC$ (white)

④ [draw EP_3 (blue)]
$q := EP_3 \cap \ell$

Claim: $q \in Q$! i.e. we've constructed a point on Q

cubic polys.

$$\left.\begin{array}{l} f_1 \longleftrightarrow C_1 = \cup(\text{yellow}) \\ f_2 \longleftrightarrow C_2 = \cup(\text{blue}) \\ f_3 \longleftrightarrow C_3 = \cup(\text{red}) \end{array}\right\} \text{ all thru } \mathcal{S}.$$

$\Downarrow$

N.B.: • $\mathcal{S}$ is generic (Cayley–Bacharach theorem)
• what if P_i "at ∞" (need projective coordinates)

$$0 = \alpha f_1 + \beta f_2 + \gamma f_3$$

- If $\alpha = 0$, $C_2 = C_3 \Rightarrow \ell = AB, CD, \text{or } EP_3 \Rightarrow \ell \ni A, C, \text{or } E$ ⨳
- So $f_1 = -\frac{\beta}{\alpha} f_2 - \frac{\gamma}{\alpha} f_3$ and $C_2, C_3 \ni q \underset{\text{use equation}}{\Rightarrow} C_1 \ni q \Rightarrow q \in Q \text{ or } P_1 P_2$
- If $q \in P_1 P_2$, then $P_1 P_2, EP_3, \ell$ "collapse" to same line $\Rightarrow \ell \ni E$ ⨳

$\Rightarrow q \in Q$. $\square$

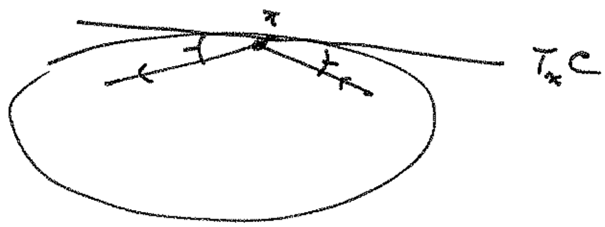
Varying choice of l gives arbitrary $q \in Q$, & $ABCDEq$ = arbitrary hexagon inscribed in Q

Theorem (Pascal): Intercepts of opposite edges of a hexagon inscribed in a conic, lie on a line.

III.) Billiard trajectories

Now let $C :=$ ellipse

A billiard trajectory wrt C is a polygonal path inscribed in C with equal angles of incidence & reflection at C:



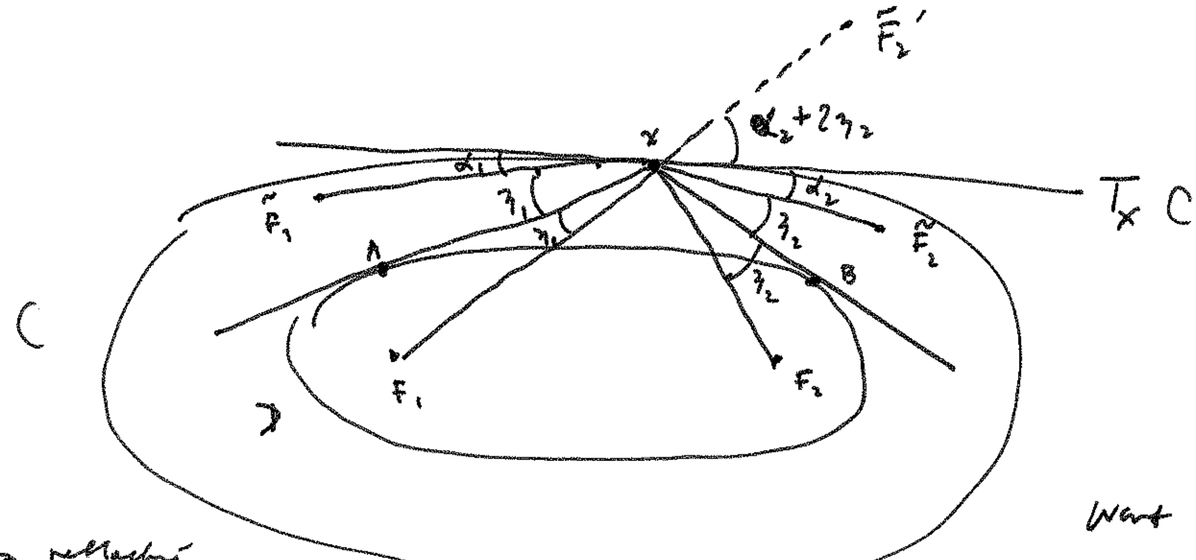
we want N-periodic, such trajectories.

Well, by this point your buddy has come back with rope + 2 stakes & you manage to find the foci of C, so you can now trace out all confocal ellipses D inside C.

A Poncelet trajectory wrt (C, D) "confocal pair" is a polygonal path inscribed in C and circumscribed about D. How will this help?

Theorem (Flatto, 2003): Confocal PT's are BT's (for C).

Pf:



~ denotes reflection in a segment

Want to show $\alpha_1+\eta_1=\alpha_2+\eta_2$

Since for $p \in L$, $|pF_1| + |pF_2|$ is minimized by $p = x$ (defn. of ellipse!), $x \in \tilde{F}_2' F_1$. $\Rightarrow \alpha_2 + 2\eta_2 = \alpha_1 + 2\eta_1$. (So suff. to show $\eta_1 = \eta_2$.)

