Putnam Practice: Va Tech Contest Practice

These are problems from past Va Tech contests. This year’s contest is on Saturday Oct. 22, from 9-11:30 in Compton 241. In addition to these problems, we will discuss (9) and (10) from Problem Set 3 (Pell’s equation).

1. Let \( m, n \) be positive integers and let \([a]\) denote the residue class mod \( mn \) of the integer \( a \) (thus \( \{r\} \mid r \text{ is an integer} \) has exactly \( mn \) elements). Suppose the set \( \{[ar]\} \mid r \text{ is an integer} \) has exactly \( m \) elements. Prove that there is a positive integer \( q \) such that \( q \) is prime to \( mn \) and \( [nq] = [a] \).

2. Evaluate
\[
\int_0^{\pi/2} \frac{\cos^4 x + \sin x \cos^3 x + \sin^2 x \cos^2 x + \sin^3 x \cos x}{\sin^4 x + \cos^4 x + 2 \sin x \cos^3 x + 2 \sin^2 x \cos^2 x + 2 \sin^3 x \cos x} \, dx.
\]

3. Find nonzero complex numbers \( a, b, c, d, e \) such that
\[
a + b + c + d + e = 1 \\
a^2 + b^2 + c^2 + d^2 + e^2 = 15 \\
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = -1 \\
\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} = 15 \\
abcde = -1
\]

4. Define \( f(n) \) for \( n \) a positive integer by \( f(1) = 3 \) and \( f(n + 1) = 3^{f(n)} \). What are the last two digits of \( f(2012) \)?

5. Define a sequence \( (a_n) \) for \( n \) a positive integer inductively by \( a_1 = 1 \) and
\[
a_n = \prod_{\substack{d | n \\{ d \neq 1 \}}} a_d.
\]

Thus \( a_2 = 2, a_3 = 3, a_4 = 2, \) etc. Find \( a_{999000} \).

6. Let \( A_1, A_2, A_3 \) be \( 2 \times 2 \) matrices with entries in \( \mathbb{C} \) (the complex numbers). Let \( \text{tr} \) denote the trace of a matrix (so \( \text{tr} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = a + d \)). Suppose \( \{A_1, A_2, A_3\} \) is closed under matrix multiplication (i.e., given \( i, j \), there exists \( k \) such that \( A_i A_j = A_k \)), and \( \text{tr}(A_1 + A_2 + A_3) \neq 3 \). Prove that there exists \( i \) such that \( A_i A_j = A_j A_i \) for all \( j \) (here \( i, j \) are 1, 2, or 3).

7. Let \( d \) be a positive integer and let \( A \) be a \( d \times d \) matrix with integer entries. Suppose \( I + A + A^2 + \cdots + A^{100} = 0 \) (where \( I \) denotes the \( d \times d \) identity matrix, so \( I \) has 1’s on the main diagonal, and 0 denotes the zero matrix, which has entries all
0). Determine the positive integers \( n \leq 100 \) for which \( A^n + A^{n+1} + \cdots + A^{100} \) has determinant \( \pm 1 \).

(8) For \( n \) a positive integer, define \( f_1(n) = n \) and then for \( i \) a positive integer, define \( f_{i+1}(n) = f_i(n)^{f_i(n)} \). Determine \( f_{100}(75) \) mod 17 (i.e., determine the remainder after dividing \( f_{100}(75) \) by 17, an integer between 0 and 16). Justify your answer.

(9) Prove that \( \cos(\pi/7) \) is a root of the equation \( 8x^3 - 4x^2 - 4x + 1 = 0 \), and find the other two roots.

(10) Let \( \triangle ABC \) be a triangle with sides \( a, b, c \) and corresponding angles \( A, B, C \) (so \( a = BC \) and \( A = \angle BAC \) etc.). Suppose that \( 4A + 3C = 540^\circ \). Prove that \( (a - b)^2(a + b) = bc^2 \).

(11) Let \( A, B \) be two circles in the plane with \( B \) inside \( A \). Assume that \( A \) has radius 3, \( B \) has radius 1, \( P \) is a point on \( A \), \( Q \) is a point on \( B \), and \( A \) and \( B \) touch (i.e., are tangent) so that \( P \) and \( Q \) are the same point. Suppose that \( A \) is kept fixed and \( B \) is rolled once around the inside of \( A \) so that \( Q \) traces out a curve starting and finishing at \( P \). What is the area enclosed by this curve?

(12) Define a sequence by \( a_1 = 1, a_2 = 1/2, \) and \( a_{n+2} = a_{n+1} - \frac{a_n a_{n+1}}{2} \) for \( n \) a positive integer. Find \( \lim_{n \to \infty} na_n \).