

PUTNAM PRACTICE PROBLEMS
FOR FRIDAY, NOVEMBER 4, 2016

Integrals

① Evaluate $\int_0^{\infty} \frac{dx \ln x}{x^2+1}$

② Evaluate $\int_0^1 \frac{x-1}{\ln x} dx$

③ Evaluate $\int_2^4 \frac{dx \sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$

④ Evaluate $\int_0^1 dx \frac{x^4(1-x)^4}{1+x^2}$

⑤ Evaluate $\int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \sin(x^2+y^2)$

⑥ Evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$

⑦ If $f(x) = \int_1^x \frac{\ln t}{1+t} dt$, evaluate $f(x) + f(\frac{1}{x})$

⑧ Evaluate $\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$

⑨ Evaluate $\int_0^{\pi} dx \frac{x \sin x}{1 + \cos^2 x}$

Hint: Show that $\int_0^{\pi} dx x f(\sin x) = \frac{\pi}{2} \int_0^{\pi} dx f(\sin x)$

⑩ Evaluate $\int_0^{\infty} \frac{dx}{\sqrt{x}} e^{-1985(x+\frac{1}{x})}$

Hint: If $f(x)$ is real and continuous, show that

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} f(x - \frac{1}{x}) dx$$

⑪ Evaluate $\int_0^{\pi/2} dx \ln(\sin x)$

⑫ Evaluate $\int_0^{\infty} \frac{dx}{x} [\arctan(\pi x) - \arctan(x)]$

⑬ Evaluate $\int_0^{2\pi} d\theta (a \cos \theta + b \sin \theta)^{1/3}$

⑭ Find the volume of

(a) $x^2 + y^2 \leq 1, x^2 + z^2 \leq 1$ in $\mathbb{R}^3$

(b) $x^2 + y^2 \leq 1, x^2 + z^2 \leq 1, y^2 + z^2 \leq 1$ in $\mathbb{R}^3$

(c) $x^2 + y^2 + z^2 + w^2 \leq 1$ in $\mathbb{R}^4$

⑮ Evaluate $\int_0^1 dx \int_0^{1-x} dy e^{(x+y)^2}$

⑯ Let $I(x) = \int_x^{\infty} e^{-s^2} ds$

Evaluate $\int_0^{\infty} dx e^{x^2} [I(x)]^2$

⑰ In the (x, y) plane, if R is the set of points inside and on a convex polygon, let $D(x, y)$ be the distance from (x, y) to the nearest point of R .

Show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-D(x, y)} = a + bL + cA,$

where $A = \text{area of } R, L = \text{perimeter of } R.$

Find $a, b,$ and $c.$

⑱ EVALUATE $\int \frac{x^2 dx \ln x}{x^2 - 2x + 2}$