Likewise, $B \in \tilde{F}_2 F_1$ & $A \in F_2 \tilde{F}_1$, so

$$|\tilde{F}_2 F_1| = |\tilde{F}_2 B| + |BF_1| = |F_2 B| + |BF_1| = |F_2 A| + |AF_1| = |F_2 A| + |A\tilde{F}_1| = |\tilde{F}_1 F_2|$$

$\Rightarrow$ $\tilde{F}_2 p F_1 \cong F_2 p \tilde{F}_1$ as triangles $\Rightarrow \eta_1 = \eta_2$, done. $\square$

But this still doesn't say why this will solve the central problem: while I only get to select from 7 values of N, I can choose any starting point. Without an idea, no (finite) amount of practice can prepare you for this!

IV) Poncelet's porism.

According to my dictionary, a "porism" is a "proposition that uncovers the possibility of finding such conditions as to make a specific problem capable of innumerable solutions". Sounds promising.

First let's translate "circuminscribed trajectories" into more set theoretic language, and relax the constraint that D be contained with C.

A P.T. wrt (C, D) is a sequence $\{(x_i, L_i)\}_{i \geq 0}$ s.t.
- $L_i \cap C = \{x_i, x_{i+1}\}$
- $L_i \in D^*$ (tangent to D).

In fact, we won't be able to avoid passing to the complexified conics $\hat{C}, \hat{D}$ consisting of all complex solutions to $f_C = 0, f_D = 0$. Define the incidence correspondence

$$\mathcal{E} := \{ (x, L) \in \hat{C} \times \hat{D}^* \mid x \in L \},$$

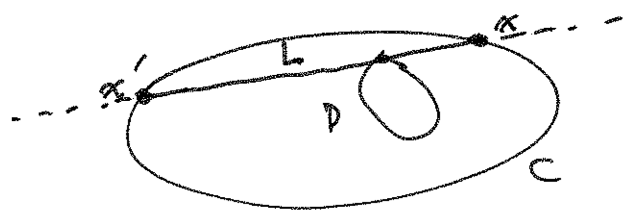
set of lines tangent to D

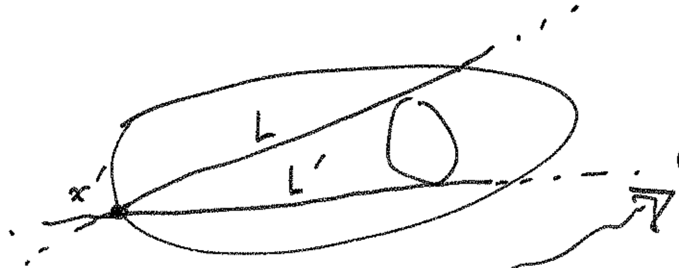
and consider the 2 involutions

$\iota_1 : (x, L) \mapsto (x', L)$

g

$\iota_2 : (x', L) \mapsto (x', L')$

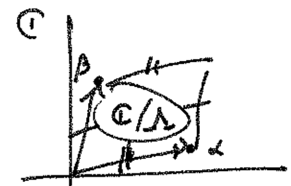




The argument for this is based on the fact that $\mathcal{E} \twoheadrightarrow \hat{C} \cong S^2$ is a double cover branched over the 4 points $\hat{C} \cap \hat{D}$, hence has genus 1 (i.e. is a torus).

Set $g := \iota_2 \circ \iota_1 : \mathcal{E} \to \mathcal{E}$. Clearly, iterating g yields a Poncelet trajectory.

Now suppose (want prove) $\mathcal{E} \cong \mathbb{C}/\Lambda$ (coord. z) $\Lambda = \mathbb{Z}\langle \alpha, \beta \rangle$



$\cong$ topologically

Then the only possibilities for an involution to be complex analytic & 1-1 are $z \mapsto \zeta - z \pmod{\Lambda}$

So $\underbrace{z \xmapsto{\iota_1} \zeta_1 - z \xmapsto{\iota_2} \zeta_2 - (\zeta_1 - z) = \underbrace{\zeta_2 - \zeta_1}_{\gamma} + z}_{g}$

Hence, g = translation by $\gamma \pmod{\Lambda}$, and $g^N(z_0) \underset{\Lambda}{\equiv} z_0 \Leftrightarrow \underline{\underline{N\gamma \in \Lambda}}$,

(x_0, L_0)

a condition which is independent of the starting point. We have proved

(can return to real setting here)

Theorem (Poncelet): If one P.T. wrt (C, D) is N-periodic, they all are.

[Cayley: method of computing D to get N-periodicity.]

Now if you can find (by practicing) N ($=3, 4, \dots, 9$)-periodic billiard trajectories from one starting point, just locate + trace out the confocal ellipses they are tangent to, & label them 3 thru 9. Then, regardless of the starting point, you can aim your shot tangent to the N^{th} ellipse & get an N-periodic billiard trajectory!

Thank You

10

V) References

http://enriques.mathematik.uni-mainz.de/intgeo/poncelet.html

http://www.borcherds.co.uk/geogebra/PonceletsPorismTwoConicsOrder$\overset{6}{5}$.html

EllipsesOrder3

——— Thank you ———