Title: Fourier Transforms & Bernoulli Polynomials | (Math Club, Sept. 24 2012) ①

Fourier series

They are an example of mathematics motivated by physics. Fourier had introduced the heat equation

$$\frac{\partial f}{\partial t} = \alpha^2 \frac{\partial^2 f}{\partial x^2} \quad [+h] \qquad \left(\begin{array}{c}\text{periodic (period } 2\pi) \\ \text{w./initial conditions} \\ f(x,0) = f(x)\end{array}\right)$$

and noticed that if you could represent the initial conditions by trigonometric series

recall $= \cos(nx) + i\sin(nx)$

$$(*) \qquad f(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx} \qquad (a_n \in \mathbb{C})$$

then

$$f(x,t) = \sum_{n \in \mathbb{Z}} \left(a_n e^{-\alpha^2 n^2 t}\right) e^{inx}$$

solved it. Notice that the highest frequencies are smoothed most rapidly. Fourier's idea that $(*)$ should be possible even for discontinuous $f(x)$ — e.g. the square wave function obtained by "shoving an iron ring halfway into a fire" — caused controversy and led to the rejection of his 1807 paper. Eventually he became president of the professional society that rejected his paper and had it published in their prestigious journal. (In fact, the so-called "Fourier series" of a periodic function — even a continuous one — need not converge everywhere to the function, unless the function is everywhere differentiable.)

Discrete Fourier Transform: the finite version, done via FFT algorithm on computer (e.g. MATLAB)

$\Phi(N) :=$ functions $f : \mathbb{Z}/N \to \mathbb{C}$

$\Phi(N)^0 :=$ those with $f(\bar{0}) = 0$

$\Phi(N)_0 :=$ those with $\sum_{n=0}^{N-1} f(\bar{n}) = 0$

DFT: $$\hat{f}(k) = \sum_{n=0}^{N-1} f(n)\, e^{-\frac{2\pi i k n}{N}}$$

IDFT: $$f(m) \stackrel{?}{=} \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}(k)\, e^{\frac{2\pi i m k}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} f(n)\, e^{\frac{2\pi i}{N}(m-n)k}$$

$$= \sum_{n=0}^{N-1} f(n) \underbrace{\frac{1}{N}\sum_{k=0}^{N-1}\left(e^{\frac{2\pi i}{N}(m-n)}\right)^k}_{\substack{=1 \text{ if } m=n \\ =0 \text{ if } m\neq n}}$$

$$= f(m)$$

Properties: (1) $\sum |f(n)|^2 = \frac{1}{N}\sum |\hat{f}(k)|^2$ Parseval

(2) $\widehat{(f\star g)}(k) = \overline{\hat{f}(k)}\,\hat{g}(k)$ where $(f\star g)(n) := \sum_{\ell=0}^{N-1} \overline{f(\ell)}\, g(n+\ell)$

(3) $\widehat{f'}(n) = \frac{2\pi i}{N}(k\hat{f}(k))$ $\left[\text{from } f(n) = \frac{1}{N}\sum_k \hat{f}(k)\, e^{\frac{2\pi i k n}{N}}\right]$

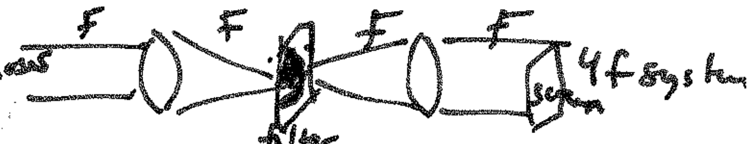
(4) $f \in \begin{Bmatrix}\Phi^0 \\ \Phi_0\end{Bmatrix} \iff \hat{f} \in \begin{Bmatrix}\Phi_0 \\ \Phi^0\end{Bmatrix}$

e.g. if $f(\bar{0}) = 0$, $\sum_k \hat{f}(k) = \sum_k\sum_n f(n)\, e^{-\frac{2\pi i n k}{N}} = \sum_n f(n) \underbrace{\sum_k \left(e^{-\frac{2\pi i n}{N}}\right)^k}_{} = 0$

Applications of Fourier Transforms / Series

(Since they decompose a function into its constituent frequencies, we expect that they should be very useful for studying waves.)

