PUTNAM PRACTICE PROBLEMS
FOR 2 DECEMBER 2015

FUNCTIONAL EQUATIONS AND PROBABILITY

1. Prove that there is a unique function \( f \) from the set \( \mathbb{R}^+ \) of positive real numbers to \( \mathbb{R}^+ \) such that
\[ f(f(x)) = 6x - f(x) \]

2. \( f(x) \) is continuous and \( \int_1^x ds f(s) = \int_1^x ds f(s) \). Find \( f(x) \).

3. If \( \left[ f(x) \right]^2 f(\frac{1-x}{1+x}) = 6^x \), for all \( x \neq 1 \) and \( c \) is a positive constant, find \( f(x) \).

4. \( f(x) \) is defined for all real \( x \) except \( x = 0 \) and \( x = 1 \). If \( f(x) + f\left(\frac{x-1}{x}\right) = 1 + x \) for \( x \neq 0, 1 \), find \( f(x) \).

5. Find all polynomials \( P(x) \) such that \( P(0) = 0 \) and \( P(x+1) = \left[ P(x) \right]^2 + 1 \) for \( x \) integer.

6. \( f(x) \) is defined for all integers and takes only real values. If \( f(x)f(y) = f(x+y) + f(x-y) \), \( f(0) \neq 0 \), \( f(1) = 5/2 \), find \( f(n) \).

7. Find a polynomial of degree 5 such that \( P(x) + 10 \) is divisible by \( (x+2)^3 \) and \( P(x) - 10 \) is divisible by \( (x-2)^3 \).

8. If \( 32 \int_0^x f(t) dt = \int_0^{2x} f(x) dt \) and \( f'''(x) \) exists for all \( x \), find \( f \).

9. \( f(x) \) is continuous and real valued, \( f(1) = 2 \), and \( f(\sqrt{x^2 + y^2}) = f(x)f(y) \) for all \( x \) and \( y \). Show that \( f(x) = 2 \sqrt{x} \).

10. Find all \( f \) such that \( f \) is differentiable and \( f(x+y) = f(x)f(y) \).
PROBABILITY

(1) Show that the probability of winning the following game is better than 1 chance in 3:
Your opponent has a box of papers on which a number is written. All numbers are different. You win if you guess the biggest number. Rules:
You pick a piece of paper and look at the number. You then say either
(1) "This is the biggest number"
    If you are right, you win. If not, you lose.
-OR-
(2) "This is not the biggest number"
    If you are right, you pick another piece of paper. If not, you lose.
(2) You break a stick at an arbitrary point into 2 pieces. You then break the longer piece at a random point into 2 pieces. What is the chance that the 3 pieces form a triangle?
(3) If \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \), find the probability that a positive integer chosen at random is divisible by a perfect square.