

- 13.6. On the unit circle centered at the origin ($x^2 + y^2 = 1$) we pick three points at random. We cut the circle into three arcs at those points. What is the expected length of the arc containing the point $(1, 0)$?
- 13.7. In a laboratory a handful of thin 9-inch glass rods had one tip marked with a blue dot and the other with a red. When the laboratory assistant tripped and dropped them onto the concrete floor, many broke into three pieces. For these, what was the average length of the fragment with the blue dot?
- 13.8. We pick n points at random on a circle. What is the probability that the center of the circle will be in the convex polygon with vertices at those points?

14. Miscellany.

- 14.1. (Putnam 1986) What is the units (i.e., rightmost) digit of $\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor$?
- 14.2. (IMO 1975) Prove that there are infinitely many points on the unit circle $x^2 + y^2 = 1$ such that the distance between any two of them is a rational number.
- 14.3. (Putnam 1988) Prove that if we paint every point of the plane in one of three colors, there will be two points one inch apart with the same color. Is this result necessarily true if we replace "three" by "nine"?
- 14.4. Imagine an infinite chessboard that contains a positive integer in each square. If the value of each square is equal to the average of its four neighbors to the north, south, west and east, prove that the values in all the squares are equal.
- 14.5. (Putnam 1990) Consider a paper punch that can be centered at any point of the plane and that, when operated, removes precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?
- 14.6. (Putnam 1984) Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

- 14.7. (Putnam 1974) Call a set of positive integers "conspiratorial" if no three of them are pairwise relatively prime. What is the largest number of elements in any conspiratorial subset of integers 1 through 16?
- 14.8. (Putnam 1984) Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F ,

- (i) $x * z = y * z$ implies $x = y$ (right cancellation holds), and
 - (ii) $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).
- 14.9. (Putnam 1995) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x . Prove that for any two partitions π and π' , there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S .]
- 14.10. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as average of two distinct elements of S . For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S ?
- 14.11. Suppose that a sequence $a_1, a_2, a_3, \dots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.
- 14.12. On a table there is a row of fifty coins, of various denominations (the denominations could be of any values). Alice picks a coin from one of the ends and puts it in her pocket, then Bob chooses a coin from one of the ends and puts it in his pocket, and the alternation continues until Bob pockets the last coin. Prove that Alice can play so that she guarantees at least as much money as Bob.
- 14.13. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function such that $f \circ f$ has a fixed point, i.e., there is some real number x_0 such that $f(f(x_0)) = x_0$. Prove that f also has a fixed point.