- signal processing
  - analyzing frequencies in signal
  - computer generated holography



$\hat{f} \longmapsto f$

  - image recognition (want $f \star g$ big "blip" at origin)

$g \longmapsto \hat{g}\cdot\overline{\hat{f}} \longmapsto f \star g$

cf. property (2) above

  - noise removal (from old recordings, aerial photos)
  - data ~~image~~ compression (why nice digital photos are only 1 MB not 10)
    apply DFT to 8×8 pixel sets.

(Really gets into wavelet transforms, which is also used in CT & MRI. Work by V. Wickerhauser on wavelets led to a form of compression used by FBI to encode fingerprint images.)

- solve PDE's (cf. property (3) above)
- designing earthquake-proof buildings: make sure vibration modes of building don't intersect frequencies in domain of FT of seismograph.
- find structure of large biochemical molecules (e.g. DNA) by x-ray diffraction
- improving radio reception, sonar systems.
- efficient mult. of large #'s (variant of (2) above, applied to polynomial multiplication)
  + wealth of other # thy. apps.
- music, voice (used in vocal labs), bird song study
- astronomy

We'll look at a cool number-theoretic application now.

Bernoulli #'s

$$B_n = 1, -\tfrac{1}{2}, \tfrac{1}{6}, 0, -\tfrac{1}{30}, 0, \ldots$$

$$\sum_{k\ge 0} B_k \frac{t^{k-1}}{k!} = \frac{1}{e^t-1}$$

$$\frac{1}{e^t-1} - \frac{1}{t} = \frac{t-e^t+1}{t(e^t-1)} \xrightarrow{d/dt} \frac{1-e^t}{e^t-1+te^t} \to \frac{-e^t}{e^t+te^t+e^t} \to -\frac{1}{2}$$

$$= \left(\frac{1}{t} - \frac{1}{2}\right) + \sum_{n\neq 0}\left(\frac{1}{t-2\pi i n} + \frac{1}{2\pi i n}\right)$$

add $\frac{1}{2\pi i n}$ to make converge near 0

Liouville type argu

$$= \quad '' \quad + \sum_{n>0} \frac{2t}{t^2-(2\pi i n)^2}$$

$$= \quad '' \quad - \sum_{n>0} \frac{2t}{(2\pi i n)^2} \sum_{k\ge 0} \frac{t^{2k}}{(2\pi i)^{2k} n^{2k}}$$

$$\underset{m=k+1}{=} \quad '' \quad - \sum_{m\ge 1} \frac{t^{2m-1}}{(2\pi i)^{2m}} \Bigg(2 \underbrace{\sum_{n>0} \frac{1}{n^{2m}}}_{\zeta(2m)} \Bigg)$$

$$\Rightarrow \quad B_{2m} = \frac{-(2m)!\,2\zeta(2m)}{(2\pi i)^{2m}}$$

$$\Rightarrow \quad \zeta(2m) = \frac{(2\pi)^{2m}|B_{2m}|}{2\,(2m)!}$$

(for $m \ge 1$)

$$\Rightarrow \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \ldots$$

Bernoulli polynomials

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j \, x^{k-j}$$

eg $B_3 = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, $B_4 = x^4 - 2x^3 + x^2 - \frac{1}{30}$

$$\sum_{k=0}^{\infty} B_k(x)\frac{t^{k-1}}{k!} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j} B_j \, x^{k-j}\frac{t^{k-1}}{k!} = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\binom{j+l}{j} B_j \, x^{l}\frac{t^{j+l-1}}{(j+l)!}$$

$l = k-j$

$= \frac{1}{j!\,l!}$

$$= \sum_{j}\frac{B_j t^{j-1}}{j!}\sum_{l}\frac{(tx)^l}{l!} = \frac{e^{tx}}{e^t - 1} .$$

