Putnam Practice: Va Tech Contest Practice

These are problems from past Va Tech contests. This year's contest is on Saturday Oct. 22, from 9-11:30 in Compton 241. In addition to these problems, we will discuss (9) and (10) from Problem Set 3 (Pell's equation).

(1) Let m, n be positive integers and let $[a]$ denote the residue class mod mn of the integer a (thus $\{[r] \mid r \text{ is an integer}\}$ has exactly mn elements). Suppose the set $\{[ar] \mid r \text{ is an integer}\}$ has exactly m elements. Prove that there is a positive integer q such that q is prime to mn and $[nq] = [a]$.

(2) Evaluate

$$\int_0^{\pi/2} \frac{\cos^4 x + \sin x \cos^3 x + \sin^2 x \cos^2 x + \sin^3 x \cos x}{\sin^4 x + \cos^4 x + 2\sin x \cos^3 x + 2\sin^2 x \cos^2 x + 2\sin^3 x \cos x} dx.$$

(3) Find nonzero complex numbers a, b, c, d, e such that

$$\begin{aligned} a + b + c + d + e &= 1 \\ a^2 + b^2 + c^2 + d^2 + e^2 &= 15 \\ \tfrac{1}{a} + \tfrac{1}{b} + \tfrac{1}{c} + \tfrac{1}{d} + \tfrac{1}{e} &= -1 \\ \tfrac{1}{a^2} + \tfrac{1}{b^2} + \tfrac{1}{c^2} + \tfrac{1}{d^2} + \tfrac{1}{e^2} &= 15 \\ abcde &= -1 \end{aligned}$$

(4) Define $f(n)$ for n a positive integer by $f(1) = 3$ and $f(n+1) = 3^{f(n)}$. What are the last two digits of $f(2012)$?

(5) Define a sequence (a_n) for n a positive integer inductively by $a_1 = 1$ and

$$a_n = \frac{n}{\prod_{\substack{d \mid n \\ 1 \le d < n}} a_d}.$$

Thus $a_2 = 2$, $a_3 = 3$, $a_4 = 2$, etc. Find a_{999000}.

(6) Let A_1, A_2, A_3 be 2×2 matrices with entries in $\mathbb{C}$ (the complex numbers). Let tr denote the trace of a matrix (so $tr \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$). Suppose $\{A_1, A_2, A_3\}$ is closed under matrix multiplication (i.e., given i, j, there exists k such that $A_i A_j = A_k$), and $tr(A_1 + A_2 + A_3) \neq 3$. Prove that there exists i such that $A_i A_j = A_j A_i$ for all j (here i, j are 1, 2,or 3).

(7) Let d be a positive integer and let A be a $d \times d$ matrix with integer entries. Suppose $I + A + A^2 + \cdots + A^{100} = 0$ (where I denotes the $d \times d$ identity matrix, so I has 1's on the main diagonal, and 0 denotes the zero matrix, which has entries all

0). Determine the positive integers $n \leq 100$ for which $A^n + A^{n+1} + \cdots + A^{100}$ has determinant ± 1.

(8) For n a positive integer, define $f_1(n) = n$ and then for i a positive integer, define $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75)$ mod 17 (i.e. determine the remainder after dividing $f_{100}(75)$ by 17, an integer between 0 and 16). Justify your answer.

(9) Prove that $\cos(\pi/7)$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$, and find the other two roots.

(10) Let $\triangle ABC$ be a triangle with sides a, b, c and corresponding angles A, B, C (so $a = BC$ and $A = \angle BAC$ etc.). Suppose that $4A + 3C = 540°$. Prove that $(a - b)^2(a + b) = bc^2$.

(11) Let A, B be two circles in the plane with B inside A. Assume that A has radius 3, B has radius 1, P is a point on A, Q is a point on B, and A and B touch (i.e. are tangent) so that P and Q are the same point. Suppose that A is kept fixed and B is rolled once around the inside of A so that Q traces out a curve starting and finishing at P. What is the area enclosed by this curve?

(12) Define a sequence by $a_1 = 1$, $a_2 = 1/2$, and $a_{n+2} = a_{n+1} - \frac{a_n a_{n+1}}{2}$ for n a positive integer. Find $\lim_{n\to\infty} n a_n$.