Similar reasoning to previous $= \left(\frac{1}{t} - \frac{1}{2} + x\right) + \sum_{n\neq 0}\left(\frac{e^{-2\pi i n x}}{t + 2\pi i n} - \frac{e^{-2\pi i n x}}{2\pi i n}\right)$

$$= \quad \text{"} \quad + \sum_{n\neq 0}\frac{e^{-2\pi i n x}}{2\pi i n}\left(\sum_{\substack{j\geq 0\\ j\geq 1}}\frac{(-1)^j t^j}{(2\pi i n)^j} \cancel{-1}\right)$$

$$\underset{j=k-1}{=} \quad \text{"} \quad + \sum_{k\geq 2}\frac{(-1)^{k-1}t^{k-1}}{(2\pi i)^k}\sum_{n\neq 0}\frac{e^{-2\pi i n x}}{n^k}$$

$$\Longrightarrow \quad B_k(x) = \frac{(-1)^{k-1}k!}{(2\pi i)^k}\sum_{m\neq 0}\frac{e^{-2\pi i m x}}{m^k} \quad (\text{for } m \geq 2) .$$

L-functions

Very important in number theory, the basic examples are obtained by taking $F \in \bar{\Phi}(N)$ and defining

$$\tilde{L}(F,k) := \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{F(\bar{n})}{n^k}.$$

(We know how to compute these so far only for $F \equiv 1$ and k even.)

They have a beautiful connection to the DFT and Bernoulli polynomials. Let $f \in \bar{\Phi}(N)$. Then

$$\sum_{a=0}^{N-1} f(a)\, B_{\ell+2}\left(\frac{a}{N}\right) = \frac{(-1)^{\ell+1}(\ell+2)!}{(2\pi i)^{\ell+2}} \sum_{a=0}^{N-1} f(a) \sum_{m \neq 0} \frac{e^{-2\pi i m a/N}}{m^{\ell+2}}$$

$$= \left(\quad " \quad \right) \cdot \sum_{m \neq 0} \frac{1}{m^{\ell+2}} \underbrace{\sum_{a=0}^{N-1} f(a)\, e^{-2\pi i m \frac{a}{N}}}_{\hat{f}(m)}$$

$$\Longrightarrow \tilde{L}(\hat{f}, \ell+2) = \frac{(2\pi i)^{\ell+2}}{(-1)^{\ell+1}(\ell+2)!} \sum_{a=0}^{N-1} f(a)\, B_{\ell+2}\left(\frac{a}{N}\right).$$

(need not be $\mathbb{Q}$-valued)

Theorem: If f is $\mathbb{Q}$-valued, then $\tilde{L}(\hat{f}, \ell+2) \in (2\pi i)^{\ell+2}\mathbb{Q}$.

eg $N = 4$, $f = 0, 1, 0, -1$
$\hat{f} = 0, -2i, 0, 2i$

$$\rightsquigarrow \tilde{L}(\hat{f}, 3) = 4i \sum_{m \geq 0} \frac{(-1)^m}{(2m+1)^3}$$

$$\parallel$$

$$\frac{(2\pi i)^3}{3!}\left(\underbrace{1 \cdot B_3\left(\tfrac{1}{4}\right)}_{3/64} - \underbrace{1 \cdot B_3\left(\tfrac{3}{4}\right)}_{-3/64}\right) = \frac{-i\pi^3}{8}$$

$$\Longrightarrow \sum_{m \geq 0} \frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{32}.$$

Concluding Remarks

① What about $\overbrace{\sum_{m\geq 0} \frac{(-1)^m}{(2m+1)^2}}^{\text{Catalan's constant } =: G}$? $\overbrace{\sum_{m\geq 1} \frac{1}{m^3}}^{\zeta(3)}$? The problem here is that the related Σ sums have cancelling $\pm n$ terms. So these two remain somewhat mysterious. It is known that $\zeta(3) \notin \mathbb{Q}$, but this isn't even known for G or $\zeta(5)$!!

② The "Theorem" is the "1-dimensional" analogue of a result for functions on $\mathbb{Z}/N \times \mathbb{Z}/N$, which allows one to define something called the <u>horospherical map</u>. With a significant amount of work, it led to a proof of a version of the Hodge Conjecture for modular curves by A. Beilinson (Fields medalist at Chicago).